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Poles of a local zeta function and Newton polygons

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This article continues the investigation begun in [7] of a local zeta function defined by Igusa [3] as follows. Let K be a nonarchimedean local field of characteristic zero, R its ring of integers, and \( f \in R[x_1, \ldots, x_n] \). Let \( \Phi \) be a Schwartz-Bruhat function on \( K^n \) and \( s \) a complex number with \( \text{Re}(s) > 0 \). Define

\[
Z_f(s, \Phi) = \int_{K^n} |f|^2 \Phi \, dx
\]

where \( |dx| \) is the usual Haar measure on \( K^n \), and \( | \cdot | \) is the usual absolute value on \( K \).

As in [7], the aim of this article is to detect the poles of the meromorphic continuation of \( Z_f(s, \Phi) \) into \( \text{Re}(s) < 0 \). The only strategy seemingly available to accomplish this task is to take a resolution \( \pi: X \to K^n \) of \( f \) and study the resolution data \( \{(N_i, n_i)\} \) in which \( N_i = \text{multiplicity of } f \text{ along divisor } D_i \), \( n_i - 1 = \text{multiplicity of } \text{det}(d\pi) \text{ along } D_i \). The set of ratios \( \{-n_i/N_i\} \cup \{-1\} \) contains the poles of \( Z_f(s, \Phi) \) as observed in [3], but for the known examples most of these ratios are not actually poles. The problem is to determine the actual poles.

Here this is accomplished for a certain class of reducible plane curves with exactly one singularity at the point \( (0, 0) \in K \times K \) which is “toroidal”. Toroidal singularities are a useful class because a resolution can be easily constructed via a monomial transformation \( \pi \), obtained from a study of the Newton polygon associated to a defining function for the curve in a given coordinate system. The poles of \( Z_f(s, \Phi) \) are determined by the singularity of \( f \) so we shall assume without loss of generality that \( \Phi \) is the characteristic function of \( R \times R \). We then fix \( f \) and denote \( Z_f(s, \Phi) \) by \( Z(s) \).

Section 1 recalls this “toroidal resolution” following [6]. In Section 2 arithmetical information is obtained on the numerical data. Section 3 shows why most of the ratios \( -n_i/N_i \) cannot be poles of \( Z(s) \). Moreover the small number of its genuinely possible poles is described in terms of
the polygon for \( f \). Section 4 contains the affirmative result describing conditions imposed both upon \( f \) and the polygon itself which imply the non-vanishing of the residue of \( Z(s) \) at each of the ratios in this smaller set of good candidates for poles of the zeta function. A simple description of the largest pole is given in terms of the polygon which is identical to that given in [10] by Varchenko when the local field is \( \mathbb{R} \) and Vasiliev [12] when the local field is \( \mathbb{C} \). It is also interesting to note here that the negative of the value obtained by Varchenko and Vasiliev also has an interpretation. As shown by Ehlers and Lo in [2], it is the smallest of the exponents associated to the mixed Hodge structure on the vanishing cohomology of the Milnor fiber. This follows from Varchenko [9].

### 1. Toroidal resolutions

For more detailed descriptions of toroidal resolution [1], [6], [8] or [11] may be consulted. Here, a brief summary of techniques and results will be given for the case of two variables only.

For fixed coordinates \((x_1, x_2)\) in \( \mathbb{R} \) and \( f \in \mathbb{R}[x_1, x_2] \) define \( \text{Supp}(f) = \{ I = (i_1, i_2) \in \mathbb{N}^2: x_1^i_1 x_2^i_2 = x^I \text{ appears with a non-zero coefficient in the expression for } f \} \). Define \( S = \bigcup_{I \in \text{Supp}(f)} (I + \mathbb{R}^2_+) \) and \( \Gamma_+(f) = \) the boundary of the convex hull of \( S \). \( \Gamma = \Gamma_+(f) \) is the Newton polygon for \( f \) with respect to the \((x_1, x_2)\) coordinates. We may write \( f = f_{I} + (\text{higher order terms}) \) where \( f_I(x_1, x_2) = \sum_{I \in \Gamma} a_I x^I \) is the “principal part” of \( f \).

One can dualize \( \Gamma \) as follows. In the dual space \((\mathbb{R}^*_+)^2\) of covectors in the first quadrant define the “first meet locus” of a covector \( a \) as \( \mathbb{C} = \{ x \in \text{conv}(S): a \cdot x = \mu(a) \} \) where \( \mu(a) = \inf \{ a \cdot y: y \in S \} \). \( \mathbb{C} = \mathbb{C}(a) \) is either a closed face of \( \Gamma \) or a vertex. On the set of covectors define an equivalence relation by \( a_1 \sim a_2 \) iff \( \mathbb{C}_{a_1} = \mathbb{C}_{a_2} \). The equivalence classes consist of a finite set of open cones whose boundaries are also equivalence classes consisting of covectors dual to a one dimensional face of \( \Gamma \). There is an evident refinement process allowing one to obtain a finite set of closed cones of the form \( \langle a_1, a_2 \rangle = \{ a: a = a_1 a_1^* + \beta a_2^*; \alpha, \beta \geq 0 \} \) such that

\[
\det \begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix} = \pm 1 \quad \text{and}
\]

ii) \( a_1, a_2 \) are “primitive”, that is, if \( a' = (a_1', a_2') \), then \( \text{g.c.d.}(a_1', a_2') = 1 \) if \( i = 1, 2 \).

To each “unit” cone \( \sigma = \langle a_1, a_2 \rangle \) one associates an affine chart \( K_x K = K_x K_{\sigma} \) and defines a map \( \pi(\sigma): K_x K_{\sigma} \to K_x K \) by \( x_i \pi(\sigma)(y_1, y_2) = y_1^{a_1} y_2^{a_2}, i = 1, 2 \); i.e. if \( P = (p_1, p_2) \) denotes the point on \( \Gamma \) corresponding to the monomial \( x_1^{p_1} x_2^{p_2} \) we have \( x_1^{p_1} x_2^{p_2} \pi(\sigma) = y_1^{p_1} y_2^{p_1} a_1^* a_2^* \). A smooth variety over \( K \), denoted \( X(\sigma) \), is obtained by patching \( K_x K_{\sigma_1} \) and \( K_x K_{\sigma_2} \) if and only if \( \sigma_1 \cap \sigma_2 \neq \emptyset \) by the evident relation \( z \in K_x K_{\sigma_1} \) is identified with \( w \in K_x K_{\sigma_2} \) if and only if
If $\sigma_1 = \langle c_1, c_2 \rangle$, $\sigma_2 = \langle a_1, a_2 \rangle$ and if $A_1$ denotes the matrix $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $A_2$ denotes the matrix $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$; then $A_1$ determines $\pi(\sigma_1)$ and $A_2^{-1}$ determines $\pi(\sigma_2)^{-1}$. Orientation is chosen so that $\det A_i = 1$ for $i = 1, 2$. Thus $A_1 A_2^{-1}$ determines $\pi(\sigma_2)^{-1} \circ \pi(\sigma_1)$.

Now $\sigma_1 \cap \sigma_2 \neq \emptyset$ so either i) $a_1 = c_2$ or ii) $a_2 = c_1$. We examine $A_1 A_2^{-1}$ in each case

i) $A_1 A_2^{-1} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix}$ where $k = \det \begin{pmatrix} c_1 \\ a_2 \end{pmatrix} \geq 1$.

ii) $A_1 A_2^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & k' \end{pmatrix}$ where $k' = \det \begin{pmatrix} a_1 \\ c_2 \end{pmatrix} \geq 1$.

These compatibility relations precisely determine the charts needed to describe the preimage of $R \times R$ in $X(\Gamma)$. Index the cones used in the partition of $(\mathbb{R}_+^*)^2$ used to construct $X(\Gamma)$ so that $\sigma_1 = \langle (1, 0), a_1 \rangle$, $\sigma_2 = \sigma_2 = \langle a_1, a_2 \rangle, \ldots, \sigma_N = \langle a_{N-1}, (0, 1) \rangle$. If $(y_1, y_2)$ are coordinates in $K^2(\sigma_i)$ and $(y'_1, y'_2)$ coordinates in $K^2(\sigma_{i+1})$, then by the above we see that the monomial transformation $(\pi(\sigma_{i+1})^{-1} \circ \pi(\sigma_i))(y_1, y_2) = (y'_1, y'_2)$. Hence if we assume $(y_1, y_2)$ lies in $R \times R \subset K^2(\sigma_i)$, then since $y'_2 = y_2^{-1}$ in the overlap, we can shrink $y'_2$ in $K^2(\sigma_{i+1})$ from lying in $R$ to lying in $P$, the unique maximal ideal of $R$. As a result $\pi^{-1}(R \times R)$ may be covered by $R \times R$ in $K^2(\sigma_{2i})$ and $R \times P$ in $K^2(\sigma_{2i+1})$. We shall use this in order to examine the pullback of $Z(s)$ in $X(\Gamma)$.

Having constructed $X(\Gamma)$ and $\pi: X(\Gamma) \to K^2$ a proper birational modification of $K^2$, one can then resolve $f$ via $\pi$ by imposing a non-degeneracy condition on $f_\Gamma$, as in [5]:

**Non-degeneracy condition**: For each closed face of $\Gamma$, the functions $(x_1 f_{x_1})_\Gamma$ and $(x_2 f_{x_2})_\Gamma$, consisting of the monomials in $x_1 f_{x_1}$ lying on $\Gamma$, have no common zeroes in $(K^*)^2$.

[6] discusses how this condition implies that at each point in $\pi^{-1}(0, 0)$ there are local coordinates in which $f$ is written in normal crossing form. In particular, it follows from the assumption of the origin being an isolated singularity of $f$ in $R \times R$ that the strict transform of $f$ is nonsingular in $\pi^{-1}(R \times R) \subset X(\Gamma)$, and can only have simple roots on any component of the exceptional divisor inside $\pi^{-1}(R \times R)$. This observation implies the genericity of the non-degeneracy condition in the following sense:
PROPOSITION 1: In fixed coordinates \((x_1, x_2)\) and for a fixed polygon \(\Gamma\), let \(\mathcal{F}_\Gamma = \{ f \in K[x_1, x_2]: \Gamma = \Gamma_\cdot (f) \}\). Then \(\mathcal{G}_\Gamma = \{ g \in \mathcal{F}_\Gamma: g \text{ is non-degenerate with respect to } \Gamma \} \) is an inductive limit of Zariski open subsets of \(\mathcal{F}_\Gamma\).

PROOF: We refer to [5] for the proof of an entirely analogous statement.

The results in Sections 3 and 4 concern subsets of polynomials in \(\mathcal{G}_\Gamma\). \(\mathcal{G}_\Gamma\) is a useful class of polynomials for which to analyze \(Z(s)\) because of the ease of constructing the resolution.

2. Relationships on the numerical data

The data needed in the next sections concerns the set of divisors in the exceptional locus \(\pi^{-1}(0, 0)\) of \(\pi\). Note that to each covector \(a\), used in the refinement of the partition of \((\mathbb{R}^*_+)^2\) described in Section 1, one associates a divisor \(D_a\). To each \(D_a\) correspond two integers \((m(a), |a|)\) and a ratio \(p_a = -|a|/m(a)\), where \(m(a) = \inf\{ x \cdot a: x \in \Gamma \}\) and \(|a| = a_1 + a_2\) if \(a = (a_1, a_2)\). The set \(\mathcal{A} = \{(m(a), |a|)\}_a\) is called the numerical data of the resolution.

From [3] one knows that the set \(\{ p_a \}_a\) is the set of possible poles for the meromorphic continuation of \(Z(s)\) into \(\text{Re}(s) < 0\). In terms of \(\Gamma\) one may easily interpret \(p_a\), as described in [10]. The support line to \(\Gamma\) in the direction determined by \(a\) is defined by \(x \cdot a = m(a)\). The point of intersection with the diagonal \(x = y = t\) is at the value \(t_a = m(a)/|a|\). Thus \(p_a = -1/t_a\). The graph of \(t_a\) versus \(a_2/a_1\) for covectors \((a_1, a_2)\) dual to faces is represented by Diagram one in the case where there is one face of \(\Gamma\) intersecting the diagonal and \(b = (b_1, b_2)\) is dual to that face. The case where two faces of \(\Gamma\) intersect the diagonal is represented by Diagram two where \(b = (b_1, b_2)\) and \(b' = (b'_1, b'_2)\) denote the vectors dual to the two faces.

The following proposition summarizes the preceding discussion.

PROPOSITION 2: Let \(b\) be a covector dual to any face of \(\Gamma\) not containing the intersection with the diagonal as a vertex. Let \(\sigma_1 = \langle a^1, b \rangle, \sigma_2 = \langle b, a^2 \rangle\) be two unit cones oriented so that \(\det \begin{pmatrix} a^1 \\ b \end{pmatrix} = \det \begin{pmatrix} b \\ a^2 \end{pmatrix} = 1\). Then \(p_b \neq p_{a^i}\) for \(i = 1, 2\).

Of more interest is the following:

PROPOSITION 3: With the orientation of Proposition 2, if \(a^1, b, \) and \(a^2\) are covectors not dual to any face of \(\Gamma\) and satisfying the property that there is a
unique vertex $P = (p_1, p_2)$ with $P = K_b = K_{a^1} = K_{a^2}$, then

$$\frac{|a^1| + |a^2|}{m(a^1) + m(a^2)} = \frac{|b|}{m(b)}.$$  

PROOF: Consider the expression $m(b) |a^i| - |b| m(a^i)$. Let $\mathbf{1} = (1, 1)$. By
the property defining $P$ the difference equals

$$(b \cdot P)(a' \cdot \bar{1}) - (a' \cdot P)(b \cdot \bar{1})$$

$$= \det \begin{pmatrix} b \cdot P & b \cdot \bar{1} \\ a' \cdot P & a' \cdot \bar{1} \end{pmatrix}$$

$$= \det \begin{pmatrix} b \\ a' \end{pmatrix} \det \begin{pmatrix} p_1 & 1 \\ p_2 & 1 \end{pmatrix}$$

$$= \begin{cases} p_2 - p_1 & \text{if } i = 1 \\ p_1 - p_2 & \text{if } i = 2. \end{cases}$$

The proposition is equivalent to showing $m(b)|a^1| - m(a^1)|b| = m(a^2)|b| - m(b)|a^2|$ which is immediate from the above formula.

**Corollary 1:** In the situation of Proposition 2, if $K_{a^1} = \{I\}$, $K_{a^2} = \{J\}$, where $I = (I_1, I_2)$ and $J = (J_1, J_2)$ are the vertices of the face $\tau_b$ dual to $b$, then

$$\frac{|a^1| + |a^2|}{m(a^1) + m(a^2)} = \frac{|b|}{m(b)} + \frac{k|b|}{m(b)(m(a^1) + m(a^2))}$$

for some integer $k \geq 1$.

**Proof:** Proposition 3 implies $m(b)|a^1| - m(a^1)|b| = I_2 - I_1$ and $m(b)|a^2| - m(a^2)|b| = J_2 - J_1$. Thus the numerator of $(|a^1| + |a^2|/m(a^1) + m(a^2)) - |b|/m(b)$ is equal to $I_2 - I_1 + J_1 - J_2$. Now $(I_2 - J_2)/(I_1 - J_1)$ is the slope of the face $\tau_b$, so

$$\frac{I_2 - J_2}{I_1 - J_1} = -\frac{b_1}{b_2}.$$ 

Using the fact that $b$ is primitive and that $I_2 > J_2$ we have that $I_2 - J_2 = kb_1$ and $I_1 - J_1 = -kb_2$ for some positive integer $k$. Therefore $I_2 - I_1 + J_1 - J_2 = k(b_1 + b_2) = k|b|$ which gives us the desired statement. \textit{q.e.d.}

**Corollary 2:** If $b$ is a covector dual to a face $\tau$ then using the notation of Corollary 1 we have

(a) If $b = (b_1, 1)$ is dual to a face with a vertex $P = (0, p_2)$, then

$$m(b)|a^1| - m(a^1)|b| = m(b).$$

(b) If $b = (1, b_2)$ is dual to a face with a vertex $P = (p_1, 0)$ then

$$m(b)|a^2| - m(a^2)|b| = m(b).$$
(c) In all other cases we have

\[ m(b) |a'| - m(a') |b| < m(b) \]

**Proof:** These statements follow from the expressions for \( m(b) |a'| - m(a') |b| \) and the fact that \( m(b) = b \cdot P \).

### 3. On the vanishing of possible poles for \( Z(s) \)

In Section 2 we described the set of possible poles for \( Z(s) \) associated to any \( f \in \mathcal{G}_r \). This section eliminates most of these ratios from candidacy in the set of poles for \( Z(s) \). We think of \( \text{Res}_{s=p} Z(s) \) for a possible pole \( p \) as a sum of integrals along all divisors \( D_a \) in \( \pi^{-1}(R \times R) \) where \( D_a \) has numerical data \( (m(a), |a|) \) and \( p = p_a = -|a|/m(a) \) (in the notation of section 1). The theorem we shall prove here says that when the covector \( a \) is dual to a vertex only of \( \Gamma \), then the contribution to \( \text{Res}_{s=p} Z(s) \) along \( D_a \) is zero.

We let \( q = \text{card}(R/P) \), and recall that \( |x| = q^{-\text{ord} x} \) where \( \text{ord} \) denotes the usual order of a fixed element in \( P - P^2 \).

**Theorem 1:** Let \( p \) be a possible pole of \( Z(s) \) in the sense that there is at least one divisor \( D_a \) with the numerical data \( (m(a), |a|) \) such that \( p = -|a|/m(a) \). Moreover assume \( p \neq -1 \). If \( b \) is a covector in a refinement of the partition dual to \( \Gamma \) such that \( K_b \) is only a vertex of \( \Gamma \) and \( p = p_b \), then the contribution to \( \text{Res}_{s=p} Z(s) \) along divisor \( D_b \) with numerical data \( (m(b), |b|) \) is zero.

**Proof:** We examine the pullback of \( Z(s) \) in the charts containing the divisor \( D_b \). Let \( a^1 \) and \( a^2 \) be the covectors oriented about \( b \) as described above so that \( \sigma_1 = \langle a^1, b \rangle \) and \( \sigma_2 = \langle b, a^2 \rangle \) are unit cones and \( K_{a^1} = K_{a^2} = K_b \). Let \( \pi(\sigma_1) : K^2(\sigma_1) \rightarrow K^2 \) and \( \pi(\sigma_2) : K^2(\sigma_2) \rightarrow K \) be the associated charts. In \( K^2(\sigma_1) \) we have \( (f \circ \pi(\sigma_1))(y_1, y_2) = y_1^{m(a^1)} y_2^{m(b)} f_{\sigma_1}(y_1, y_2) \) and in \( K^2(\sigma_2) \) we have \( (f \circ \pi(\sigma_2))(y'_1, y'_2) = y'_1^{m(b)} y'_2^{m(a^2)} f_{\sigma_2}(y'_1, y'_2) \) where \( f_{\sigma_1} \) and \( f_{\sigma_2} \) denote the strict transforms of \( f \) in each chart. Thus the integrals \( Z_1(s) \) and \( Z_2(s) \) will contribute to the pole at \( s = -|b|/m(b) \) where

\[
Z_1(s) = \int_{R} \int_{R} |y_1|^{m(a^1)s + |a^1| - 1} |y_2|^{m(b)s + |b| - 1} |f_{\sigma_1}|^s |dy_1| |dy_2|
\]

and

\[
Z_2(s) = \int_{P} \int_{R} |y'_1|^{m(b)s + |b| - 1} |y'_2|^{m(a^2)s + |a^2| - 1} |f_{\sigma_2}|^s |dy'_1| |dy'_2|.
\]
We now examine $f_{\sigma_1}$ and $f_{\sigma_2}$ more closely. Under the change of coordinates given by $\pi(\sigma_1)$ a monomial corresponding to the point $p = (p_1, p_2)$, i.e. $x_1^a x_2^b$, is transformed to the monomial $y_1^{m(a)} y_2^{m(b)}$. $f_{\sigma_1}$ is obtained by factoring $y_1^{m(a)} y_2^{m(b)}$ from each monomial. Since $b$ is not dual to a face of the polygon, $m(b)$ is determined by a unique vertex of the polygon, and this vertex also determines the value of $m(a)$. For all other points of the polygon we have $b \cdot p > m(b)$; therefore $f_{\sigma_1}$ has the form $c + y_2 g(y_1, y_2)$, where $c$ is the coefficient of the monomial term corresponding to the vertex of the polygon that comprises $K_b$. Under the change of coordinates given by $\pi(\sigma_2)$ a monomial $x_1^a x_2^b$ is transformed to the monomial $y_1^{a^1} y_2^{a^2}$. Thus a similar argument shows that $f_{\sigma_2}$ has the form $c + y_1 h(y_1, y_2)$. Thus if $\text{ord } c = e$ we see that $|f_{\sigma_1}|$ is constant on $R \times P^{e+1}$ and $|f_{\sigma_2}|$ is constant on $P^{e+1} \times P$.

We split the domain of integration for $Z_i(s)$ into $R \times (R - P^{e+1})$ and $R \times P^{e+1}$. First consider the domain of integration $R \times (R - P^{e+1})$. We can write this as a disjoint union of $U_i$ where each $U_i$ is a coset of the form $R \times (a_i + P^{e+1})$. We have that $|y_2|$ is constant on each coset. Consider separately the cases where $f_{\sigma_1} \neq 0$ on $U_i$ and $f_{\sigma_1} = 0$ on $U_i$.

If $f_{\sigma_1} \neq 0$ on $U_i$ we can write $U_i$ as a disjoint union of cosets $U_{ij}$ modulo $P^k \times P^k$ for $k$ sufficiently large such that $|f_{\sigma_1}|$ is also constant on these cosets. Then suppose a given coset is $(c_i + P^k) \times (d_j + P^k)$ and $|f_{\sigma_1}| = q^{-l}$ on this coset. Then the integral over this coset will contribute a term of the form

$$C q^{-(\text{ord } a, m(b) s | b | - 1)} q^{-(\text{ord } c, m(a) s | a^1 | - 1)} q^{-l s}$$

for some positive constant $C$ if $c_i \notin P^k$ and

$$C q^{-(\text{ord } a, m(b) s | b | - 1)} q^{-l s} \frac{q^{-(m(a) s | a^1 | k)}}{1 - q^{-m(a) s | a^1 |}}$$

for some positive constant $C$ if $c_i \in P^k$.

Now consider the case where $f_{\sigma_1} = 0$ on $U_i$. Because $(0, 0)$ is the only singularity of $f$ in $R \times R$, we must have at each point $(y_1, y_2)$ in $U_i$ either $\partial f_{\sigma_1}/\partial y_1(y_1, y_2) \neq 0$ or $\partial f_{\sigma_1}/\partial y_2(y_1, y_2) \neq 0$. Refine $U_i$ into smaller cosets $U_{ij}$ so that on each $U_{ij}$ one of $|\partial f_{\sigma_1}/\partial y_1|$ or $|\partial f_{\sigma_1}/\partial y_2|$ is a non-zero constant. Suppose $|\partial f_{\sigma_1}/\partial y_2| = q^{-m}$ on $U_{ij}$. Write $U_{ij}$ as a disjoint union of cosets modulo $P^l \times P^l$ for $l > m$. Let $D$ be one of these cosets. Then the change of coordinates given by $\tilde{y}_1 = y_1$, $\tilde{y}_2 = f_{\sigma_1}(y_1, y_2)$ maps $D$ homeomorphically to its image, cf. [4], Lemma 5. Moreover for large enough $k$ this image can be written as a disjoint union of cosets modulo $P^k \times P^k$. Then $|d\tilde{y}_1| |d\tilde{y}_2| = q^m |d\tilde{y}_1| |d\tilde{y}_2|$ so an integral over
one of these cosets has the form
\[ q^m q^{-(\text{ord } a_i)(m(b)s + |b| - 1)} \]
\[ \times \int_{\mathbb{C} + \mathbb{P}^k} \int_{\mathbb{C} + \mathbb{P}^k} |\tilde{y}_1|^{m(a')s + |a'| - 1} |\tilde{y}_2|^s |d\tilde{y}_1| |d\tilde{y}_2| \]
which contributes a term of the form (1) if \( c_i \notin P^k \) and \( d_i \notin P^k \), where \( l = \text{ord } a_i \); a term of the form (2) if \( c_i \in P^k \) and \( d_i \notin P^k \); a term of the form
\[ C q^{-(\text{ord } a_i)(m(b)s + |b| - 1)} q^{-(\text{ord } c_i)(m(a')s + |a'| - 1)} \frac{q^{-(s+1)k}}{1 - q^{-(s+1)}} \] (3)
for a positive constant \( C \) if \( c_i \notin P^k \) and \( d_i \in P^k \) and a term of the form
\[ C q^{-(\text{ord } a_i)(m(b)s + |b| - 1)} q^{-(m(a')s + |a'|)k} \frac{q^{-(s+1)k}}{(1 - q^{-(s+1)}) (1 - q^{-(m(a')s + |a'|)k})} \] (4)
for a positive constant \( C \) if \( c_i \in P^k \) and \( d_i \in P^k \).

By our assumptions \( |b|/m(b) \neq 1 \) and by Proposition 2 \( |b|/m(b) \neq |a'|/m(a') \). Therefore the terms in (1), (2), (3) and (4) do not contribute to the pole at \(-|b|/m(b)\). By splitting the domain of integration for \( Z_2(s) \) into \( \mathbb{P}^{e+1} \times \mathbb{P} \) and \((R - \mathbb{P}^{e+1}) \times \mathbb{P}\) and applying a similar argument we see that the integral over \((R - \mathbb{P}^{e+1}) \times \mathbb{P}\) does not contribute to the residue of the pole at \( s = -|b|/m(b) \).

Therefore the contributions to the residue at \( s = -|b|/m(b) \) come from the integrals
\[ q^{-es} \int_{\mathbb{P}^{e+1}} \int_{\mathbb{R}} |y_1|^{m(a')s + |a'| - 1} |y_2|^{m(b)s + |b| - 1} |d y_1| |d y_2| \]
and
\[ q^{-es} \int_{\mathbb{P}^{e+1}} |y_1|^{m(b)s + |b| - 1} |y_2|^{m(a^2)s + |a^2| - 1} |d y_1| |d y_2|. \]

Evaluating these integrals gives us \((1 - q^{-1})^{2} q^{-es} \) times
\[ q^{-(m(b)s + |b|)(e+1)} \]
\[ \frac{1 - q^{-(m(a')s + |a'|)k}}{(1 - q^{-(m(b)s + |b|)}) (1 - q^{-(m(a')s + |a'|)k})} \]
\[ + \frac{q^{-(m(b)s + |b|)k} q^{-(m(a^2)s + |a^2|)k}}{(1 - q^{-(m(b)s + |b|)}) (1 - q^{-(m(a^2)s + |a^2|)k})} \]
which gives a contribution to the residue of \((1 - q^{-1})^2 q^{-es}\) times

\[
\frac{1}{1 - q^{-\left(m(a^2)s + |a^2|\right)}} + \frac{q^{-(m(a^2)s + |a^2|)}}{1 - q^{-\left(m(a^2)s + |a^2|\right)}}
\]

for \(s = -\frac{|b|}{m(b)}\). But this expression will be zero if and only if

\[
\frac{|a_1^1 + a_2^1|}{m(a^1) + m(a^2)} = \frac{|b|}{m(b)}
\]

which is true by Proposition 3. q.e.d.

4. The existence of poles of \(Z(s)\)

We first obtain a result analogous to the one obtained in [10] finding the largest root of the Bernstein-Sato polynomial for a real-valued function.

**Theorem 2:** Let \(b\) be a vector dual to a face of the Newton polygon that intersects the diagonal \(x = y\). Assume \(|b|/m(b) \neq 1\). Then \(Z(s)\) has a simple pole at \(-\frac{|b|}{m(b)}\) and this is the pole closest to the origin.

**Proof:** As before let \(b = (b_1, b_2)\) and let \(a^1 = (a_1^1, a_2^1)\) and \(a^2 = (a_1^2, a_2^2)\) denote the vectors such that \(a^1\) and \(b\) form the basis for the unit cone \(\sigma_1\) and \(a^2\) and \(b\) form the basis for the unit cone \(\sigma_2\) where we assume \(\det \left( a^1 \right) = \det \left( b \right) = 1\). Then in the charts \(K^2(\sigma_1)\) and \(K^2(\sigma_2)\) the pullbacks of \(Z(s)\) are

\[
Z_1(s) = \int_{R^1} \int_{R^2} |y_1|^{m(a^1)s + |a^1| - 1} |y_2|^{m(b) + |b| - 1} |f_{\sigma_1}|^s |dy_1| |dy_2|
\]

and

\[
Z_2(s) = \int_{R^1} \int_{R^2} |y_1|^{m(b)s + |b| - 1} |y_2|^{m(a^2)s + |a^2| - 1} |f_{\sigma_2}|^s |dy_1| |dy_2|
\]

where \(f_{\sigma_1}(0, 0) \neq 0\) and \(f_{\sigma_2}(0, 0) \neq 0\).

Consider \(Z_1(s)\). We can decompose \(R \times R\) into cosets of \(P^k \times P^k\) for \(k\) sufficiently large so that in each coset at least one of the following is true:

a) \(|f_{\sigma_1}|\) is a non-zero constant
b) \(|y_2|\) is a non-zero constant
c) \(|y_1|\) is a non-zero constant.

Consider first of all a coset \((c_i + P^k) \times (a_i + P^k)\) of type (a). The
integral over this coset has the form
\[ q^{-ls} \int_{a_i + P^k} \int_{c_i + P^k} [f_1 y_1 | m(a')s + a'i | - 1 | y_2 | m(b)s + |b| - 1 | d y_1 | | d y_2 | \]

where \( |f_{ao}| = q^{-l} \). In the following, terms of type (1), (2), (3) or (4) refer to those appearing in Theorem 1. This integral contributes a term of type (1) if \( a_i \notin P^k \) and \( c_i \notin P^k \), a term of type (2) if \( c_i \in P^k \) and \( a_i \notin P^k \), a term of type (3) for a positive constant \( C \) if \( c_i \notin P^k \) and \( a_i \in P^k \); and a term of type (4) for a positive constant \( C \) if \( c_i \in P^k \) and \( a_i \in P^k \). If we consider a coset of type (b) then using an argument similar to that given in Theorem 1 we will get terms of the form (1), (2), (3), or (4). Similarly, if we integrate over cosets of type (c) we get terms entirely similar to (1), (2), (3) and (4) where we replace \( a^i \) by \( a^2 \). We can similarly determine the form of all contributions to \( Z_2(s) \).

First of all suppose the diagonal intersects the Newton polygon in a vertex \((d, d)\) of the polygon. Let \( b' \) and \( b^2 \) denote the covectors dual to the two faces having \((d, d)\) as a common vertex, where we assume \( \det(b_1) > 0 \). Let \((I_1, I_2)\) denote the other vertex of the face dual to \( b^1 \) and \((J_1, J_2)\) denote the other vertex of the face dual to \( b^2 \). First consider the case where \( b = b^1 \), and let \( a^1, a^2 \) be the vectors associated to \( b \) as above. Then as the proof of Corollary 1 to Proposition 3 shows we have
\[ m(b)|a^1| - m(a^1)|b| = 0 \text{ and } m(b)|a^2| - m(a^2)|b| = I_1 - I_2 > 0. \]

Similarly if \( b = b^2 \) and \( a^1 \) and \( a^2 \) are again associated to \( b \) as above we have
\[ m(b)|a^1| - m(a^1)|b| = J_2 - J_1 > 0 \text{ and } m(b)|a^2| - m(a^2)|b| = 0. \]

If the diagonal does not intersect the Newton polygon in a vertex let \((I_1, I_2)\) and \((J_1, J_2)\) denote the vertices of the face intersecting the diagonal, where we assume \( I_1 < J_1 \). Then the proof of Corollary 1 to Proposition 3 shows that if \( b, a^1, a^2 \) are as above we have
\[ m(b)|a^1| - m(a^1)|b| = I_2 - I_1 > 0 \text{ and } m(b)|a^2| - m(a^2)|b| = J_1 - J_2 > 0. \]

Thus in all cases the ratios \( m(b)|a^1| - m(a^1)|b| \geq 0 \) and we also have \( m(b) - |b| \geq 0 \). Hence all of the terms of the form (1)-(6) give contributions to the residue at \( s = -|b|/m(b) \) which are greater than or equal to

\[ Cq^{-ls} q^{-l (m(a)|a^1| - m(a^1)|b| - 1)} \frac{q^{-l (m(b)|b| - m(b)|b|)} k}{1 - q^{-l (m(b)|b| - m(b)|b|)}} \]

for a positive constant \( C \) if \( c_i \in P^k \) and \( a_i \in P^k \). If we consider a coset of type (b) then using an argument similar to that given in Theorem 1 we will get terms of the form (1), (2), (3), or (4). Similarly, if we integrate over cosets of type (c) we get terms entirely similar to (1), (2), (3) and (4) where we replace \( a^1 \) by \( a^2 \). We can similarly determine the form of all contributions to \( Z_2(s) \).
zero. Since there must be terms of type (5) we see that the total contribution to the residue at $s = -|b|/m(b)$ is strictly positive. $\text{q.e.d.}$

**Remark:** An evident extension of this result to polyhedra ($n > 2$) can also be easily inferred from the proof. The essential point for the case in which the polyhedra has a unique face $\tau$ intersecting the diagonal is the following. Let $b$ be the covector dual to $\tau$ and let $\sigma = \langle a^1 = b, a^2, \ldots, a^n \rangle$ be a cone containing $b$ as one of its spanning vectors. Set $s_0 = -|b|/m(b)$. Then the values $m(a')s_0 + |a'|$ are all strictly positive for $i \neq 1$. As such, the contribution to the residue of $Z(s)$ at $s = s_0$ in $\mathbb{K}^n(\sigma)$ along the divisor with numerical data $(m(b), |b|)$ is always positive. Summing up over all the cones containing $b$ gives a positive value to the residue at $s_0$.

In order to extend this result to smaller poles, that is values $-|b|/m(b)$ where $b$ is dual to a face of $\Gamma$ not intersecting the diagonal, it appears necessary to impose additional technical conditions both on the polygon $\Gamma$ and functions $f \in \mathcal{F}$. These are:

**Condition 1:** No two support lines containing faces of $\Gamma$ should intersect on the diagonal $x = y$.

**Condition 2:** The coefficients of the principal part $f$ are all in $R - P$.

**Condition 3:** Let $\sigma = \langle b, a \rangle$ where $b$ is dual to a face. Let $D_b$ be the divisor in the exceptional locus corresponding to $b$. In the chart $K^2(\sigma)$, if the strict transform $f_0$ of $f$ has the property $\deg(f_0 |_{D_b}) = s$, then there must be $s$ numbers $\lambda_1, \ldots, \lambda_s$ in $R$ satisfying the property that $\lambda_i \neq \lambda_j \mod P$ for $i \neq j$ and $(f_0 |_{D_b})(\lambda_i) = 0$ for all $i$.

**Remark:** Conditions (2) and (3) are actually independent of the choice of resolution (or equivalently the refinement into unit cones of the original partition dual to the polygon). To see this fix a covector $b = b_1$ dual to the face $\tau$, with vertices $I = (I_1, I_2)$, $J = (J_1, J_2)$ such that $J_2 > J_1$.

The conditions $\det\begin{pmatrix} x_1 \\ b_1 \end{pmatrix} = 1$, $x = (x_1, x_2)$ with $x_1, x_2 \geq 0$, determine a ray on the first quadrant, all integral points of which correspond to possible covectors with which $b$ could be paired to form a unit cone refining the original partition. Of course not all covectors lead to valid choices as some might have first meet loci on $\Gamma$ in vertices other than $I$.

Since $x_2x_1^{-1} - b_2b_1^{-1} = -(x_1b_1)^{-1}$, as $x_1$ increases the direction of $(x_1, x_2)$ approaches that of $b$. Thus there is a smallest positive integer $\xi_1$ such that

\begin{align*}
  i) \quad & \xi_2 = \xi_1 \left( \frac{b_2}{b_1} \right) - \frac{1}{b_1} \in \mathbb{N} \\
  ii) \quad & \text{If } (\xi_1, \xi_2) = a^0, \text{ the first meet locus } K_{a^0} = \{ I \}.
\end{align*}
Observe that if \( a = (a_1, a_2) \) is any integral point with \( b_2/b_1 > a_2/a_1 > \xi_2/\xi_1 \) and \( \det \begin{pmatrix} a \\ b \end{pmatrix} = 1 \), there is some positive integer \( r \) such that \( a = a^0 + r \cdot b \).

Let \( f_\tau = \sum_{k \in \tau, \gamma} a_k x_1^k x_2^{k'} \) be the principal part of \( f \) for the face \( \tau \). Let \( \sigma = \langle a, b \rangle \) be a unit cone in a refinement of the original partition with \( \det \begin{pmatrix} a \\ b \end{pmatrix} = 1 \). Then one sees that

\[
\left( f_\tau \circ \pi(\sigma) \right)(y_1, y_2) = y_1^{m(a)} y_2^{m(b)} \sum_{K \in \tau} a_K y_1^{K \cdot a - M(a)}.
\]

The right hand side is the part of the strict transform \( f_\sigma \) upon which conditions (2) and (3) are imposed.

Now observe that \( m(a) = m(a^0) + rm(b) \) and \( K \cdot a = K \cdot a^0 + rm(b) \) since \( K \in \tau \). Thus, \( K \cdot a - m(a) = K \cdot a^0 - m(a^0) \) is independent of \( r \) and depends only on \( b \) since \( a^0 \) only depends on \( b \). Identical reasoning is used for the conditions \( \det \begin{pmatrix} b \\ x \end{pmatrix} = 1 \), \( x = (x_1, x_2) \) with \( x_1, x_2 \geq 0 \) and then applied to cones \( \sigma = \langle b, a \rangle \) with \( \det \begin{pmatrix} b \\ a \end{pmatrix} = 1 \).

This shows that conditions (2) and (3) are independent of the refinement and so, the resolution \( \pi: X(\Gamma) \rightarrow K^2 \).

As seen next, conditions (1)–(3) imply that the residue of \( Z(s) \) at \( s = -|b|/m(b) \) cannot be zero. On the other hand lifting any of these conditions seems to produce an expression for the residue for which showing it is non-zero is difficult.

**Theorem 3:** Let \( b \) be a covector dual to a face of a polygon \( \Gamma \) satisfying condition (1) and such that if \( b = (b_1, 1) \) (resp. \( b = (1, b_2) \)) then neither vertex of the face dual to \( b \) has the form \( (0, p_2) \) (resp. \( (p_1, 0) \)). Then for any function \( f \in \mathcal{G}_r \) satisfying conditions (2) and (3), the ratio \( s = -|b|/m(b) \) is a pole of \( Z(s) \).

**Proof:** As before let \( a^1 \) and \( a^2 \) denote the vectors such that \( a^1 \) and \( b \) form the basis for the unit cone which we shall denote by \( \sigma_1 \), and \( a^2 \) and \( b \) form the basis for the unit cone which we shall denote by \( \sigma_2 \). As before we have that the contribution to the pole at \( -|b|/m(b) \) will come from the pullback of the integral for \( Z(s) \) in the charts \( K^2(\sigma_1) \) and \( K^2(\sigma_2) \). In \( K^2(\sigma_1) \) we have

\[
(f \circ \pi(\sigma_1))(y_1, y_2) = y_1^{m(a^1)} y_2^{m(b)} f_{\sigma_1}(y_1, y_2)
\]

and in \( K^2(\sigma_2) \) we have

\[
(f \circ \pi(\sigma_2))(y_1', y_2') = y_1'^{m(b)} y_2'^{m(a^2)} f_{\sigma_2}(y_1', y_2').
\]
where \( f_{01} \) and \( f_{02} \) denote the strict transforms of \( f \) in each chart. Thus the integrals \( Z_1(s) \) and \( Z_2(s) \) will contribute to the pole at \( s = -|b|/m(b) \) where

\[
Z_1(s) = \int_R \int_R |y_1|^{|m(a)|s + |a^1| - 1} |y_2|^{|m(b)|s + |b| - 1} |f_{01}|^s |d\,y_1| |d\,y_2|
\]

and

\[
Z_2(s) = \int_P \int_R |y_1'|^{|m(b)|s + |b| - 1} |y_2'|^{|m(a^2)|s + |a^2| - 1} |f_{02}|^s |d\,y_1'| |d\,y_2'|
\]

We examine the form of \( f_{01} \) in more detail. Since \( b \) is a covector dual to a face we have that for any point \( P \) on the face \( b \cdot P = m(b) \). The values of \( m(a^1) \) and \( m(a^2) \) are determined by opposite vertices of this face. We again recall that a monomial of the form \( x^{f^1}x^{f^2} \) is transformed into \( y_1^{m(a^1)} y_2^{m(b)} \) in \( K^2(a_1) \); and \( f_{01} \) is obtained by factoring \( y_1^{m(a^1)} y_2^{m(b)} \) from each monomial. Therefore \( f_{01} \) has the form \( c + p(y_1) + y_2 h(y_1, y_2) \) where \( c \) is the coefficient of the monomial term corresponding to the vertex of the polygon that determines \( m(b) \) and \( m(a^1) \), \( p(y_1) \) comes from the other monomials on the face for which \( b \) is the covector, and \( y_2 h(y_1, y_2) \) comes from all the other monomials. Similarly \( f_{02} \) has the form \( d + p'(y_2') + y_1' h(y_1', y_2') \) where \( d \) is the coefficient of the monomial term that determines \( m(b) \) and \( m(a^2) \).

We first consider the integral for \( Z_1(s) \). By an argument similar to that given in Theorem 1 we can reduce the domain of integration to \( R \times P \) without affecting the residue. We then split this domain of integration into \( (R - P) \times P \) and \( P \times P \). On \( P \times P \) we have that \( |f_{01}| = 1 \), hence the integral over this domain is

\[
(1 - q^{-1})^2 \frac{q^{-|m(a)|s + |a^1|} q^{-|m(b)|s + |b|}}{(1 - q^{-|m(a)|s + |a^1|})(1 - q^{-|m(b)|s + |b|})}
\]

which gives a contribution to the residue of

\[
(1 - q^{-1})^2 \frac{q^{-|m(a)|s + |a^1|}}{1 - q^{-|m(a)|s + |a^1|}}.
\]

In order to consider the domain of integration \( (R - P) \times P \) we observe that condition 3 allows us to write \( c + p(y_1) = \prod_{i=1}^r (a_i y_1 - b_i) \). By our assumptions we have \( a_i \) and \( b_i \) are in \( R - P \) for all \( i \) by condition 2 and if we let \( \lambda_i = b_i a_i^{-1} \) for \( 1 \leq i \leq r \), we have that the \( \lambda_i \) are distinct modulo \( P \) by condition 3. We write \( R - P \) as a disjoint union of cosets.
modulo $P$ so that the $\lambda_i$ are in distinct cosets. Let \( \{\lambda_1, \ldots, \lambda_r, \theta_1, \ldots, \theta_t\} \)
\denote a complete set of coset representatives modulo $P$ in \( R - P \).

We first consider a domain of integration of the form \( (\theta_i + P) \times P \).
\Then \( |y_1| = 1 \) on \( \theta_i + P \) and by replacing \( y_1 \) by \( \theta_i + y_1 \) we get an integral of the form

\[
\int_{P \times P} |y_2|^{m(b)s + |b|-1} \times \prod_{j=1}^{r} (a_j(\theta_i + y_1) - b_j) + y_2 h(\theta_i + y_1, y_2) \, |d\, y_1| \, |d\, y_2|.
\]

Observing that \( a_j \theta_i - b_j \not\in P \) for any \( j \) we have that the integral over this domain is equal to

\[
\int_{P \times P} |y_2|^{m(b)s + |b|-1} |d\, y_1| \, |d\, y_2|,
\]

which gives a contribution to the residue of

\[
(1 - q^{-1})q^{-1}.
\]

Now consider a domain of integration of the form \( (\lambda_i + P) \times P \). Then \( |y_1| = 1 \) on this coset so by replacing \( y_1 \) by \( \lambda_i + y_1 \) we get an integral of the form

\[
I(q^{-s}) = \int_{P \times P} |y_2|^{m(b)s + |b|-1} \times a_i y_1 \prod_{j=1}^{r} (a_j(\lambda_i + y_1) - b_j) + y_2 h(\lambda_i + y_1, y_2) \, |d\, y_1| \, |d\, y_2|.
\]

We wish to calculate \( \lim_{s \to -|b|/m(b)} (1 - q^{-(m(b)s + |b|)}) I(q^{-s}) \). Let

\[
I_n(q^{-s}) = \int_{P \times P^n} |y_2|^{m(b)s + |b|-1} \times a_i y_1 \prod_{j=1, j \neq i}^{r} (a_j(\lambda_i + y_1) - b_j) + y_2 h(\lambda_i + y_1, y_2) \, |d\, y_1| \, |d\, y_2|.
\]

Since \( (1 - q^{-(m(b)s + |b|)}) I_n(q^{-s}) \) converges uniformly to \( (1 -
\[ q^{-\left(m(b)s + |b|\right)}I(q^{-s}) \] as \( n \to \infty \) in a neighborhood of \(-|b|/m(b)\); we have that the contribution to the residue is equal to \( \lim_{n \to \infty} \lim_{s \to -|b|/m(b)} (1 - q^{-\left(m(b)s + |b|\right)}) I_n(q^{-s}). \) By reasoning similar to before, we can reduce the domain of integration from \((P - P^n) \times P\) to \((P - P^n) \times P^n\) without affecting the contribution to the residue at \( s = -|b|/m(b) \) of \( I_n(q^{-s}). \) Then, again observing that \( q \lambda_j - b_j \notin P \) for any \( j \neq i \) we have that the integral over this domain is equal to

\[ \int_{P^n} \int_{P^n} |y_2|^{m(b)s + |b|-1} |y_1|^s |d\ y_1| |d\ y_2|. \]

So \( I_n(q^{-s}) \) gives a residue contribution of

\[ (1 - q^{-1})^2 \frac{q^{-\left(s+1\right)} - q^{-\left(s+1\right)n}}{1 - q^{-\left(s+1\right)}}. \]

So taking the limit as \( n \to \infty \) we see that the contribution to the residue at \( s = -|b|/m(b) \) from \( I(q^{-s}) \) is

\[ (1 - q^{-1})^2 \frac{q^{-\left(s+1\right)}}{1 - q^{-\left(s+1\right)}}. \] (9)

Now consider the integral \( Z_2(s). \) Again by an argument similar to that given in Theorem 1 we can reduce the domain of integration to \( P \times P \) without affecting the residue at \( s = -|b|/m(b). \) Then \( |f_{s_2}| = 1 \) on this domain so the integral over this domain is

\[ (1 - q^{-1})^2 \frac{q^{-\left(m(a^2)s + |a^2|\right)} - q^{-\left(m(b)s + |b|\right)}}{(1 - q^{-\left(m(a^2)s + |a^2|\right)})(1 - q^{-\left(m(b)s + |b|\right)})} \]

which gives a contribution to the residue of

\[ (1 - q^{-1})^2 \frac{q^{-\left(m(a^2)s + |a^2|\right)}}{(1 - q^{-\left(m(a^2)s + |a^2|\right)})} \] (10)

for \( s = -|b|/m(b). \)

Combining the contributions to the residue from (7), (8), (9) and (10) gives

\[ (1 - q^{-1})^2 \left( \frac{q^{a_1}}{1 - q^{a_1}} + \frac{q^{a_2}}{1 - q^{a_2}} + \frac{rq^{a_3}}{1 - q^{a_3}} + \frac{tq^{-1}}{1 - q^{-1}} \right) \]
where \( \alpha_i = -(m(a')s + |a'|) \) for \( i = 1, 2 \); \( \alpha_3 = -(s + 1) \), and \( s = -|b|/m(b) \).

If \( q = p^f \), then we consider the above expression in the fully ramified extension of \( Q_p \) obtained by adjoining \( \pi \) where \( \pi^{m(b)} = p \). We then consider the expansion of the above expression in integral powers of \( \pi \).

Let \( \gamma_j = m(b)\alpha_j \) for \( 1 \leq j \leq 3 \). We have

\[
\frac{q^{\alpha_j}}{1 - q^{\alpha_j}} = \pi^{\gamma_i f} + \pi^{2\gamma_i f} + \ldots
\]

if \( \alpha_j > 0 \), and

\[
\frac{q^{\alpha_j}}{1 - q^{\alpha_j}} = -1 - \pi^{-\gamma_j f} - \pi^{-2\gamma_j f} - \ldots
\]

if \( \alpha_j < 0 \). We observe that \( \alpha_3 = |b| - m(b) \). Thus \( \alpha_3 \geq 0 \) only in the cases excluded in the statement of the theorem or the case covered by Theorem 2. Therefore we may assume \( \alpha_3 < 0 \). Also we observe that the proof of Corollary 1 to Proposition 3 shows that \( \alpha_1 \) and \( \alpha_2 \) have opposite sign.

We write \( r \) in terms of its \( p \)-adic expansion \( r = a_0 + a_1 p + \ldots + a_{(f-1)} p^{(f-1)} \) where \( 0 \leq a_i \leq (p - 1) \) and recall that \( r + t = q - 1 \).

Thus in the case where \( \alpha_1 < 0 \) and \( \alpha_2 > 0 \) the expression for the residue becomes

\[
-(1 + \pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \ldots) + \pi^{\gamma_2 f} + \pi^{2\gamma_2 f} + \ldots
\]

\[
-\sum_{i=0}^{f-1} a_i \pi^{m(b)}(1 + \pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \ldots)
\]

\[
+ \left(1 + \sum_{i=0}^{f-1} a_i \pi^{m(b)} - \pi^{m(b)}f\right)(1 + \pi^{m(b)}f + \pi^{2m(b)}f + \ldots).
\]

By cancelling leading terms the above expression becomes

\[
-(\pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \ldots) + \pi^{\gamma_2 f} + \pi^{2\gamma_2 f} + \ldots
\]

\[
-\sum_{i=0}^{f-1} a_i \pi^{m(b)}(\pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \ldots)
\]

\[
+ \sum_{i=0}^{f-1} a_i \pi^{m(b)}(\pi^{m(b)}f + \pi^{2m(b)}f + \ldots).
\]
By Corollary 1 to Proposition 3 we have that \(-\gamma_3 = \gamma_1 + \gamma_2 + m(b) + (k - 1) |b|\) for \(k \geq 1\). The assumptions of the theorem assure us that by Corollary 2 to Proposition 3 we have \(\gamma_1 > -m(b)\) and hence \(-\gamma_3 > \gamma_2\). Corollary 1 to Proposition 3 also assures us that \(\gamma_1 \neq -\gamma_2\) hence the lowest order term in the above expression is either \(-\pi^{-\gamma_1}f\) or \(\pi^{\gamma_2}f\). Therefore the residue is non-zero at \(s = -|b|/m(b)\). The proof in the case where \(\alpha_1 > 0\) and \(\alpha_2 < 0\) is entirely similar. q.e.d.

We give an example to illustrate the possibility of an exceptional case where the residue is zero excluded by the above theorem. Let \(f(x_1, x_2) = x_2^5 + x_1x_2^4 + x_1^3x_2^2 + x_1^6\). Then the vector \(b = (2, 1)\) is dual to the face with vertex \((0, 6)\). Take \(a_1 = (1, 0)\) and \(a_2 = (1, 1)\). Then we have a possible pole at \(-1/2\) but in the above expression for the residue we have \(r = 1, -\gamma_1 = m(b) = 6, -\gamma_3 = \gamma_2 = 3\) which gives a residue of zero.

### 5. Examples

Examples of curves satisfying the conditions in Theorem 2 can be easily found. We illustrate the preceding theory with one such class of examples.

Let \(\{(p_i, q_i)\}_{i=1}^N\) be a set of pairs of relatively prime positive integers such that

\[
\begin{align*}
\text{i)} & \quad \frac{-q_N}{p_N} \leq \frac{-q_{N-1}}{p_{N-1}} \leq \ldots \leq \frac{-q_1}{p_1} \\
\text{ii)} & \quad q_N \geq q_{N-1} \geq \ldots \geq q_1 \quad \text{and} \quad p_N \geq p_{N-1} \geq \ldots \geq p_1.
\end{align*}
\]

Let \(\{((\alpha_i, \beta_i))\}_{i=1}^N\) be a set of pairs of elements each belonging to \(R - \{0\}\).
Set \(f_\Gamma(x_1, x_2) = \prod_{i=1}^N(\alpha_i x_1^{\beta_i} + \beta_i x_2^{\beta_i})\). Let \(H(x_1, x_2)\) be any polynomial such that the polygon for \(H\) lies completely above the polygon \(\Gamma\) for \(f_\Gamma\).
Set \(f = f_\Gamma + H\). We consider the curve defined by \(f\).

The polygon \(\Gamma\) can be easily described for the above class of curves. Each distinct ratio \(-q_i/p_i\) is associated to a face \(\tau_i\) of the polygon having that ratio as its slope, and these comprise all of the faces of the polygon.

The primitive covector \(b' = (q_i, p_i)\) is dual to the face \(\tau_i\). The number of integral points on each face \(\tau_i\) is one more than the number of pairs \((p_k, q_k)\) which are equal to the pair \((p_i, q_i)\). The \(b'\) determine a partition of \((\mathbb{R}_+^*)\) which we refine as indicated in Section 1.

Consider a fixed \(b'\) and let \(a^1\) and \(a^2\) denotes the vectors such that \(\sigma_1 = \langle a^1, b'\rangle\) and \(\sigma_2 = \langle b', a^2\rangle\) form unit cones for the partition where we assume \(\det(a^1) = \det(b', a^2) = 1\). We consider the pullback of \(f_\Gamma\) in \(K^2(\sigma_1)\). Let \(i = i_1, \ldots, i_t\) be such that \((p_{i_l}, q_{i_l})\) = \((p_i, q_i)\) for \(1 \leq l \leq t\). Let
$j_1, \ldots, j_u$ denote those $j$ such that $\det \begin{pmatrix} b^{j_l} \\ b^j \end{pmatrix} > 0$ for $1 \leq l \leq u$ and let $j_{u+1}, \ldots, j_v$ denote those $j$ such that $\det \begin{pmatrix} b^{j_l} \\ b^j \end{pmatrix} < 0$ for $u < l \leq v$. Then

$$(f_1 \circ \pi(\sigma_1))(y_1, y_2) = y_1^{m(a^j)} y_2^{m(b^j)}$$

$$\times \prod_{l=1}^t (\alpha_i y_1 + \beta_i)(c + y_2 h_{\sigma_i}(y_1, y_2))$$

where $c = \prod_{l=1}^u \alpha_i \prod_{l-u+1}^v \beta_i$. Similarly one can show that in $K^2(\sigma_2)$ we have

$$(f_1 \circ \pi(\sigma_2))(y_1', y_2') = y_1'^{m(b^j)} y_2'^{m(a^j)}$$

$$\times \prod_{l=1}^t (\alpha_i + \beta_i y_2')(c + y_1' h_{\sigma_i}(y_1', y_2'))$$

In order to consider a possible pole of $Z(s)$ at $s = -|b'|/m(b')$ we impose Condition 1 of Theorem 2, i.e. that $-|b|/m(b) \neq -|b'|/m(b')$, thus localizing the calculation of the residue to the divisor $D_{b'}$. In the case where $t = 1$ we need to consider the contribution from the integrals

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^j)s + |a|^j |a|^{-1} |y_2|^{m(b^j)s + |b|^j |b|^{-1} |\alpha_i y_1 + \beta_i| c + y_2 h_{\sigma_i}(y_1, y_2) |^s |d y_1| |d y_2|$$

and

$$Z_2(s) = \int_P \int_R |y_1'|^{m(b^j)s + |b|^j |b|^{-1} |y_2'|^{m(a^j)s + |a|^j |a|^{-1} |\alpha_i + \beta_i y_1'|^s |c + y_1' h_{\sigma_i}(y_1', y_2') |^s |d y_1'| |d y_2'|$$

at $s = -|b'|/m(b')$. These integrals have the exact form as those for irreducible curves which are shown to give a non-zero residue in Theorem 2 in [7].

In the cases where $t > 1$ it is necessary to examine

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^j)s + |a|^j |a|^{-1} |y_2|^{m(b^j)s + |b|^j |b|^{-1} |p_1(y_1)|^s |c + y_2 h_{\sigma_i}(y_1, y_2) |^s |d y_1| |d y_2|$$

and

$$Z_2(s) = \int_P \int_R |y_1'|^{m(b^j)s + |b|^j |b|^{-1} |y_2'|^{m(a^j)s + |a|^j |a|^{-1} |p_1(y_1')|^s |c + y_1' h_{\sigma_i}(y_1', y_2') |^s |d y_1'| |d y_2'|$$
and

\[ Z_2(s) = \int \int_{\mathbb{R}} |y_1'|^{m(b_i)} |b_i|^{-1} |y_2'|^{m(a^2)} |a^2|^{-1} |p_2(y_1')|^s \cdot c \]

\[ + y_1' h_{\alpha_2}(y_1', y_2') | y_1'|^s |dy_1'| |dy_2'| \]

where \( p_1(y_1) \) and \( p_2(y_1') \) are polynomials of degree \( > 1 \) where the roots of \( p_1(y_1) \) are \(-\beta_{i_l}/\alpha_{i_l}\) and the roots of \( p_2(y_1') \) are \(-\alpha_{i_l}/\beta_{i_l}\) for \( 1 \leq l \leq t \). The imposition of Condition 2 is the statement that the \( \alpha_{i_l} \) and \( \beta_{i_l} \) should be in \( R - P \) for \( 1 \leq l \leq t \) and the imposition of Condition 3 is the statement that \( \alpha_{i_k}/\beta_{i_k} \neq \alpha_{i_l}/\beta_{i_l} \) modulo \( P \) for \( k \neq l \ 1 \leq k \leq t, 1 \leq l \leq t \). Under these restrictions Theorem 2 shows that the residue is non-zero at \( s = -|b'|/m(b') \).

References


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