

COMPOSITIO MATHEMATICA

HENERI A. M. DZINOTYIWEYI

**Sizes of quotient spaces of certain function
algebras on topological semigroups**

Compositio Mathematica, tome 55, n° 3 (1985), p. 303-311

http://www.numdam.org/item?id=CM_1985__55_3_303_0

© Foundation Compositio Mathematica, 1985, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

SIZES OF QUOTIENT SPACES OF CERTAIN FUNCTION ALGEBRAS ON TOPOLOGICAL SEMIGROUPS

Heneri A.M. Dzinotyiweyi *

Introduction

Let S be a locally compact topological semigroup, $C(S)$ the space of all bounded complex-valued continuous functions on S , $LWUC(S)$ the space of all left weakly uniformly continuous functions in $C(S)$ and $M_a(S)$ the convolution measure algebra of absolutely continuous bounded complex-valued Radon measures on S .

When S is a closed subsemigroup of a locally compact topological group such that S is neither compact nor discrete, we showed that the quotient space $C(S)/LWUC(S)$ is nonseparable in [9]. In this paper, we will extend this result to a more general class of topological semigroups.

For a locally compact topological group G , $M_a(G)$ can be identified with the usual group algebra, $L^1(G)$, of G —see e.g. Hewitt and Ross [13]. When G is nondiscrete it is known that the quotient space $L^\infty(G)/C(G)$ and the radical of the Banach algebra $L^\infty(G)^*$ are nonseparable—see E.E. Granirer [10] and S.L. Gulick [12]. Motivated by these results, we will show that for a large class of nondiscrete topological semigroups S we have $M_a(S)^*/C(S)$ and the radical of $M_a(S)^{**}$ nonseparable; the actual setting of our results being more general.

Definitions and notations

Let A and B be any subsets of a semigroup S and x any element of S . We take AB , $A^{-1}B$, $x^{-1}B$ and $A^{-1}x$ to denote $\{ab: a \in A \text{ and } b \in B\}$, $\{y \in S: ay \in B \text{ for some } a \in A\}$, $\{x\}^{-1}B$ and $A^{-1}\{x\}$ (respectively). By symmetry the definitions of BA^{-1} , Bx^{-1} and xA^{-1} must be clear. By a *right cancellative semigroup* we mean a semigroup S such that whenever $yx = zx$ then $y = z$, for all x , y and z in S .

* This research was partially supported by a Fulbright Research Fellowship.

Throughout this paper, a semigroup, S , endowed with a Hausdorff topology with respect to which the semigroup operation $(x, y) \rightarrow xy$ is a jointly continuous mapping of $S \times S$ into S , is called a *topological semigroup*.

Let S be a locally compact topological semigroup for the remainder of this section. For every function f in $C(S)$ and x in S , we define the functions ${}_x f$ and f_x in $C(S)$ by

$${}_x f(y) := f(xy) \text{ and } f_x(y) := f(yx) (y \in S).$$

Let

$$LWUC(S) := \{f \in C(S) : \text{the map } x \rightarrow {}_x f \text{ of } S \text{ into } C(S) \text{ is weakly continuous}\},$$

$$WAP(S) := \{f \in C(S) : \text{the set } \{{}_x f : x \in S\} \text{ is relatively weakly compact}\},$$

$$AP(S) := \{f \in C(S) : \text{the set } \{{}_x f : x \in S\} \text{ is relatively norm compact}\}.$$

These spaces of functions have been studied widely - see e.g. [4] and [5]. If A is a subset of $C(S)$ and E of S we write $A|_E := \{f|_E : f \in A\}$ where $f|_E$ denotes the restriction of a function f to E .

Let $M(S)$ be the set of all bounded complex-valued Radon measures on S . It is well known that $M(S)$ is a Banach algebra with respect to the usual total variation norm, $\|\cdot\|$, and convolution multiplication given by

$$\nu * \mu(E) := \int \mu(x^{-1}E) d\nu(x) = \int \nu(Ex^{-1}) d\mu(x),$$

for all Borel subsets E of S and measures ν, μ in $M(S)$. For each μ in $M(S)$ and x in S we take $|\mu|$ to be the Radon measure arising from the total variation of μ and \tilde{x} the point mass at x . Let $M_a(S) := \{\mu \in M(S) : \text{the maps } x \rightarrow |\mu|(x^{-1}(C)) \text{ and } x \rightarrow |\nu|(Cx^{-1}) \text{ of } S \text{ into } \mathbb{R} \text{ are continuous, for every compact subset } C \text{ of } S\}$

The set $M_a(S)$ has been studied in many publications; for S , it plays a role analogous to that of $L^1(G)$ for a locally compact topological group G - see e.g. [1], [2], [15] and [16]. In particular, we have the following result proved in [1] and [2]

THEOREM 1: *We have $M_a(S)$ an L -ideal of $M(S)$ (–that is: $M_a(S)$ is a norm-closed subalgebra of $M(S)$ such that for all $\mu \in M(S)$ and $\nu \in M_a(S)$ we have $\nu^*\mu, \mu^*\nu \in M_a(S)$ and if $\mu \ll |\nu|$ then $\mu \in M_a(S)$).*

For each $\mu \in M(S)$, let $\text{supp}(\mu) := \{x \in S: \text{if } V \text{ is an open neighbourhood of } x \text{ then } |\mu|(V) > 0\}$.

Following A.C. and J.W. Baker we say S is a *foundation semigroup* if S coincides with the closure of $U\{\text{supp}(\mu): \mu \in M_a(S)\}$.

For ease of reference we quote the following result proved by Sleijpen [15].

THEOREM 2: *Let S be a foundation semigroup with identity element 1 and $S_1 := \{x \in S: 1 \in \text{int}(X^{-1}x \cap xX^{-1}) \text{ whenever } X \text{ is a neighbourhood of } x\}$. Then S_1 is dense in S and if V is an open neighbourhood in S then Vv^{-1} is a neighbourhood of 1, for all $v \in V \cap S_1$.*

The main results

Our next theorem is a generalization, to a larger class of semigroups, of a result we proved before—see [9, Theorem 2.5]. The proof employed contains a mixture of techniques we employed in [9] and those used in the proof of Baker and Butcher [3, Theorem 3]. The proof we give is also much simpler compared with that in [9].

THEOREM 3: *Let S be a normal, locally compact and right cancellative topological semigroup. Suppose S is neither countably compact nor discrete and $C^{-1}D$ is compact for all compact subsets C and D of S . Then, for some closed subset \bar{X} of S we have that $(C(S) \setminus LWUC(S))_{|\bar{X}}$ contains a linear isometric copy of l^∞ and so the quotient space $C(S)/LWUC(S)$ is nonseparable.*

PROOF. Since S is nondiscrete, we can find a relatively compact infinite set $\{s_n: n \in \mathbb{N}\}$ in S . As $C := cl(\{s_n: n \in \mathbb{N}\})$ is compact, we can choose a sequence $\{t_n\}$ in S with no cluster point and such that

$$t_{n+1} \notin \bigcup_{i=1}^n C^{-1}(Ct_i) \quad \text{for all } n \in \mathbb{N} \tag{1}$$

Choose infinite subsequences $T_k := \{t_{k1}, t_{k2}, \dots\}$ of $T := \{t_1, t_2, \dots\}$ such that

$$\bigcup_{k=1}^\infty T_k = T$$

and

$$T_k \cap T_{k'} = \emptyset$$

if and only if $k \neq k'$.

Let $X_k := \{s_m t_{k_n} : m, n \in \mathbb{N}\}$, $X := \{s_m t_n : m, n \in \mathbb{N}\}$ and note that our construction of the T_k 's and T imply

$$(a) \quad \bar{X}_k = \{ct_{k_n} : c \in C \text{ and } n \in \mathbb{N}\} \text{ (–see proof of [3, Theorem 3])},$$

$$(b) \quad \bar{X}_k \cap \bar{X}_{k'} = \emptyset \quad \text{if and only if } k \neq k',$$

$$(c) \quad \bigcup_{k=1}^{\infty} \bar{X}_k = \bar{X}.$$

Next we define the functions $f_k: \bar{X}_k \rightarrow \mathbb{R}$ by

$$f_k(s_m t_{k_n}) := \begin{cases} 1 & \text{if } m < n \\ -1 & \text{if } m \geq n \end{cases}$$

$$f_k(ct_{k_n}) := -1 \quad \text{if } c \in C \setminus \{s_m : m \in \mathbb{N}\}.$$

then (as similarly shown in [3, page 105],) f_k is continuous, for all $k \in \mathbb{N}$.

Corresponding to each element $\{d_{k'}\}$ in l^∞ , let $F_{(d_{k'})}$ be the function defined on \bar{X} by

$$F_{(d_{k'})}(x) := d_l f_l(x)$$

if and only if $x \in \bar{X}_l$ for some $l \in \mathbb{N}$.

By items (b) and (c) we have $F_{(d_{k'})}$ well-defined as a function. From items (b) and (c) we have that each \bar{X}_k is both closed and open in the space \bar{X} . Consequently $F_{(d_{k'})}$ is continuous, by the continuity of the f_l 's.

Now noting that

$$(*) \quad F_{(d_{k'})}(s_m t_{k_n}) = \begin{cases} d_k & \text{if } m < n \\ -d_k & \text{if } m \geq n, \end{cases}$$

[3, Theorem 5] and Tietze's Extension Theorem imply the existence of a function $\bar{F}_{(d_{k'})}$ in $C(S) \setminus LWUC(S)$ such that

$$\bar{F}_{(d_{k'})}|_{\bar{X}} = F_{(d_{k'})} \text{ and } \|\bar{F}_{(d_{k'})}\|_S = \|F_{(d_{k'})}\|_{\bar{X}} = \|\{d_{k'}\}\|_\infty.$$

Thus the (clearly) linear map $\{d_{k'}\} \rightarrow \overline{F}_{(d_{k'})_1 X}$ of l^∞ into $(C(S) \setminus LWUC(S))_{|\overline{x}}$ is isometric.

Since l^∞ is nonseparable, it follows that $C(S) \setminus LWUC(S)$ and hence the quotient space $C(S)/LWUC(S)$ is nonseparable.

For our next results, recall that the norm of $M_a(S)^*$ is given by

$$\|h\|_{M_a(S)^*} := \sup\{|h(\nu)| : \nu \in M_a(S) \text{ with } \|\nu\| = 1\}.$$

For a locally compact topological group G , $M_a(G)^*$ is simply $L^\infty(G)$.

THEOREM 4: *Let S be a nondiscrete and right cancellative foundation semigroup with an identity element 1. Then the quotient spaces $M_a(S)^*/C(S)$ and $M_a(S)^*/LWUC(S)$ contain isometric linear copies of l^∞ .*

PROOF. Let W be a compact neighbourhood of 1 and corresponding to each function g in $C(S)$ let G be the function in $C(W \times W)$ given by

$$G(x, y) := g(xy) \quad \text{for all } x, y \in W.$$

Then a simple compactness argument shows that the set $\{G(x, \cdot) : x \in W\}$ is relatively (norm and hence) weakly compact in $C(W)$. (Here each $G(x, \cdot)$ is given by $G(x, \cdot)(y) := G(x, y)$ for all x, y in W and $C(X)$ denotes the space of all bounded complex-valued continuous functions on a topological space X .)

Since S is not discrete, 1 is not isolated and so we can find a sequence $\{V_k\}$ of disjoint open neighbourhoods contained in W . Choose $v_k \in V_k \cap S_1$ and note that $V_k v_k^{-1}$ is a neighbourhood of 1, by Theorem 2. So there is a sequence $\{U_k\}$ of open neighbourhoods of 1 such that

$$U_k^2 \subset V_k v_k^{-1} \quad \text{for all } k \text{ in } \mathbb{N}.$$

By [8, Lemma 4.2] we can choose sequences $\{C_{k_1}, C_{k_2}, \dots\}$ and $\{D_{k_1}, D_{k_2}, \dots\}$ of non- $M_a(S)$ -negligible compact subsets of U_k such that, for all n, m, i and j in \mathbb{N} ,

$$C_{k_n} D_{k_m} \cap C_{k_i} D_{k_j} = \emptyset \quad \text{whenever } n < m \text{ and } i > j.$$

By right cancellation we have

$$C_{k_n} D_{k_m} v_k \cap C_{k_i} D_{k_j} v_k = \emptyset \quad \text{whenever } n < m \text{ and } i > j. \tag{1}$$

In the notation of [8, page 166], take $A = M_a(S)$ and set $d_a := d_{M_a(S)}$. We can choose sequences of points $\{c_{k_n}\}$ and $\{e_{k_n}\}$ such that

$$c_{k_n} \in d_a(C_{k_n}) \quad \text{and} \quad e_{k_n} \in d_a(D_{k_n}v_k). \quad (2)$$

Let

$$E_k := \bigcup_{i=1}^{\infty} \bigcup_{i < j} C_{k_i} D_{k_j} v_k \quad \text{and} \quad F_k := \bigcup_{j=1}^{\infty} \bigcup_{i > j} C_{k_i} D_{k_j} v_k.$$

We define the function h_k on S by

$$h_k := X_{E_k} - X_{F_k},$$

where X_A denotes the characteristic function of a subset A of S . Since E_k and F_k are disjoint σ -compact subsets of S , we also have that h_k is a functional in $M_a(S)^*$ (where $h_k(\nu) := \int h_k(x) d\nu(x)$, for all ν in $M_a(S)$).

We claim that, in the norm of $M_a(S)^*$,

$$\|h_k + g\|_{M_a(S)} \geq 1 \quad \text{for all } g \text{ in } C(S) \quad (3)$$

If not, then for some (real-valued) function g in $C(S)$ we can find $\epsilon > 0$ such that

$$\|h_k + g\|_{M_a(S)} \leq 1 - \epsilon.$$

In particular, for positive measures ν_α, μ_β in $M_a(S)$ such that $\|\nu_\alpha\| = \|\mu_\beta\| = 1$, $\text{supp}(\nu_\alpha) \subseteq C_{k_n}$ and $\text{supp}(\mu_\beta) \subseteq D_{k_n}v_k$, we have

$$|h_k(\nu_\alpha^* \mu_\beta) + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon. \quad (4)$$

Recalling our definition of h_k , (4) implies that

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(\nu_\alpha^* \mu_\beta)| \leq 1 - \epsilon. \end{cases}$$

Letting the net (ν_α) converge in the weak*-topology to \bar{c}_{k_n} and (μ_β) to \bar{e}_{k_m} we thus get that (, since g is continuous on W),

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(c_{k_n} e_{k_m})| \leq 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(c_{k_n} e_{k_m})| \leq 1 - \epsilon. \end{cases}$$

It follows that

$$G(c_{k_n}, e_{k_m}) = g(c_{k_n}e_{k_m}) \begin{cases} < -\epsilon & \text{if } n < m \\ > \epsilon & \text{if } n > m \end{cases}$$

and so $G(x, \cdot): x \in W$ is not relatively weakly compact, by Grothendieck's Theorem [11]. This contradicts the observation at the beginning of our proof. By this conflict, claim (3) holds.

Since the V_k 's are pairwise disjoint and $C_{k_n}D_{k_m}v_k \subset V_k$ for all n, m and k in \mathbb{N} , we have that

$$\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + C(S)$$

defines a linear mapping of l^∞ into $M_a(S)^*/C(S)$. Noting that $\|h_k\|_{M_a(S)} = 1$, item (3) implies that

$$\|\{t_k\}\|_\infty = \|\sum_{k=1}^{\infty} t_k h_k + C(S)\|_{M_a(S)^*/C(S)}$$

and so the mapping $\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + C(S)$ is isometric.

Similarly the mapping $\{t_k\} \rightarrow \sum_{k=1}^{\infty} t_k h_k + LWUC(S)$ of l^∞ into $M_a(S)^*/LWUC(S)$ is linear and isometric. This completes our proof. (The idea of embedding l^∞ used here is inspired by [6] and [9].)

The second dual of $M_a(S)$, namely $M_a(S)^{**}$, can be turned into a Banach algebra with Arens product \circ defined as follows: For $\phi \in M_a(S)^{**}$, $h \in M_a(S)^*$ and $\nu \in M_a(S)$ we define $\nu \circ h$, $h \circ \nu$ and h^ϕ in $M_a(S)^*$ by

$$\nu \circ h(\mu) := h(\nu * \mu), \quad h \circ \nu(\mu) := h(\mu * \nu) \quad \text{and}$$

$$h^\phi(\mu) := \phi(\mu \circ h) \quad \text{for all } \mu \text{ in } M_a(S).$$

For all ϕ, ψ in $M_a(S)^{**}$ we have $\phi \circ \psi$ given by

$$\phi \circ \psi(h) := \phi(h^\psi).$$

$R_a(S)$, the radical of $M_a(S)^{**}$, is the intersection of all maximal modular left (or right) ideals (See [14] page 55).

Let G be a locally compact topological group. When G is nondiscrete and abelian, Civin and Yood [7] showed that $R_a(G)$ is infinite dimensional and later S. Gulick [12] showed that $R_a(G)$ is even nonseparable.

In [10], Granirer showed that $R_a(G)$ is nonseparable whenever G is nondiscrete or G is discrete and amenable. We generalize the former result to the semigroup situation.

THEOREM. *Let S be a nondiscrete and right cancellative foundation semigroup with an identity element. Then there exists a subspace P of $M_a(S)^*$ such that P^* is a linear isometric copy of $(l^\infty)^*$ and the restriction of the radical of $M_a(S)^{**}$ to P is P^* . In particular the radical of $M_a(S)^{**}$ is nonseparable.*

PROOF. (c.f. [10] for a related proof in the group case.) Let

$$A := \{ \phi \in M_a(S)^{**} : \phi(f) = 0 \text{ for all } f \text{ in } LWUC(S) \}.$$

For all $\psi \in M_a(S)^{**}$, $\nu \in M_a(S)$ and $h \in M_a(S)^*$, we have that $\nu \circ h \in LWUC(S)$, by the left handed version of [9, Lemma 4.1]; consequently $h^\phi(\nu) := \phi(\nu \circ h) = 0(\phi \in A)$ and so

$$\phi \circ \psi(h) := \psi(h^\phi) = 0(\phi \in A).$$

Thus A is a right ideal of $M_a(S)^{**}$ such that

$$A \circ M_a(S)^{**} = \{0\}.$$

Hence $A \subset R_a(S)$ —see e.g. Richart [14, Theorem 2.3.5(ii)].

Now by Theorem 4, there exists an isometric linear map Π of $l^\infty M_a(S)^*/LWUC(S)$. So for some closed subspace P of $M_a(S)^*$, we have $\Pi(l^\infty)$ dense in $P/LWUC(S)$. The inverse map Π^{-1} therefore extends to a unique isometric linear map $\tau: P/LWUC(S) \rightarrow l^\infty$. Hence the dual $\tau^*: (l^\infty)^* \rightarrow (P/LWUC(S))^*$ is an isometric linear map that is onto. But $A = LWUC(S)^\perp \subset M_a(S)^{**}$ can be identified with $(M_a(S)^*/LWUC(S))^*$. Hence each element of $(P/LWUC(S))^*$ can be identified with the restriction of some element of A to P . This completes our proof.

The case for a cancellative discrete topological semigroup S that is amenable can be similarly handled as in the equivalent group case -see [10, page 323]. To what extent one can drop the right cancellation requirement on S , in Theorems 4 and 5, remains an open problem.

We proved related results for other spaces of functions in [9]. In particular we showed that if S is a C -distinguished topological semigroup such that $M_a(S)$ is nonzero and S is not relatively neo-compact, then $WUC(S)/WAP(S)$ contains an isometric linear copy of l^∞ . (See [9] for definition of terms.) There seems to be some relationship between sizes of quotient spaces and the existence of continuous projections. We have the following conjecture.

CONJECTURE: *Let S be as stated in the preceding paragraph. Then there does not exist bounded linear projections from $WUC(S)$ onto $WAP(S)$ or from $WUC(S)$ onto $AP(S)$.*

When $S = \mathbb{R}$ —the usual additive group of reals with usual topology then this conjecture is true—see e.g. [17]. we are indebted to Professor W.G. Bade for drawing our attention to the reference [17].

References

- [1] A.C. and J.W. BAKER: Algebras of measures on a locally compact semigroup II. *J. Lond. Math. Soc.* 2 (1970) 651–659.
- [2] A.C. and J.W. BAKER: Algebras of measures on a locally compact semigroup III. *J. Lond. Math. Soc.* 4 (1972) 685–695.
- [3] J.W. BAKER and R.J. BUTCHER: The Stone-Cech compactification of a topological Semigroup. *Math. Proc. Camb. Phil.* 80 (1976) 103–117.
- [4] J.F. BERGLUND, H.D. HUNGHEHN and P. MILNES: *Compact right topological semigroups and generalizations of almost periodicity*. Springer-Verlag lecture notes 663 (1978).
- [5] R.B. BURCHEL, *Weakly almost periodic functions on semigroups*. New York: Gordon and Breach (1970).
- [6] Ching CHOU: Weakly almost periodic functions and almost convergent functions on a group. *Trans. Amer. Math. Soc.* 206 (1975) 175–200.
- [7] P. CIVIN and B. YOOD: The second conjugate space of a Banach algebra as an algebra. *Pac. J. math.* 11 (1961) 847–870.
- [8] H.A.M. DZINOTYIWEYI: Weakly almost periodic functions and the irregularity of multiplication in semigroup algebras. *Math. Scand.* 46 (1980) 157–172.
- [9] H.A.M. DZINOTYIWEYI: Nonseparability of Quotient spaces of function algebras on topological semigroups. *Trans. Amer. Math. Soc.* 272 (1982) 223–235.
- [10] E.E. GRANIRER: The radical of $L^\infty(G)^*$. *Proc. Amer. Math. Soc.* 41 (1973) 321–324.
- [11] A. GROTHENDIECK: Critères de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.* 74 (1952) 168–186.
- [12] S.L. GULICK: Commutativity and ideals in the biduals of topological algebras. *Pac. J. Math.* 18 (1966) 121–137.
- [13] E. HEWITT and K.A. ROSS: *Abstract harmonic analysis*. Vol. I. Berlin: Springer-Verlag (1963).
- [14] C. RICKART: *General theory of Banach algebras*. University series in higher mathematics. Princeton, N.J.: Van Nostrand, (1960).
- [15] G.G. SLEIJPEN: Locally compact semigroups and continuous translations of measures. *Proc. Iond. Math. Soc.* 37 (1978) 75–97.
- [16] G.L.G. SLEIJPEN: *Emaciated sets and measures with continuous translations*. *Ibid* 98–119.
- [17] B.B. WELLS, Jr.: Uncomplemented function algebras. *Studia mathematica* T. XXX11 (1969) 41–46.

(Obaltum 11-VIII-1983 & 30-I-1984.)

Department of mathematics
 University of Zimbabwe
 Box MP 167, Mount Pleasant
 Harare
 Zimbabwe