COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 55, nº 3 (1985), p. 295-302
<http://www.numdam.org/item?id=CM_1985__55_3_295_0>

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REMARKS ON CHAIN CONDITIONS IN PRODUCTS

Stevo Todorčević

Introduction

We say that a partially ordered set $\mathfrak{O}$ satisfies the κ-chain condition if every set of pairwise incompatible elements of $\mathfrak{O}$ has size $< \kappa$. A topological space X is said to satisfy the κ-c.c. if every family of pairwise disjoint open subsets of X has size $< \kappa$. In this note we give several remarks on the well-known problem which asks for which cardinals κ the κ-chain condition is a productive property (see [6; §3]). This problem for the case $\kappa = \aleph_1$ was first asked by E. Marczewski ([14]), and later by D. Kurepa ([11], [12]) in a more general form. Kurepa [11] showed that the countable chain condition (i.e., the $\aleph_1$-c.c.) of a Suslin continuum is not a productive property. He also showed ([13]) that any product of κ^+-c.c. spaces satisfies the $(2^\kappa)^+$-*c.c.* The first examples of non-productive κ^+-*c.c.* posets, assuming $2^\kappa = \kappa^+$, were constructed by F. Galvin and R. Laver ([8]). In [7], W. Fleissner showed that it is consistent with ZFC that $2^{\aleph_0}$ is large and there exists a c.c.c. space X such that X^2 does not satisfy the $2^{\aleph_0}$-c.c. In this note we show without additional set-theoretical assumptions that for class-many cardinals κ the κ-chain condition is not a productive property. In particular, we show that the $\mathrm{cf}2^{\aleph_0}$-c.c. is not productive. This result was announced in [20] where the same result for the $2^{\aleph_0}$-c.c. was incorrectly claimed. Our observation uses "entangled" linear orders, and gives a quite general method for constructing non-productive κ-c.c. posets.

§1. Sierpiński's construction

In this section we generalize a classical construction of rigid linear suborderings of $\mathbb{R}$ given by W. Sierpiński [19] from the case $n = 1$ to the case of any finite n. Using a simple diagonalization argument Sierpiński ([19]) constructed a one-to-one sequence $E = \{ r_\alpha : \alpha < 2^{\aleph_0} \}$ of real numbers with the following property:

(1) For any continuous function f from a G_δ subset of $\mathbb{R}$ into $\mathbb{R}$ there is an $\alpha < 2^{\aleph_0}$ such that

$$\forall \beta \geqslant \alpha \;\; f''\{ r_\gamma : \gamma < \beta \} \cap E \subseteq \{ r_\gamma : \gamma < \beta \}.$$

Using the well-known Lavrentiev Extension Theorem ([10; §35]) he showed that (1) implies the following property of the set E:

(2) If f is a homeomorphism between two sets of reals, then

$$\{\alpha < 2^{\aleph_0} : r_\alpha \in \operatorname{dom}(f) \& f(r_\alpha) \neq r_\alpha\} \text{ is not cofinal in } 2^{\aleph_0}.$$

Sierpiński's result was motivated by a problem of M. Fréchet about the number of non-homeomorphic topological spaces. We refer the reader to §§35 and 40 of [10] for further information and generalizations of this result. In this note we shall be interested in the following property of the set E which easily follows from (2) and which has been quite often used in the theory of uncountable order types (see [4]):

(3) If f is an one-to-one monotonic function from a set of reals into the reals, then $\{\alpha < 2^{\aleph_0} : r_\alpha \in \operatorname{dom}(f) \& f(r_\alpha) \neq r_\alpha\}$ is not confinal in $2^{\aleph_0}$.

Since a function may be regarded as a set of ordered pairs, the statement (3) suggests a stronger statement where the pairs are replaced by the n-tuples from E. It turns out that in order to get this stronger property one has only replace (1) by a stronger statement which includes any continuous function f from a G_δ subset of $\mathbb{R}^n$ into $\mathbb{R}$ for any $n < \omega$. Since later we intend to give some applications of this result, let us prove it in the following more general form. But first we need some definitions. Let L be an infinite linearly ordered set and let $\{r_\alpha : \alpha < \theta\}$ be a one-to-one enumeration of L. Then for any $n < \omega$ and $x \in L^n$ by $D(x)$ we denote the set $\{\alpha < \theta : \exists i < n\ x_i = r_\alpha\}$. We say that $A \subseteq L^n$ is cofinal (in θ) iff $\forall \alpha < \theta\ \exists x \in A\ \alpha < D(x)$.

THEOREM 1: *Assume L is a linear ordering of size* 2^λ *with a dense subset D of size* $\leqslant \lambda$. *Then there is a one-to-one sequence* $E = \{r_\alpha : \alpha < 2^\lambda\}$ *of elements of L such that*:

(4) *For every* $n < \omega$, *for any cofinal set* $A \subseteq E^n$ *consisting of one-to-one n-tuples, and for every* $s \in {}^n2$, *there exist* $x, y \in A$ *such that*

$$\forall i < n\ (x_i < y_i \leftrightarrow s_i = 0).$$

PROOF: Let us assume that L is a dense linear ordering and let $\mathbb{K}$ be the Dedekind completion of L. The following fact is an easy generalization of a similar fact for $\mathbb{R}$ ([10; § 35]).

LEMMA 1: *Let f be a continuous function from a subset of* $\mathbb{K}^n$ *into* $\mathbb{K}$ *where* $n < \omega$. *Then f can be continuously extended to a* G_λ *subset of* $\mathbb{K}^n$.

PROOF: Let $A = \operatorname{dom}(f)$. For $p \in \mathbb{K}^n$ we define

$$\omega(p) = \bigcap \{\overline{f''(A \cap I)} : I \text{ is open in } \mathbb{K}^n \& p \in I\}.$$

Thus for each $p \in \mathbb{K}^n$, $\omega(p)$ is a (possibly empty) compact subset of $\mathbb{K}^n$. Let $A^* = \{p \in \mathbb{K}^n : |\omega(p)| = 1\}$. Then $A \subseteq A^*$ and f extends continuously on A^*. Using the fact that $\mathbb{K}$ has a dense subset of size $\leqslant \lambda$ one easily shows that A^* is a G_λ subset of $\mathbb{K}^n$.

Since there exist only 2^λ continuous functions from G_λ subsets of $\mathbb{K}^n$ into $\mathbb{K}$, $(n < \omega)$, we can easily choose a one-to-one sequence $E = \{r_\alpha : \alpha < 2^\lambda\} \subseteq L$ with the following property:

(5) For any continuous function f from a G_λ subset of $\mathbb{K}^n$ into $\mathbb{K}$, $(n < \omega)$ there is an $\alpha < 2^\lambda$ such that

$$\forall \beta \geqslant \alpha \;\; f''\{r_\gamma : \gamma < \beta\}^n \cap E \subseteq \{r_\gamma : \gamma < \beta\}.$$

We shall show that this E satisfies (4). Assume not, and let m be the minimal $n < \omega$ for which (4) fails. Clearly $m > 1$. Let $A \subseteq E^m$ be a cofinal set of disjoint one-to-one m-tuples and let $s \in {}^m 2$ be such that no $x, y \in A$ satisfy $\forall i < m \; (x_i < y_i \Leftrightarrow s_i = 0)$. Using a permutation of coordinates of elements of A and of s we may assume that

$$(6) \;\; \forall x \in A \; \left(x = \langle r_{\alpha_0}, \dots, r_{\alpha_{m-1}} \rangle \Rightarrow \alpha_0 < \dots < \alpha_{m-1}\right)$$

Let $B = \{x \restriction (m-1) : x \in A\} \subseteq E^{m-1}$. Define $f : B \to \mathbb{K}$ by $f(z) = r$ iff $\exists x \in A \; x = z\hat{\ }r$. By (5), (6) and Lemma 1 it follows directly that f cannot be continuous on a cofinal subset of B. So the following lemma gives a contradiction. For $p \in K^{m-1}$, $\omega(p)$ is defined for our f and B as in the proof of Lemma 1.

LEMMA 2: *The set* $B_0 = \{z \in B : |\omega(z)| \geqslant 2\}$ *is not cofinal in* 2^λ.

PROOF: Otherwise, by going to a cofinal subset of B_0 and by symmetry, we may assume that there is a $d \in D$ such that

$$(7) \;\; \forall z \in B_0 \; \exists r \in \omega(z) \;\; f(z) < d < r.$$

Assume for definiteness that $s_{m-1} = 0$; the case $s_{m-1} = 1$ is considered similarly. By the minimality of m we can find $u, v \in B_0$ so that $\forall i < m-1 \; (u_i < v_i \Leftrightarrow s_i = 0)$. Let

$$I = \left\{z \in \mathbb{K}^{m-1} : \forall i < m-1 \; (u_i < z_i \Leftrightarrow s_i = 0)\right\}.$$

Then I is an open subset of $\mathbb{K}^{m-1}$ which contains v. Note that by the choice of A and s it follows that $\forall z \in I \cap B$, $f(z) < f(u) < d$. Hence $\omega(v) \subseteq \overline{f''B \cap I} < d$. But this contradicts (7) and finishes the proof.

§2. Entangled linear orderings

This section begins with a slight generalization of a notion of Avraham, Rubin and Shelah ([1], [2]) and ends with some applications of the result of §1 which were mentioned in the introduction.

Let L be an infinite linear ordering, let κ be an infinite cardinal and let $n < \omega$. Then by $(L)^n$ we denote the set of all $\langle x_0, \ldots, x_{n-1} \rangle \in L^n$ such that $x_0 < \ldots < x_{n-1}$. We say that L is (κ, n)-entangled iff for every $A \subseteq (L)^n$ of size κ and for every $s \in {}^n2$ there exist $x, y \in A$ such that $\forall i < n\ (x_i < y_i \Leftrightarrow s_i = 0)$. L is κ-entangled iff L is (κ, n)-entangled for every $n < \omega$. L is an entangled linear ordering iff L is κ-entangled for $\kappa = \aleph_1$.

Note that every linear order is $(\kappa, 0)$- and $(\kappa, 1)$-entangled. Note also that L is $(\kappa, 2)$-entangled iff for every one-to-one monotonic function from a subset of L into L, we have that $|\{r \in \mathrm{dom}(f) \colon f(r) \neq r\}| < \kappa$. Clearly if L is $(\kappa, 2)$-entangled then every family of disjoint non-trivial intervals of L is of size $< \kappa$. If L is $(\kappa, 3)$-entangled then L moreover has a dense subset of size $< \kappa$. Thus every $(\aleph_1, 3)$-entangled linear ordering is isomorphic with a set of reals. The next easy folklore result shows that uncountable $\aleph_1$-entangled sets of reals can exist in certain models of set theory. On the other hand, in the model of Baumgartner [3] there is no uncountable $(\aleph_1, 2)$-entangled linear ordering.

THEOREM 2: *If E is any set of Cohen or random reals, then E is $\aleph_1$-entangled.*

To state the result of §1 using the new terminology let $\mathrm{ded}(\lambda, \theta)$ denote the fact that there is a linear ordering of size θ with a dense subset of size $\leqslant \lambda$. Sierpiński [19] showed that $\mathrm{ded}(\lambda, \lambda^+)$ always holds and that $\forall \theta < \lambda\ 2^\theta \leqslant \lambda$ implies $\mathrm{ded}(\lambda, 2^\lambda)$. Thus $\mathrm{ded}(\lambda, 2^\lambda)$ holds if, for example, $2^\lambda = \lambda^+$ or if λ is a strong limit cardinal. In [16], W. Mitchell constructed a model of ZFC in which $\mathrm{ded}(\aleph_1, 2^{\aleph_1})$ fails. Theorem 1 can also be written in the following form.

THEOREM 3: *Assume* $\mathrm{ded}(\lambda, 2^\lambda)$. *Then for every $\kappa \leqslant 2^\lambda$ with $cf\ \kappa = cf\ 2^\lambda$, there is an κ-entangled linear ordering of size κ which moreover has a dense subset of size $\leqslant \lambda$.*

COROLLARY 4: *For every $\kappa \leqslant 2^{\aleph_0}$ with $cf\ \kappa = cf\ 2^{\aleph_0}$ there is an κ-entangled set of reals of size κ.*

COROLLARY 5: (*Sierpiński*) *For every $\kappa \leqslant 2^{\aleph_0}$ with $cf\ \kappa = cf\ 2^{\aleph_0}$ there is an $(\kappa, 2)$-entangled set of reals of size κ.*

The next proposition contains our basic observation which connects entangled linear orders and chain conditions. We shall state later a slightly stronger result of this kind.

THEOREM 6: *Let κ be a regular infinite cardinal and let L be a κ-entangled linear order of size $\lambda \geqslant \kappa$. Then there exist partially ordered sets $\mathfrak{O}_0$ and $\mathfrak{O}_1$ such that $\mathfrak{O}_0$ and $\mathfrak{O}_1$ satisfy the* κ-c.c. *but $\mathfrak{O}_0 \times \mathfrak{O}_1$ does not satisfy the* λ-c.c.

PROOF: Let $\{r_\alpha : \alpha < \lambda\}$ be a one-to-one enumeration of L and let $E = \{\langle r_\alpha, r_{\alpha+1}\rangle : \alpha \text{ even } < \lambda\}$. We consider E as a subposet of $L \times L$ with the product ordering. Let

$$\mathfrak{O}_0(E) = \{ p \in [E]^{<\aleph_0} : p \text{ is a chain of } E\},$$

$$\mathfrak{O}_1(E) = \{ p \in [E]^{<\aleph_0} : p \text{ is an antichain of } E\},$$

considered as posets under the ordering $\supseteq$. Let A be a subset of $\mathfrak{O}_0(E)$ or $\mathfrak{O}_1(E)$ of size κ. By the standard Δ-system argument we may assume that the elements of A are disjoint and of the same cardinality $n < \omega$. Since L is κ-entangled, L contains a dense subset D of size $< \kappa$. Every member of A can be separated by a set of $2n$ intervals with end-points in D, and by regularity of κ we may assume that the separating set is the same for each $p \in A$. Now a simple application of the $(\kappa, 2n)$-entangledness of L gives two compatible members of A. The product $\mathfrak{O}_0(E) \times \mathfrak{O}_1(E)$ is not λ-c.c. since $\{\langle\{e\}, \{e\}\rangle : e \in E\}$ is a pairwise incompatible subset of $\mathfrak{O}_0(E) \times \mathfrak{O}_1(E)$. This completes the proof.

Note the following direct way of getting topological spaces which satisfy the conclusion of Theorem 6. Let X_0 be the set of all chains of E and let X_1 be the set of all antichains of E. We consider X_0 and X_1 as subspaces of $\{0, 1\}^E$ under the standard identification via characteristic functions. Then X_0 and X_1 are compact Hausdorff κ-c.c. spaces such that $X_0 \times X_1$ is not λ-c.c. It is clear that the proof of Theorem 6 also shows that each (finite) power of $\mathfrak{O}_0(E)$ and of $\mathfrak{O}_1(E)$ is again a κ-c.c. poset.

By Theorems 3 and 6 it follows directly that ded$(\lambda, 2^\lambda)$ implies the existence of two cf 2^λ-c.c. posets $\mathfrak{O}_0$ and $\mathfrak{O}_1$ such that $\mathfrak{O}_0 \times \mathfrak{O}_1$ is not cf 2^λ-c.c. Thus in particular, the cf $2^{\aleph_0}$-c.c. is not productive. We shall later state a more general result of this kind, and in order to do this we need some definitions of Galvin [8].

Let $\lambda \geqslant \kappa \geqslant \aleph_0$ be cardinals and let $H \subseteq [\lambda]^2$. Then we let $\mathfrak{O}(\lambda, H) = \{F \in [\lambda]^{<\aleph_0} : [F]^2 \subseteq H\}$; $\mathfrak{O}(\lambda, H)$ is partially ordered by $\supseteq$. A set $K \subseteq [\lambda]^2$ is called κ-big if, given any $n < \omega$ and any $H_1, \ldots, H_n \subseteq [\lambda]^2$, if $K \subseteq H_1 \cap \ldots \cap H_n$, and if each of H_ι is the union of less than κ

rectangles, then $\mathfrak{D}(\lambda, H_1) \times \ldots \times \mathfrak{D}(\lambda, H_n)$ satisfies the κ-c.c. A set $K \subseteq [\lambda]^2$ is big iff K is λ-big. Galvin [8] showed that $2^\kappa = \kappa^+$ implies the existence of κ^+ disjoint big subsets of $[\kappa^+]^2$ and that MA implies the existence of $\aleph_0$ disjoint big subsets of $[2^{\aleph_0}]^2$. The next proposition shows that entangled linear orderings can also be used in constructing disjoint big subsets of $[\lambda]^2$.

THEOREM 7: *Let κ be regular and infinite and let L be a κ-entangled linear order of size $\lambda \geqslant \kappa$. Then there exist $\aleph_0$ pairwise disjoint κ-big subsets of* $[\lambda]^2$.

PROOF: Let $\{r_\alpha : \alpha < \lambda\}$ be a one-to-one enumeration of L, and let Λ denote the set of all limit ordinals $< \lambda$. For α, $\beta \in \Lambda$ and $n < \omega$, we define

$$\{\alpha, \beta\} \in K_n \text{ iff } r_{\alpha+n+1} > r_{\beta+n+1} \& \forall i \leqslant n \;\; r_{\alpha+i} < r_{\beta+i}.$$

It is clear that the proof of Theorem 6 also shows that K_n, $(n < \omega)$ is a family of $\aleph_0$ disjoint κ-big subsets of $[\Lambda]^2$. This finishes the proof.

COROLLARY 8: *Assume* ded$(\lambda, 2^\lambda)$. *Then there exist $\aleph_0$ pairwise disjoint big subsets of* $[cf\ 2^\lambda]^2$.

Note that the proof of Theorem 7 also gives the following proposition where κ is not necessarily a regular cardinal.

THEOREM 9: *If there exists a κ-entangled linear ordering of size $\lambda \geqslant \kappa$, then* $\lambda \not\to [\kappa]^2_{\aleph_0}$.

Note that Corollary 4 and Theorem 9 have as an immediate consequence $\kappa \not\to [\kappa]^2_{\aleph_0}$ for all $\kappa \leqslant 2^{\aleph_0}$ with cf $\kappa =$ cf $2^{\aleph_0}$, which is a well-known result of S. Shelah ([9]).

Galvin [8] used families of disjoint big subsets of $[\lambda]^2$ in constructing several very strong counter-examples to productiveness of the κ-chain condition. Using Theorem 7 and Galvin's ideas we can also get some of his general results. However, here we mention only one instance of such a general result, but let us note that it is possible to prove an analogue of Theorem 4.7 from Galvin [8].

THEOREM 10: *Assume ded$(\lambda, 2^\lambda)$. Then for every $n < \omega$ there is a partially ordered set $\mathfrak{D}$ such that $\mathfrak{D}^n$ satisfies the cf* 2^λ-c.c. *but $\mathfrak{D}^{n+1}$ does not.*

PROOF: Let L be a cf 2^λ-entangled linear ordering which exists by Theorem 3. Let $\{r_\alpha : \alpha < \text{cf } 2^\lambda\}$ be an one-to-one enumeratin of L and let Λ be the set of all limit ordinals $< \text{cf } 2^\lambda$. Let $K_0, \ldots, K_n$ be the disjoint big subsets of $[\Lambda]^2$ defined in the proof of Theorem 7. Clearly, we may assume that $n \geqslant 1$. For $i \leqslant n$, let $H_i = \bigcup\{K_j : j \leqslant n \& j \neq i\}$. Let $\mathfrak{O}_i = \mathfrak{O}(\lambda, H_i)$ for $i \leqslant n$. Since K_i's are big sets any product of $\leqslant n$ of the posets $\mathfrak{O}_0, \ldots, \mathfrak{O}_n$ satisfies the cf 2^λ-c.c., but clearly $\mathfrak{O}_0 \times \ldots \times \mathfrak{O}_n$ is not a cf 2^λ-c.c. poset. Hence $\mathfrak{O} = \mathfrak{O}_0 \oplus \ldots \oplus \mathfrak{O}_n$ satisfies the conclusion of the Theorem.

COROLLARY 11: *For every $n < \omega$ there is a partially ordered set $\mathfrak{O}$ such that $\mathfrak{O}^n$ satisfies the cf $2^{\aleph_0}$-c.c. but $\mathfrak{O}^{n+1}$ does not.*

THEOREM 12: *Let $\mathfrak{B}_\kappa$ be the standard poset for adding $\kappa > \aleph_0$ Cohen or random reals. Then $\mathfrak{B}_\kappa$ forces that for every $n < \omega$ there is a poset $\mathfrak{O}$ such that $\mathfrak{O}^n$ satisfies the* c.c.c. *but $\mathfrak{O}^{n+1}$ does not satisfy the* κ-c.c.

PROOF: By Theorem 2 and the proof of Theorem 10.

Theorem 12 for the case of Cohen reals was first proved by Fleissner [7]. This theorem also holds for $\kappa = 1$ (or $\kappa = \aleph_0$) replacing the clause "$\mathfrak{O}^{n+1}$ does not satisfy the κ-c.c." by "$\mathfrak{O}^{n+1}$ does not satisfy the c.c.c.". This has been proved by J. Roitman [17].

The first construction of an uncountable entangled set of reals under the assumption of CH was (implicitly) given by E. Michael [15]. This fact was first pointed out by E.S. Berney (unpublished) who used it in a construction of an uncountable Boolean algebra with no uncountable antichain, a result which has been also independently proved by R. Bonnet [5]. Michael's construction is a very nice generalization of the classical construction of concentrated sets of reals from the case $n = 1$ to the case of any finite n. His argument uses the Baire category theorem and can also be done with MA instead of CH.

In [3], J. Baumgartner gave, assuming CH, a first generalization of the classical construction ([10; §35]) of a function f: $\mathbb{R} \to \mathbb{R}$ which is not monotonic on any uncountable subset of $\mathbb{R}$ from the case $n = 1$ to the case of any finite n. Baumgartner's construction is more flexible than Michael's, and our §1 owes much to Baumgartner's construction.

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(Oblatum 28-VI-1983)

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