COMPOSITIO MATHEMATICA

W. H. SCHIKHOF

Borel's theorem for C^∞-functions on a non-archimedean valued field

Compositio Mathematica, tome 55, nº 3 (1985), p. 289-294
<http://www.numdam.org/item?id=CM_1985__55_3_289_0>

Compositio Mathematica 55 (1985) 289–294.

BOREL'S THEOREM FOR C^∞-FUNCTIONS ON A NON-ARCHIMEDEAN VALUED FIELD

W.H. Schikhof

In this note we prove the following theorem (for explanations see below).

THEOREM 1: *Let K be a non-archimedean nontrivially valued field. Let λ_0, λ_1, $\lambda_2, \dots$ be any sequence in K. Then there exists a C^∞-function f: $K \to K$ such that $D_n f(0) = \lambda_n$ for all $n \in \{0, 1, 2, \dots\}$.*

DEFINITION ([1], [2]): For each $n \in \mathbb{N}$ let $\nabla^n K := \{(\xi_1, \xi_2, \dots \xi_n) \in K^n$: if $i \neq j$ then $\xi_i \neq \xi_j\}$. Let f: $K \to K$.

Define $\Phi_n f$: $\nabla^{n+1} K \to K$ inductively as follows. $\Phi_0 f := f$ and, for $n \in \mathbb{N}$,

$$\Phi_n f(\xi_1, \xi_2, \dots, \xi_{n+1}) := -(\xi_1 - \xi_2)^{-1}(\Phi_{n-1} f(\xi_2, \xi_3, \xi_4, \dots, \xi_{n+1})$$
$$- \Phi_{n-1} f(\xi_1, \xi_3, \xi_4, \dots, \xi_{n+1}))$$
$$((\xi_1, \xi_2, \dots, \xi_{n+1}) \in \nabla^{n+1} K).$$

Let $n \in \mathbb{N} \cup \{0\}$. f is a *C^n-function* if $\Phi_n f$ can be extended to a continuous function $\overline{\Phi}_n f$ on K^{n+1}. For such a C^n-function we set (for $n \in \mathbb{N} \cup \{0\}$, $x, y \in K$)

$$D_n f(x) := \overline{\Phi}_n f(x, x, \dots, x)$$

$$T_{n+1} f(x, y) := \sum_{j=0}^{n} (x-y)^j D_j f(y)$$

$$R_{n+1} f(x, y) := f(x) - T_{n+1} f(x, y).$$

f is a *C^∞-function* if f is a C^n-function for each $n \in \mathbb{N} \cup \{0\}$.
REMARK 1: The "ordinary" definition of a C^n-function (f is n times differentiable and $f^{(n)}$ is continuous) does not lead to nice properties. The stronger definition of above restores somewhat the damage caused by the absence of the Mean Value Theorem.

Remark 2: The following statements can be obtained by elementary and straightforward arguments (see [2]).

(i) A C^{n+1}-function is also a C^n-function. (Locally) analytic functions are C^∞-functions, "f is a C^n-function" is a local property.

(ii) (Taylor) if f is a C^n-function then

$$R_n f(x, y) = (x-y)^n \overline{\Phi}_n f(x, y, y, \ldots, y) \quad (x, y \in K)$$

so that

$$\lim_{(x, y) \to (a, a)} \frac{R_n f(x, y)}{(x-y)^n} = D_n f(a) \quad (a \in K)$$

Further, we have $n!\, D_n f = f^{(n)}$.

(iii) (Polynomials). Let $\ast$ denote the function $x \mapsto x$ $(x \in K)$. Then $D_j \ast^n = \binom{n}{j} \ast^{n-j} (0 \leqslant j \leqslant n)$. For a polynomial function f defined by $f(x) = \lambda_0 + \lambda_1 x + \ldots \lambda_m x^m (x \in K)$ we have $D_j f(0) = \lambda_j$ $(0 \leqslant j \leqslant m)$ and $\Phi_{m+1} f = 0$.

Remark 3: Observe that the characteristic of K is allowed to be $\neq 0$. For this reason we prefer to work with D_n rather than the less informative n-th derivative.

The next theorem reduces the number of variables involved.

Theorem 2. ([2], 10.7): *Let K be as in Theorem 1, let $n \in \mathbb{N}$. The following conditions on a C^{n-1}-function f: $K \to K$ are equivalent.*

(α) *f is a C^n-function.*

(β) *For each $a \in K$, $\lim_{(x, y) \to (a, a)} (x-y)^{-n} R_n f(x, y)$ exists.*

For the proof of Theorem 1 we need two estimates on polynomial functions $K \to K$.

Lemma 1: *Let Q: $x \mapsto \lambda_0 + \lambda_1 x + \ldots + \lambda_s x^s$ be a polynomial function. Then*

$$|D_j Q(x)| \leqslant \max(|\lambda_0|, |\lambda_1|, \ldots, |\lambda_s|)$$

$$(j \in \mathbb{N} \cup \{0\}, \ x \in K, \ |x| \leqslant 1).$$

If, in addition, $\lambda_0 = \lambda_1 = \ldots = \lambda_m = 0$ for some $m \leqslant s$ then

$$|D_j Q(x)| \leqslant |x|^{m+1-j} \max(|\lambda_{m+1}|, \ldots, |\lambda_s|)$$

$$(j \in \mathbb{N} \cup \{0\}, \ j \leqslant m+1, \ x \in K, \ |x| \leqslant 1).$$

PROOF: Everything follows from the formula

$$D_j Q(x) = \sum_{i=j}^{s} \lambda_i \binom{i}{j} x^{i-j}.$$

LEMMA 2: *Let* P: $x \mapsto \lambda_0 + \lambda_1 x + \ldots + \lambda_m x^m$ *be a polynomial function and let* $n \in \mathbb{N} \cup \{0\}$, $n \leqslant m$. *Then*

$$|(x-y)^{-n} R_n P(x, y) - \lambda_n| \leqslant \max(|x|, |y|) \cdot$$

$$\max(|\lambda_0|, \ldots, |\lambda_m|) \ (x, y \in K, |x| \leqslant 1, |y| \leqslant 1, x \neq y).$$

PROOF: We have $\Phi_{m+1} P = 0$ so that

$$R_{m+1} P(x, y) = (x-y)^{m+1} \overline{\Phi}_{m+1} P(x, y, y, \ldots, y) = 0$$

for all $x, y \in K$. Hence

$$P(x) = T_{m+1} P(x, y)$$

and

$$R_n P(x, y) = P(x) - T_n P(x, y) = T_{m+1} P(x, y) - T_n P(x, y)$$

$$= \sum_{j=n}^{m} (x-y)^j D_j P(y).$$

We find for $x, y \in K$, $|x| \leqslant 1$, $|y| \leqslant 1$, $x \neq y$ that

$$(*) \quad (x-y)^{-n} R_n P(x, y) - \lambda_n = D_n P(y) - \lambda_n + \sum_{j=n+1}^{m} (x-y)^{j-n} D_j P(y)$$

Now

$$D_n P(y) = \lambda_n + \lambda_{n+1} \binom{n+1}{n} y + \ldots + \lambda_m \binom{m}{n} y^{m-n}$$

so that

$$|D_n P(y) - \lambda_n| \leqslant |y| (\max |\lambda_0|, \ldots, |\lambda_m|).$$

Further, by Lemma 1 we have

$$|D_j P(y)| \leqslant \max(|\lambda_0|, \ldots, |\lambda_m|)$$

so

$$\left| \sum_{j=n+1}^{m} (x-y)^{j-n} D_j P(y) \right| \leqslant |x-y| \cdot \max(|\lambda_0|, \dots, |\lambda_m|).$$

Together with (*) this proves Lemma 2.

PROOF OF THEOREM 1: Let

$$r_n := \{ n(\max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_n|) + 1) \}^{-1} (n \in \mathbb{N}).$$

Then

$$r_1 > r_2 > \dots, \lim_{n\to\infty} r_n = 0$$

and

$$r_n \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_n|) \leqslant n^{-1}$$

for each n. Define f: $K \to K$ as follows.

$$f(x) := \begin{cases} 0 \quad \text{if } |x| > r_1 \\ P_n(x) := \lambda_0 + \lambda_1 x + \dots + \lambda_n x^n \\ \quad \text{if } n \in \mathbb{N},\ r_{n+1} < |x| \leqslant r_n \\ \lambda_0 \text{ if } x = 0 \end{cases}$$

Clearly f is a C^∞-function at a for each $a \in K$, $a \neq 0$. If $r_{m+1} < |x| \leqslant r_m$ for some $m \in \mathbb{N}$ then

$$\begin{aligned} |f(x) - \lambda_0| &= |P_m(x) - \lambda_0| = |\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_m x^m| \\ &\leqslant |x| \max(|\lambda_1|, |\lambda_2|, \dots, |\lambda_m|) \\ &\leqslant r_m \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_m|) \leqslant m^{-1}. \end{aligned}$$

We have proved the case $n = 0$ of the following statement. For each $n \in \mathbb{N} \cup \{0\}$ the function f is C^n and $D_j f(0) = \lambda_j$ $(0 \leqslant j \leqslant n)$. To prove the step from $n-1$ to n it suffices, according to Theorem 2 and Remark 2 (ii), to show that

$$\lim_{(x,y)\to(0,0)} (x-y)^{-n} R_n f(x, y) = \lambda_n.$$

In its turn, this statement follows, by continuity of R_n, from "if

$$k \geqslant n, 0 < |x| \leqslant r_k, 0 < |y| \leqslant r_k, x \neq y$$

then

$$|(x-y)^{-n}R_nf(x,\ y)-\lambda_n|<k^{-1}\text{"}$$

and it is the latter statement that we are intended to prove. So let

$$r_{m+1}<|x|\leqslant r_m,\ r_{s+1}<|y|\leqslant r_s$$

for some $m,\ s\geqslant k$. Then

$$R_nf(x,\ y)=P_m(x)-T_nP_s(x,\ y).$$

We consider two cases.

(i) $s\geqslant m$. Writing $P_s=P_m+Q$ where $Q(t)=\lambda_{m+1}t^{m+1}+\ldots+\lambda_st^s$ we obtain

$$|(x-y)^{-n}R_nf(x,\ y)-\lambda_n|=|(x-y)^{-n}(P_m(x)-T_nP_m(x,\ y))$$

$$-\lambda_n+(x-y)^{-n}(T_nP_m(x,\ y)-T_nP_s(x,\ y))|$$

$$\leqslant|(x-y)^{-n}R_nP_m(x,\ y)-\lambda_n|\vee|x-y|^{-n}|T_nQ(x,\ y)|.$$

By Lemma 2 the first part is $\leqslant\max(|x|,\ |y|)\cdot\max(|\lambda_0|,$ $|\lambda_1|,\ldots,|\lambda_m|)\leqslant r_m\max(|\lambda_0|,\ |\lambda_1|,\ldots,|\lambda_m|)\leqslant m^{-1}\leqslant k^{-1}$.
To estimate the second part we may assume that $s>m$. Then $|x-y|>$ $|y|$. Using this, the definition of T_n, and Lemma 1, we obtain

$$|x-y|^{-n}|T_nQ(x,\ y)|=|\sum_{j=0}^{n-1}(x-y)^{j-n}D_jQ(y)|$$

$$\leqslant\max_{0\leqslant j<n}|x-y|^{j-n}|y|^{m+1-j}\cdot\max(|\lambda_0|,\ldots,|\lambda_s|)$$

$$\leqslant|y|^{m+1-n}\max(|\lambda_0|,\ldots,|\lambda_s|\leqslant r_s\max(|\lambda_0|,\ldots,|\lambda_s|)\leqslant s^{-1}\leqslant k^{-1}.$$

(ii) $s<m$. Then set $P_m=P_s+Q$ where $Q(t)=\lambda_{s+1}t^{s+1}+\ldots+\lambda_mt^m$. We find

$$|(x-y)^{-n}R_nf(x,\ y)-\lambda_n|=|(x-y)^{-n}(P_s(x)-T_nP_s(x,\ y))$$

$$-\lambda_n+(x-y)^{-n}(P_m(x)-P_s(x))|$$

$$\leqslant|(x-y)^{-n}R_nP_s(x,\ y)-\lambda_n|\vee|(x-y)^{-n}Q(x)|.$$

By Lemma 2 the first part is $\leqslant \max(|x|, |y|) \cdot \max(|\lambda_0|, \ldots, |\lambda_s|) \leqslant r_s \max(|\lambda_0|, \ldots, |\lambda_s|) \leqslant s^{-1} \leqslant k^{-1}$. For the second part observe that $|x - y| > |x|$ and $|Q(x)| = |\lambda_{s+1} x^{s+1} + \ldots + \lambda_m x^m| \leqslant |x|^{s+1} \max(|\lambda_0|, \ldots, |\lambda_m|)$. Then

$$|(x-y)^{-n} Q(x)| \leqslant |x|^{s+1-n} \max(|\lambda_0|, \ldots, |\lambda_m|)$$

$$\leqslant |x| \max(|\lambda_0|, \ldots, |\lambda_m|)$$

$$\leqslant r_m \max(|\lambda_0|, \ldots, |\lambda_m|) \leqslant m^{-1} \leqslant k^{-1}.$$

References

[1] D. Barsky: Fonctions k-lipschitziennes sur un anneau local et polynômes à valeurs entières. *Bull. Soc. Math. Fr.* 101, (1973) 397–411.

[2] W.H. Schikhof: Non-archimedean calculus (Lecture notes). Report 7812, Mathematisch Instituut, Katholieke Universiteit, Nijmegen (1978).

(Oblatum 20-VII-1983)

Mathematisch Instituut
Katholieke Universiteit
Toernooiveld
Nijmegen
The Netherlands