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DUALITY THEOREMS FOR $\Gamma$-EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

Kay Wingberg

Introduction

There is a remarkable analogy between the theory of algebraic function fields in one variable over a finite field and the theory of algebraic number fields. Namely, on the one hand let $C$ be a complete, non-singular curve of genus greater than zero defined over a function field $F$, let $J$ be the Jacobian variety of $C$ and let $J_p$ be the $p$-primary subgroup of the group of points of $J$ defined over the algebraic closure $\bar{F}$ of $F$ ($p$ an odd prime distinct from the characteristic of $F$). The Frobenius automorphism of $\bar{F}/F$ induces an endomorphism of $J_p$, and a fundamental theorem of Weil asserts that the characteristic polynomial of this endomorphism is essentially the zeta function of the curve $C$. On the other hand let $\mathcal{X}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathcal{X} = K(\mu_p)$, where $K$ is a totally real number field and $\mu_p^n$ denotes the $(p^n)^{th}$ roots of unity. Let $A_n$ be the $p$-primary subgroup of the ideal class group of $\mathcal{X}_n = \mathcal{X}(\mu_{p^n+1})$ and let $A$ be the inductive limit of the groups $A_n$ and $A' = (1 - \sigma)A$, where $\sigma$ is the automorphism of the CM-field $\mathcal{X}$ induced by the complex conjugation. Iwasawa has proposed that the $\Gamma = G(\mathcal{X}_\infty/\mathcal{X})$-module $A'$ should provide an analogue of $J_p$ and the “main-conjecture” of this theory is that the characteristic polynomial of $(A')^* = \text{Hom}(A', \mathbb{Q}_p/\mathbb{Z}_p)$ should be very closely related to the $p$-adic zeta function of $K$. To prove this conjecture is extremely difficult and has been fully established for abelian number fields by Mazur/Wiles.

This idea in mind, we might expect a functional equation for the characteristic polynomial of $(A^-)^*$. But there are fields $\mathcal{X}$ with $A^- = \mathbb{Q}_p/\mathbb{Z}_p$ (for example $\mathcal{X} = \mathbb{Q}(\zeta_{37})$) and consequently there cannot in general exist a non-degenerating, skew-symmetric pairing on $(A^-)^*$ analogous to the Weil scalar product on $J_p$. One aim of this paper is to show how to obtain a functional equation in the case of CM-fields by adding certain “local factors” to $A^-$. Namely, in §9 we defined two $\Gamma$-modules $\mathcal{L}_+$ and $\mathcal{L}_-$ which have many of the desired properties. The most important result is the following: Tensoring these modules with $\mathbb{Q}_p$ we get two non-degenerating, skew-symmetric, $\Gamma$-invariant pairings and therefore a functional equation for the characteristic polynomial $h_\pm(T)$ of $\mathcal{L}_\pm$, namely:

$$\kappa(\gamma_0)^{\lambda\mp} \cdot h_\pm(T-1) = T^{2\lambda\mp} \cdot h_\pm(\kappa(\gamma_0)T^{-1} - 1),$$

where \(2\lambda^t\) is the \(\mathbb{Z}_p\)-rank of \(\mathcal{L}_x\), \(\gamma_0\) a generator of \(\Gamma\) and \(\kappa\) the cyclotomic character \(\kappa : \Gamma \rightarrow \mathbb{Z}_p^\times\). In the function field case we obtain the well known function equation for the zeta function of \(X_p\).

We will see that assuming the Leopoldt-conjecture for all the fields in the cyclotomic tower \(\mathcal{L}_x \otimes \mathbb{Q}_p\) is the symplectic \(\mathbb{Q}_p[[\Gamma]]\)-module studied by Iwasawa in [9] §11. Conjecturally this module is trivial (see Greenberg [5]).

The module \(\mathcal{L}_x\), however, is in general non-trivial and turns out to be an interesting object of consideration. We obtain the following remarkable properties: In a finite \(p\)-extension \(E/K\) the \(\mathbb{Z}_p\)-ranks of \(\mathcal{L}(E)\) and \(\mathcal{L}(K)\) are connected by a relation analogue to the Riemann-Hurwitz formula for the genus of a curve. Further, \(\mathcal{L}_x\) does not contain any finite non-trivial \(\Gamma\)-submodule. The module \(\mathcal{L}_x\) is defined as the Galois group \(G(M'/K_\infty) = G(L'(X)M(K^+)/K_\infty)/U\), where \(L'(X)\) is the maximal completely decomposed abelian \(p\)-extension of \(X_\infty\), \(M(K^+)^+\) the maximal abelian \(p\)-extension unramified outside \(p\) of the maximal totally real subfield \(K^+\) of \(K\). The module \(L(M(K^+)^+)\) is defined as the Galois group \(G(M(K^+)^+/(K^*_p)^+)\) where \(L(K)^+\) is the maximal completely decomposed abelian \(p\)-extension of \(K^+\) and \(U\) is generated by the projective limits of the roots of unity of \((K^+_n)_{\nu n}, \nu p\), considered via the reciprocity homomorphism in \(G(M(K^+)^+/(K^*_p)^+)\). In the case when none of the prime divisors of \(p\) splits in the extension \(X/K^+\) this field \(M'\) is simply the compositum of \(M(K^+)\) and the maximal unramified abelian \(p\)-extension \(L(X)\) of \(X_\infty\). If there is only one prime \(\nu\) above \(p\) in \(X_\infty\) we shall see that \(\mathcal{H}_x(T)\) is the product of the characteristic polynomials of the global group \((A^-)^*\) and the local galois group \(G(M(K^+)^+/(K^*_p)^+)\).

To obtain these results we first need a description of the \(\mathbb{Z}_p[[\Gamma \times G(X/K)]]\)-module structure of the local Galois groups \(G(X(p)_{\nu}^\prime/X_\infty)_{ab}, \nu p\), where \(X(p)_{\nu}\) denotes he maximal \(p\)-extension of the completion \(\mathcal{X}_\nu\) of \(X\) (§6). Secondly we strongly used the global duality theorem, whereas most of the fundamental results obtained by Iwasawa in [9] are proved by means of Kummer-theory. We want to show that the application of the duality theorem of Tate/ Poitou not only gives some finer results (e.g. formulas for the \(\lambda\)-invariants of the three Iwasawa-modules usually considered (§§7, 8)), but also leads to a duality theorem for certain other \(G(X_\infty/X)\)-modules in a natural way (§7).

The Riemann-Hurwitz formula for \(\mathcal{L}_x\) relies heavily on a Riemann-Hurwitz formula for the \(\mathbb{Z}_p\)-ranks \(\lambda_1(E)\) and \(\lambda_1(K)\) of the modules \(G(M(E)/E_\infty)\) and \(G(M(K)/K_\infty)\), where \(E/K\) is a finite \(p\)-extension of a totally real ground field \(K\) for which \(\mu(K) = 0\):

\[
\lambda_1(E) - 1 = (\lambda_1(K) - 1)[E_\infty : K_\infty] + \sum_{\nu p} (e_\nu - 1),
\]

where \(e_\nu\) denote the ramification indices for the extension \(E_\infty/K_\infty\). From this easily follows the result of Kida [16] and Iwasawa [11] for the \(\mathbb{Z}_p\)-rank of \((A^-)^*\). The proof in §7 is using the analogue of Riemann's
existence theorem for $p$-closed number fields [20]. In particular, it is necessary to consider non abelian extensions of $K_\infty$ resp. $E_\infty$.

The facts about finite $\Gamma$-submodules are based on certain cohomological criterions for arbitrary noetherian $\Gamma$-modules and group extensions. By these general methods we will obtain the following. Let $K_\infty/K$ be an arbitrary $\Gamma$-extension, let $K_S(p)$ be the maximal $p$-extension of $K$, which is unramified outside the finite set $S$ of primes containing all divisors of $p$ and $\infty$ and let $X_S$ be the galois group $G(K_S(p)/K_\infty)^{ab}$. Assume the weak Leopoldt-conjecture is true, then the $\mathbb{Z}_p[[\Gamma]]$-rank of $X_S$ is equal to the number of complex primes of $K$ and $X_S$ contains not any finite non-trivial $\Gamma$-submodule. If $K_\infty/K$ is the cyclotomic extension or if the Leopoldt-conjecture is valid for $K$, this is the well known result of Iwasawa [9] resp. Greenberg [6].

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## I. $\Lambda$-modules

### §1 Notations and conventions

Let $\Gamma$ denote a compact abelian group isomorphic to the additive group of $\mathbb{Z}_p$ and $\Lambda = \mathbb{Z}_p[[T]]$ the ring of all formal power series in an indeterminate $T$. The ring $\Lambda$ is local, noetherian, regular, of dimension 2 and compact in the $m$-adic topology, where $m = (p, T)$ is the maximal ideal of $\Lambda$. By a theorem of Lazard we have a homeomorphism

$$\Lambda \cong \mathbb{Z}_p[[\Gamma]] = \text{proj lim} \mathbb{Z}_p[\Gamma/\Gamma^p]$$

$$T \mapsto \gamma_0 - 1$$

$\gamma_0$ a generator of $\Gamma$, and we obtain an equivalence of category between the category of compact (discrete) $\Lambda$-modules and the category of compact (discrete) $\Gamma$-modules.

For a compact (discrete) $\Gamma$-module $M$ we have the (co-) homology groups

$$(H^0(\Gamma, M) = H_1(\Gamma, M) = M^\Gamma,$$

$$(H^1(\Gamma, M) = H_0(\Gamma, M) = M_\Gamma.)$$

If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a exact sequence of compact (discrete) $\Gamma$-modules, then there exists an exact (co-) homology sequence

$$0 \rightarrow M'^\Gamma \rightarrow M^\Gamma \rightarrow M''^\Gamma \rightarrow M'_\Gamma \rightarrow M_\Gamma \rightarrow M''_\Gamma \rightarrow 0,$$  

(1.1)
since $\Gamma$ is of cohomological dimension 1. By the lemma of Nakayama we see that a compact (discrete) $\Gamma$-module $M$ is zero if and only if $M_\Gamma = 0$ ($M_\Gamma = 0$).

Since $\Lambda$ is a local ring a compact $\Lambda$-module $M$ is $\Lambda$-free, if and only if $M$ is $\Lambda$-projective. By the above remark the homological dimension $\text{hd}_\Lambda M$ is equal or less than 2; consequently there is a free resolution of $M$

$$0 \rightarrow \Lambda^{I_2} \rightarrow \Lambda^{I_1} \rightarrow \Lambda^{I_0} \rightarrow M \rightarrow 0,$$

where $I_j$, $j = 0, 1, 2$, are sets of indices. In particular we will be interested in compact $\Lambda$-modules $M$ of homological dimension $\leq 1$.

The following criterion for $\Lambda$-freeness of a noetherian $\Lambda$-module turns out to be useful:

$$M \text{ is } \Lambda\text{-free } \Leftrightarrow M_\Gamma = 0 \text{ and } M_\Gamma \text{ is } \mathbb{Z}_p\text{-free}. \quad (1.2)$$

For the proof of the non trivial implication consider the exact sequence

$$0 \rightarrow N \rightarrow \Lambda^m \rightarrow M \rightarrow 0$$

where $m$ is the $\mathbb{Z}_p$-rank of $M_\Gamma$ and $N = \text{Ker } \varphi$ (the surjection $\varphi$ exists by Nakayama’s lemma). Now, the exact sequence

$$0 \rightarrow N_\Gamma \rightarrow \mathbb{Z}_p^m \rightarrow M_\Gamma \rightarrow 0$$

shows that $N_\Gamma = 0$ and consequently that $N = 0$.

We will need the following notations:

For an abelian local-compact group $A$ let $A^*$ be the Pontryagindual, $A(p)$ the $p$-primary part of $A$ and for $m \in \mathbb{N}$ let the groups $mA$ and $A_m$ are defined by the exact sequence

$$0 \rightarrow mA \rightarrow A \rightarrow A_m \rightarrow 0.$$

As usual we put $H'(U)$ for the cohomology group $H'(U, \mathbb{F}_p)$ of a pro-finite group $U$.

In the following $M$ will be a noetherian $\Lambda$-module. We define the generator-rank by:

$$d(M) = \dim_{\mathbb{F}_p} H_0(\Gamma, M)_p$$

and the relation-rank by:

$$r(M) = \dim_{\mathbb{F}_p} H_1(\Gamma, M)_p + \dim_{\mathbb{F}_p} H_0(\Gamma, M).$$

From Nakayama’s lemma follows the existence of a surjective homomorphism $\Lambda^{d(M)} \rightarrow M$ with kernel $N$; considering the exact sequence (1.1) we
obtain easily the equality $d(N) = r(M)$. In particular, if $h\Lambda M \leq 1$, we get an exact sequence

$$0 \to \Lambda^r(M) \to \Lambda^d(M) \to M \to 0. \quad (1.3)$$

According to the general structure theory for noetherian $\Lambda$-modules [1] VII §4.4, theorem 4 and 5, we have a quasi-isomorphism

$$M \sim E(e_0; \psi_1^{e_1}, \ldots, \psi_s^{e_s}) := \Lambda^{e_0} \oplus \Lambda/\psi_1^{e_1} \oplus \cdots \oplus \Lambda/\psi_s^{e_s},$$

where $e_i \geq 0$ and the $\psi_i = f_i(T) \cdot \Lambda$, $i = 1, \ldots, s$, are prime ideals of height 1 in $\Lambda$, that means: $f_i(T) = p$ or $f_i(T)$ is a Weierstrass-polynomial ($f_i(T) \equiv T^{\lambda_i} \mod p$). We say that $E(e_0; \psi_1^{e_1}, \ldots, \psi_s^{e_s})$ is the elementary $\Lambda$-module associated to $M$.

If $T_\Lambda(M)$ is the $\Lambda$-torsion module of $M$ and $F_\Lambda(M)$ the quotient $M/T_\Lambda(M)$, then we have the quasi-isomorphism $M \sim F_\Lambda(M) \oplus T_\Lambda(M)$ with

$$T_\Lambda(M) \sim E(\psi_1^{e_1}, \ldots, \psi_s^{e_s})$$

and $F_\Lambda(M) \sim E(e_0) = \Lambda^{e_0}$,

where $e_0$ denotes the $\Lambda$-rank $rk_\Lambda M$ of $M$. Since $rk_{\mathbb{Z}_p} T_\Lambda(M)_{\Gamma} = rk_{\mathbb{Z}_p} T_\Lambda(M)^\Gamma$ we obtain

$$rk_\Lambda M = rk_{\mathbb{Z}_p} (M_{\Gamma}) - rk_{\mathbb{Z}_p} (M^\Gamma) \quad (1.4)$$

and in particular, if $h\Lambda M \leq 1$ holds, then by (1.3) we have:

$$rk_\Lambda M = d(M) - r(M). \quad (1.5)$$

The polynomial

$$f_M(T) = \prod_{i=1}^{s} f_i(T)^{e_i}$$

of degree

$$\lambda(M) = \deg f_M(T)$$

is called the characteristic polynomial of $M$; we have the alternative description

$$f_M(T) = p^{\mu(M)} \cdot \det \left( T - (\gamma_0 - 1); M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \right),$$

where $\mu(M) = \sum_{\nu} e_{\nu}$ is given by the $\mathbb{Z}_p$-torsion module $T_{\mathbb{Z}_p}(M)$ of $M$:

$$T_{\mathbb{Z}_p}(M) \sim \bigoplus_{\nu} \Lambda/p^{e_{\nu}}. \quad (1.6)$$
We say a sequence

\[ M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \]

of noetherian \( \Lambda \)-modules is quasi-exact, if there is a quasi-isomorphism

\[ \text{Ker } \psi \sim \text{im } \varphi. \]

If \( M \) is a \( \Lambda \)-torsion module quasi-isomorph to the \( \Lambda \)-module \( N \), then it is easy to see, that there exist \( \Lambda \)-homomorphism \( \varphi' \) and \( \psi' \) such that the sequence

\[ M' \xrightarrow{\varphi'} N \xrightarrow{\psi'} M'' \]

is quasi-exact (see the proof of theorem 5. [1] VII §4.4).

For two \( \Gamma \)-modules \( M \) and \( N \) we give \( \text{Hom}_{\mathbb{Z}_p}(M, N) \) a \( \Lambda \)-module structure by defining

\[ (\gamma f)(m) = \gamma f(\gamma^{-1}m), \quad \gamma \in \Gamma, \quad f \in \text{Hom}_{\mathbb{Z}_p}(M, N), \quad m \in M. \]

Let \( \alpha(M) \) the adjoint of a \( \Lambda \)-torsion module \( M \), then we have [9] (1.3):

\[ \alpha(M) = \text{proj lim}_n \text{Hom}_{\mathbb{Z}_p}(M/\pi_n M, \mathbb{Q}_p/\mathbb{Z}_p). \]

Here the inverse limit is taken with respect to the homomorphism induced by

\[ M/\pi_n M \rightarrow M/\pi_m M \]

\[ \text{amod } \pi_n M \rightarrow (\pi_m/\pi_n) \text{amod } \pi_m M, \quad m \geq n > 0, \]

where \( \{ \pi_n \} \) is a sequence of non-zero elements in \( \Lambda \) such that \( \pi_0 \in m \), \( \pi_{n+1} \in \pi_n m \), and such that the principal divisors \( \pi_n \Lambda \), \( n > 0 \), are disjoint from the annihilator of \( M \). We get a contravariant, quasi-exact functor \( M \rightarrow \alpha(M) \) on the category of noetherian \( \Lambda \)-torsion modules. If \( M \sim M' \), then \( \alpha(M) \sim \alpha(M') \).

For a compact \( \Gamma \)-module \( M \) we define a new \( \Gamma \)-structure by \( \gamma \circ m = \gamma^{-1}m \), \( \gamma \in \Gamma \), \( m \in M \), and denote this new module by \( \hat{M} \). Obviously

\[ \hat{M} = M \]

and if we consider \( M \) as \( \Lambda \)-module we see that for \( \xi(T) \in \Lambda \) and \( m \in \hat{M} \)

\[ \xi(T) \circ m = \hat{\xi}(T) m := \hat{\xi}(T^\lambda) m, \]

1 The definition of \( \alpha(M) \) in [9] is slightly different because another \( \Gamma \)-structure for \( \alpha(M) \) is considered.
where $\hat{T} \in m$ is defined by

$$(1 + T)(1 + \hat{T}) = 1.$$ 

In particular we obtain for a $\Lambda$-torsion module $M$ a quasi-isomorphism

$$\alpha(M) \sim \hat{M},$$

and an isomorphism for an elementary $\Lambda$-torsion module $E$

$$\alpha(E) \equiv \hat{E}.$$  

We shall frequently consider the following elements of $\Lambda$

$$\omega_n = (1 + T)^{p^n} - 1, \quad n \geq 0$$

$$\xi_n = \omega_n / \omega_{n-1} = \sum_{i=0}^{p-1} (1 + T)^{i p^{n-1}}, \quad n \geq 1,$$

$$\xi_0 = \omega_0 = T.$$

For each $n \geq 0$, let $\Gamma^n$ the unique closed subgroup of $\Gamma$ with index $p^n$.

Finally we prove the following

**LEMMA 1.9:** Let $M$ be a noetherian $\Lambda$-module, then:

i) \[ \text{proj lim}_{n} \text{proj lim}_{m} (\rho^m M)_{\Gamma^n} \cong \alpha \left( \rho^m T_{\Lambda}(M) / T_{\Lambda}(M) \right)_{\Gamma^n} \]

ii) \[ \text{proj lim}_{n} \text{proj lim}_{m} (M_{\rho^m})_{\Gamma^n} \cong \alpha \left( T_{\Lambda}(M) / T_{\Lambda}(M) \right). \]

**PROOF:** First let $M \cong \Lambda_{\mathbb{Z}} \oplus T_{\Lambda}(M) / T_{\Lambda}(M) \oplus T_{\Lambda}(M)$ be an elementary $\Lambda$-module. We then have

$$\left( \rho^m M \right)_{\Gamma^n} \cong \left( \rho^m T_{\Lambda}(M) \right)_{\Gamma^n}$$

and

$$\left( M_{\rho^m} \right)_{\Gamma^n} \cong \left( \left( T_{\Lambda}(M) / T_{\Lambda}(M) \right)_{\rho^m} \right)_{\Gamma^n};$$

Since the prime divisors $\xi_n$, $n \geq 0$, (resp. $\rho\Lambda$) are disjoint from the annihilator of $T_{\Lambda}(M)$ (resp. $T_{\Lambda}(M) / T_{\Lambda}(M)$), we take in (1.7) for $\pi_n$ the element $\omega_n$ resp. $p^n$ and obtain the assertions of the lemma in this case.

To conclude in the general case we observe that

$$\alpha(\mathcal{E}) = \text{proj lim}_{n, m} (\rho^m \mathcal{E})_{\Gamma^n} \cong \text{proj lim}_{n, m} (\mathcal{E}_{\rho^m})_{\Gamma^n},$$

is zero, if $\mathcal{E}$ is a finite $\Lambda$-module.
§2 $\Lambda$-modules of homological dimension $\leq 1$

The following proposition characterizes noetherian $\Lambda$-modules of homological dimension equal or less than one.

**PROPOSITION 2.1:** Let $M$ be a noetherian $\Lambda$-module, then the following five assertions are equivalent:

i) $hd_\Lambda M \leq 1$.

ii) There exists an exact sequence $0 \to M \to E \to \mathcal{E} \to 0$, where $E$ (resp. $\mathcal{E}$) is an elementary (resp. finite) $\Lambda$-module.

iii) $M^{\Gamma n}$ is $\mathbb{Z}_p$-free for all $n \geq 0$.

iv) $M^{\Gamma n}$ is $\mathbb{Z}_p$-free for an integer $n \geq 0$.

v) $M$ does not contain any nontrivial finite $\Lambda$-submodule.

**PROOF:** If $hd_\Lambda M \leq 1$ we obtain the following commutative exact diagram

\[
\begin{array}{c}
0 \to \Lambda^{r(M)} \to \Lambda^{d(M)} \to M \to 0 \\
\downarrow \varphi \downarrow \downarrow \\
E \to E \to \mathcal{E}
\end{array}
\]

with finite $\Lambda$-modules $\mathcal{E}$ and $\mathcal{E}'$ and an elementary $\Lambda$-module $E$. By the snake lemma we get an exact sequence

\[
0 \to \Lambda^{r(M)} \to \text{Ker} \varphi \to \mathcal{E}' \to 0
\]

and therefore, since $(\text{Ker} \varphi)^{\Gamma} \leq (\Lambda^{d(M)})^{\Gamma} = 0$, the inclusion

\[
(\mathcal{E}')^{\Gamma} \hookrightarrow (\Lambda^{r(M)})^{\Gamma} \cong \mathbb{Z}_p^{r(M)}.
\]

This shows that $(\mathcal{E}')^{\Gamma} = 0$ and consequently that $\mathcal{E}' = 0$. Since $E^{\Gamma n}$ is $\mathbb{Z}_p$-free for all $n \geq 0$, ii. implies iii. The assertion iv. is obviously equivalent to v., because a finite $\Lambda$-module $\mathcal{E}$ is zero if and only if $\mathcal{E}^{\Gamma n} = 0$ for a number $n \geq 0$. Now assume v. There exists an exact sequence

\[
0 \to N \to \Lambda^{d(M)} \to M \to 0,
\]

where $N$ is a noetherian $\Lambda$-module with $N^{\Gamma} = 0$; the exact sequence

\[
0 \to M^{\Gamma} \to N^{\Gamma} \to \mathbb{Z}_p^{d(M)}
\]

shows that $N^{\Gamma}$ is $\mathbb{Z}_p$-free, since $M^{\Gamma}$ is $\mathbb{Z}_p$-free, and therefore by (1.2.) the $\Lambda$-freeness of $N$. 

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COROLLARY 2.2: Let $M$ be a noetherian $\Lambda$-module with $\text{hd}_\Lambda M \leq 1$. Then all the $\Lambda$-submodules of $M$ have homological dimension smaller or equal to 1.

Following Iwasawa, we shall call compact $\Gamma$-modules $M$ $n$-regular, if $M^{T^n} = 0$, and regular, if $M$ is $n$-regular for all $n \geq 0$.

PROPOSITION 2.3: Let $M$ be a noetherian $\Lambda$-module. Then the following assertions are equivalent

i) $M$ is regular.

ii) $M$ is $n_0$-regular for an integer $n_0 \geq \lambda(M)$.

PROOF: By 2.1 the assertion ii. follows immediately from i. Now let

$$0 \to M \to E(e_0; p_1^{e_1}, \ldots, p_s^{e_s}) \to \mathcal{E} \to 0$$

an exact sequence of $\Lambda$-modules, where $\mathcal{E}$ is finite, we see that the $n$-regularity of $M$ is equivalent to

$$(\Lambda/p_i^{e_i})^{T^n} = 0 \quad \text{for} \quad i = 1, \ldots, s.$$ 

respectively to:

$p_i \neq \xi_\nu \Lambda \quad \text{for} \quad i = 1, \ldots, s \text{ and all } \nu = n.$

Consequently $M$ is regular if and only if none of the prime divisors $p_i$, $i = 1, \ldots, s$, are equal to $\xi_\nu \Lambda$, $\nu \geq 0$. But for $\nu = n_0$ this is true by ii) and for all $\nu > n_0 \geq \lambda(M) = \deg f_M(T)$ this follows from the fact that $\deg \xi_\nu = (p - 1)p^{\nu - 1} \geq \nu$.

§3 Galois theoretical $\Lambda$-modules

We shall consider in this section $\Lambda$-modules of the following type: Let $G$ be a finitely generated pro-$p$-group with an exact sequence

$$1 \to H \to G \to \Gamma \to 1,$$

then the conjugation of $G$ on $H$ defines on $H^{ab} = H/[H, H]$ a $\Gamma$-structure. First we prove a few simple lemma's.

LEMMA 3.1: Let $A$ be a discrete $p$-primary $G$-torsion module. Then the following sequence is exact

$$0 \to H^1(\Gamma, H^1(H, A)) \to H^2(G, A) \to H^2(H, A) \to 0.$$
This is a part of the Hochschild-Serre-spectralsequence, since we have $c_{d_\Gamma}(\Gamma) = 1$. Since $(H^{ab})^1 = H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p))^*$ we get by 2.1

**Lemma 3.2:** The following assertions are equivalent:

i) $h d_\Gamma H^{ab} \leq 1$.

ii) $H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p))$ is a divisible group.

**Lemma 3.3:** Let $U$ be a pro-$p$-group. Then we have the equivalent assertions

i) $U$ is a free pro-$p$-group.

ii) $H^1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible (this means: $U^{ab}$ is $\mathbb{Z}_p$-torsionfree) and $H^2(U, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

For the proof consider the following exact cohomology sequence

$$0 \to \left( \frac{U}{U^c} \right)^* \to H^2(U) \to p^* H^2(U, \mathbb{Q}_p/\mathbb{Z}_p) \to 0 \quad (3.4)$$

induced by

$$0 \to \mathbb{Z}/p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

We want now to determine the generator- and relation-rank of $H^{ab}$. For this we set

$$\delta_i = \dim_{\mathbb{F}_p} H^i(G), \ i \geq 0,$$

$$\chi_2(G) = \delta_0 - \delta_1 + \delta_2,$$

$$t = \dim_{\mathbb{F}_p} H^2(H, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma.$$

**Proposition 3.5:** Let $h d_\Lambda H^{ab} \leq 1$; then there exists an exact sequence of $\Lambda$-modules

$$0 \to \Lambda^{\delta_2 - t} \to \Lambda^{\delta_1 - 1} \to H^{ab} \to 0,$$

and therefore

$$r g_\Lambda H^{ab} = -\chi_2(G) + t.$$

**Proof:** The generator-rank is calculated in the following way:

$$d(H^{ab}) = \dim H_0(\Gamma, H^{ab}) = \dim H_0(\Gamma, H^{ab}_p)$$

$$= \dim H^0(\Gamma, H^1(H)) = \dim H^1(G) - 1;$$
since \( H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p)) \) is divisible, we get because of 3.1

\[
r(H^{ab}) = \dim H_1(\Gamma, H^{ab})_p + \dim_p H_0(\Gamma, H^{ab})
\]

\[
= \dim_p H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p)) + \dim_p (H^{ab}_\Gamma)
\]

\[
= \dim_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) - t + \dim_p (H^{ab}_\Gamma).
\]

Finally the exact sequence

\[
0 \rightarrow H^{ab}_\Gamma \rightarrow G^{ab} \rightarrow \Gamma \rightarrow 0
\]

combined with 3.4 shows that:

\[
r(H^{ab}) = \dim_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) - t + \dim_p G^{ab} = \beta_2 - t.
\]

The assertions 1.3 and 1.5 complete the proof.

**Theorem 3.6:** The following properties are equivalent:

i) \( H^{ab} \) does not contain any nontrivial finite \( \Lambda \)-submodule and \( \text{rk}_\Lambda H^{ab} = -\chi_2(G) \).

ii) \( H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \) is divisible and \( H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \).

**Proof.** Because of 2.1 and 3.5 the assertion i. is equivalent to \( h\text{d}_\Lambda H^{ab} \leq 1 \)
and \( t = 0 \). Now, \( t = 0 \) is valid if and only if \( H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) \) is trivial and
we obtain the theorem using 3.1 and 3.2.

For the pro-\( p \)-group \( H \) we get the following

**Theorem 3.7:** The following assertions are equivalent:

i) \( H^{ab} \) is a free pro-\( p \)-group.

ii) \( H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \) is divisible, \( H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \) and \( \mu(H^{ab}) = 0 \).

**Proof:** Using 3.3 and 3.1 we must show:
\( H^{ab} \) is \( \mathbb{Z}_p \)-torsionfree if and only if \( (H^{ab})^\Gamma \) is \( \mathbb{Z}_p \)-free and \( \mu(H^{ab}) = 0 \).
But, if \( (H^{ab})^\Gamma \) is \( \mathbb{Z}_p \)-free, it follows from 2.1 ii. that the group \( pH^{ab} \) is zero if and only if \( \mu(H^{ab}) = 0 \).

**Corollary 3.8:** Suppose \( cd_p(G) \leq 2 \) and let \( U \) be an open subgroup of \( G \), \( V = H \cap U \) and \( \Gamma' = U/V \leq \Gamma \). Then \( V^{ab} \) is a noetherian \( \Gamma' \)-module and
the following assertions are equivalent:

i) \( \mu(H^{ab}) = 0 \) and \( H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \).

ii) \( \mu(V^{ab}) = 0 \) and \( H^2(V, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \).
PROOF: Since $\text{cd}_p(U) = \text{cd}_p(G) \leq 2$ the cohomology groups $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^2(U, \mathbb{Q}_p/\mathbb{Z}_p)$ are divisible. Since $\text{cd}_p(H) = \text{cd}_p(V)$ ([23] I.33 prop. 14) the corollary follows from theorem 3.7.

Next we want to study the connection between the characteristic polynomial $f_{H^{ab}}(T)$ of $H^{ab}$ and the defining relations of the group $G$. We assume that

$$H^2(G, \mathbb{Q}_p/\mathbb{Z}_0) \text{ is divisible and } H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$$

(and therefore: $hd_{\Lambda} H^{ab} \leq 1$).

Let $F$ be a free pro-$p$-group with free generators $x_0, x_1, \ldots, x_n$, $n + 1 = \ell_1$ and

$$1 \to R \to F \to G \to 1$$

a free (minimal) presentation of $G$ such that $x_0$ is mapped on a lifting of $\gamma_0 \in \Gamma$ and the images of $x_1, \ldots, x_n$ are elements of $H$. We get the commutative exact diagram

$$\begin{array}{c}
\langle x_0 \rangle \to \Gamma \\
\uparrow \hspace{1cm} \uparrow \\
1 \to R \to F \to G \to 1 \\
\| \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
1 \to R \to E \to H \to 1,
\end{array}$$

where $E$ is a free operator-pro-$p$-group over $\Gamma$ and

$$E^{ab} = \bigoplus_{i=1}^{n} \Lambda \bar{x}_i, \quad \bar{x}_i = x_i \mod [E, E];$$

(see Jannsen [14], Satz 3.4). We obtain the following exact sequence of $\Lambda$-modules

$$0 \to R[E, E] / [E, E] \to E^{ab} \to H^{ab} \to 0;$$

because $hd_{\Lambda} H^{ab} \leq 1$ we get that

$$R[E, E] / [E, E] \cong \bigoplus_{i=0}^{m} \Lambda r_i$$

is a free $\Lambda$-module of rank $m = \ell_2$ (3.5) with generators

$$r_i \equiv w_i \mod [E, E], \quad i = 1, \ldots, m,$$

where the elements $w_i = w_i(x_0, \ldots, x_n) \in R$ are defining relations for $G$. 

Since $w_i \in E$, $i = 1, \ldots, m$, we have
\[ w_i \equiv x_1^{a_{i1}} \cdots x_n^{a_{in}} \mod [E, E], \quad \alpha_{ij} \in \Lambda, \quad i = 1 \ldots m. \]

We call the matrix
\[ \mathcal{R}_G = (\alpha_{ij}) \in M_{nm}(\Lambda) \]
the relation-matrix for $G$ (relative to $(x_i)$ and $(r_i)$). By [1] VII §4.6 Cor we obtain the following

**Theorem 3.9:** If $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible, $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and $\chi_2(G) = 0$, then we have for the characteristic polynomial of the $\Lambda$-torsion module $H^{ab}$
\[ f_{H^{ab}} = \det(\mathcal{R}_G) \cdot \epsilon, \quad \epsilon \in \Lambda^\times. \]

If $H^{ab}$ is not a $\Lambda$-torsion module, it is easy to prove the following more general

**Theorem 3.10:** Suppose that $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible and $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and let $S = \bigcap (\Lambda/\mathfrak{p}_i)$, where $\{\mathfrak{p}_i\}$ is the finiteset of prime ideals of height 1 containing the annihilator of $H^{ab}$, and $\Lambda_S$ the (semi-local, principal) ring of fractions. Then there exists matrices $P \in GL_n(\Lambda_S)$ and $Q \in GL_n(\Lambda_S)$ such that
\[ P \mathcal{R}_G Q = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_m & 0 \end{pmatrix} \in M_{nm}(\Lambda), \]
and
\[ H^{ab} \sim \Lambda^{n-m} \bigoplus_{i=1}^m \Lambda/(\lambda_i); \]
consequently we have
\[ f_{H^{ab}} = \prod_{i=1}^m \lambda_i \cdot \epsilon, \quad \epsilon \in \Lambda^\times. \]

## II. $\Lambda$-modules in algebraic number theory

### §4 Class field theory and the global duality theorem

In this section we summarize the topics from class field theory that we will use in this paper and introduce notations that will be needed in chapter II.
Let \( p \geq 2 \) be a prime number,  
\( \mu_{p^n} \) the group of the \((p^n)\)th roots of unity,  
\( \mu_{p^n} = \text{inj lim} \mu_{p^n} = \mathbb{Q}_p/\mathbb{Z}_p(1) \), proj \( \mu_{p^n} = \mathbb{Z}_p(1) \),  
\( \mu_F = \mu_{p^n} \cap F \), \( F \) a field,  
\( F(p) \) the maximal \( p \)-extension of \( F \).  
\( \mathbb{Q}_\infty \) will be the unique subfield of \( \mathbb{Q}(\mu_{p^n}) \) with \( G(\mathbb{Q}_\infty | \mathbb{Q}) \equiv \mathbb{Z}_p \). Let \( K \) be a finite algebraic number field,  
\( K_\infty = \bigcup_{n=0}^\infty K_n \) a \( \Gamma \)-extension of \( K - K_\infty \) is called cyclotomic, if \( K_\infty = K \cdot \mathbb{Q}_\infty \).  
\( S_p = S_p(K) \) (resp. \( \Sigma = \Sigma(K) \)) the set of primes of \( K \) above \( p \) (resp. \( p \) or \( \infty \)),  
\( S = S(K) \) a finite set of primes of \( K \) containing \( \Sigma \),  
\( G_S = G_S(K) \) (resp. \( G_S(p) \)) the maximal \( (p) \)-extension unramified outside \( S \),  
\( H_S = H_S(K) \) (resp. \( H_S(p) \)). The abelian groups \( H^1 \) are noetherian \( \Lambda \)-modules (\( X_2, X_3 \) are \( \Lambda \)-torsion modules), [9] th. 4, 5, 8, with invariants

\[
\begin{align*}
\mu_S &= \mu_S(K) = \mu(X_S(K)), \\
\mu_r &= \mu_r(K) = \mu(X_r(K)), \\
\lambda_S &= \lambda_S(K) = \lambda(X_S(K)), \\
\lambda_r &= \lambda_r(K) = \lambda(X_r(K)), \quad r = 1, 2, 3.
\end{align*}
\]

The global duality theorem of Tate/Poitou is essential in the sequel (see [8] theorem 1): let \( A \) be a finite \( p \)-primary \( G_S \)-module, then the sequences

\[
0 \to \ker^1(G_S, A) \to H^1(G_S, A) \to \prod_{\nu \in S} H^1(G_\nu, A) \tag{4.1}
\]

\[
0 \to \ker^2(G_S, A) \to H^2(G_S, A) \to \prod_{\nu \in S} H^2(G_\nu, A) \tag{4.2}
\]

are exact, where \( A' = \text{Hom}(A, \mu_{p^n}) \), and there is a canonical non-degen-
erating pairing

\[ \ker^1(G_S, A) \times \ker^2(G_S, A') \to \mathbb{Q}_p / \mathbb{Z}_p \]  

(4.3)

which has the following functorial behaviour, [8] 5.6, 5.3: let \( K' \mid K \) be a finite extension in \( K_S \) and \( m, m' \in \mathbb{N}, m' \geq m \), then the following diagrams are commutative

\[
\begin{array}{ccc}
\ker^1(G_S(K), A) \times \ker^2(G_S(K), A') & \to & \mathbb{Q}_p / \mathbb{Z}_p \\
\text{cor} \uparrow & & \uparrow \text{res} \\
\ker^1(G_S(K'), A) \times \ker^2(G_S(K'), A') & \to & \mathbb{Q}_p / \mathbb{Z}_p \\
\end{array}
\]

(4.4)

\[
\begin{array}{ccc}
\ker^1(G_S(K), \mu_{p^m}) \times \ker^2(G_S(K), \mathbb{Z}/p^m) & \to & \mathbb{Q}_p / \mathbb{Z}_p \\
\theta_1 \downarrow & & \downarrow \theta_2 \\
\ker^1(G_S(K'), \mu_{p^{m'}}) \times \ker^2(G_S(K'), \mathbb{Z}/p^{m'}) & \to & \mathbb{Q}_p / \mathbb{Z}_p \\
\end{array}
\]

(4.5)

where \( \theta_1 \) (resp. \( \theta_2 \)) denotes the morphism induced by the \( p \)-power map (resp. the canonical projection).

Finally let

\[ \mathcal{A}(K_{n^p}) = \text{proj lim}_{m} K_{n^p}^\times / K_{n^p}^\times p^m \]

be the \( p \)-completion of the multiplicative group \( K_{n^p}^\times \) of \( K_{n^p} \), \( p \in S_p(K_n) \), and \( U^1(K_{n^p}) \) the principal units of \( K_{n^p}^\times \); for a prime \( \mathfrak{p} \mid \mathfrak{p}_1 \) we put

\[ \mathcal{A}_{\mathfrak{p}} = \text{proj lim}_{n} \mathcal{A}(K_{n^\mathfrak{p}}) \equiv \text{proj lim}_{n} G_{\mathfrak{p}}(K_n)^{ab} = G_{\mathfrak{p}}(K_\infty)^{ab}, \]

\[ U_{\mathfrak{p}}^1 = \text{proj lim}_{n} U^1(K_{n^\mathfrak{p}}), \]

and

\[ \mathcal{A} = \bigoplus_{p \in S_p(K)} \mathcal{A}_{\mathfrak{p}}, \quad U^1 = \bigoplus_{p \in S_p(K)} U_{\mathfrak{p}}^1. \]

\( E(K_n) \) (resp. \( E'(K_n) \)) denotes the units (resp. the \( S_p(K_n) \)-units) of \( K_n \),

\[ E_\infty = E(K_\infty) = \text{inj lim}_{n} E(K_n), \quad E'_\infty = E'(K_\infty) = \text{inj lim}_{n} E'(K_n) \]

and \( \overline{E(K_n)} \) (resp. \( \overline{E'(K_n)} \)) the topological closure of the images of
By the global class field theory we have the following commutative exact diagram:

\[
\begin{array}{ccc}
E(K_n) \rightarrow \prod_{\nu \in S_p(K_n)} U^0(K_{n\nu}) & \rightarrow & \prod_{\nu \in S_p(K_n)} U^1(K_{n\nu}) \\
\downarrow & & \downarrow \\
E'(K_n) \rightarrow \prod_{\nu \in S_p(K_n)} K_{n\nu} & \rightarrow & \prod_{\nu \in S_p(K_n)} A(K_{n\nu})
\end{array}
\]

and

\[
\bar{E} = \bar{E}(K_\infty) = \text{proj lim}_{n} \bar{E}(K_n), \quad \bar{E}' = \bar{E}'(K_\infty) = \text{proj lim}_{n} \bar{E}'(K_n).
\]

By the global class field theory we have the following commutative exact diagram:

\[
0 \rightarrow \bar{E} \rightarrow U^1 \rightarrow X_1 \rightarrow X_2 \rightarrow 0 \\
\downarrow \quad \downarrow \quad \| \quad \downarrow \\
0 \rightarrow \bar{E}' \rightarrow A \rightarrow X_1 \rightarrow X_3 \rightarrow 0
\]

(see for instance [18, Theorem 5.1], where the exactness of the upper row is proved on a finite level; for the other row the exactness can be proved in a similar way).

5. Application of the results of Sections 2 and 3

In the sequel the “weak Leopoldt-conjecture” will play an essential role. This conjecture states that for a $\Gamma$-extension $K_\infty | K$ the cohomology group $H^2(H_S(K), \mathbb{Q}_p/\mathbb{Z}_p)$ is zero. It is well known that this is true for the cyclotomic $\Gamma$-extension if $\mu_p \subset K$, $p > 2$ [22]. Here we will give a short proof for $p \geq 2$ and an arbitrary number field $K$.

**Proposition 5.1**: Let $K_\infty | K$ be the cyclotomic $\Gamma$-extension. Then $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

**Proof**: Let $\mathcal{A} = K(\mu_p)$; passing to the direct limit we get from (4.2) the exact sequence

\[
0 \rightarrow \text{inj lim}_{n, m} \ker^2(G_S(\mathcal{A}_n), \mu_p^m) \rightarrow H^2(H_S(\mathcal{A}_\infty), \mu_p^\infty) \\
\rightarrow \bigoplus_{\nu \in S_p(\mathcal{A}_\infty)} H^2(G_{\nu}(\mathcal{A}_\infty), \mu_p^\infty).
\]
On the one hand, we have by (4.3)

\[ \text{inj lim} \, \ker^2(G_\mathcal{S}(\mathcal{X}_n), \mu_{p^m}) \cong \left( \text{proj lim} \, \ker^1(G_\mathcal{S}(\mathcal{X}_n), \mathbb{Z}/p^m) \right)^* \]

\[ = \text{inj lim} \, \text{Cl}_S(\mathcal{X}_n) \, p^m = 0, \]

because of the finiteness of the S-ideal class group Cl_S(\mathcal{X}_n). Since the strict cohomological p-dimension of G_v(K_\infty) is 2 for a finite prime \( \nu \), on the other hand we have \( H^2(G_v(\mathcal{X}_\infty), \mu_{p^\infty}) = H^2(G_v(\mathcal{X}_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \). This is also true for \( \nu \mid \infty \): Let \( \nu \) be real and \( p = 2 \), then the group \( H^2(G_v(\mathcal{X}_\infty), \mu_{2^\infty}) = H^2(\mathbb{Z}/2, \mathbb{Q}_2/\mathbb{Z}_2) \) is zero. Consequently, we obtain

\[ H^2(\mathbb{H}_S(\mathcal{X}), \mu_{p^\infty}) = H^2(\mathbb{H}_S(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p) = 0. \]

Let \( \Delta \) be the Galois group of the extension \( \mathcal{X} \mid K \), then the order of \( \Delta \) is 1 or prime to \( p \). Hence, by the Hochschild–Serre spectral sequence,

\[ 0 = H^1(G(K_S/K_s(p)), \mathbb{Q}_p/\mathbb{Z}_p) \]

\[ \rightarrow H^2(H_S(K), \mathbb{Q}_p/\mathbb{Z}_p) \]

\[ \rightarrow H^2(\mathbb{H}_S(K), \mathbb{Q}_p/\mathbb{Z}_p) \]

and

\[ H^2(\mathbb{H}_S(K), \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\text{res}} H^2(\mathbb{H}_S(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p) \Delta = 0 \]

the proposition is proved. \( \square \)

**Lemma 5.2:** The cohomology group \( H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p) \) is divisible.

**Proof:** Since the group \( G_S \) is of cohomological dimension 2 for \( p \neq 2 \) the assertion is clear in this case. Now let \( p = 2 \); by the global duality theorem [8, Theorem 1c.] and [8, Cor. 1 of Prop. 22], there are isomorphisms

\[ H^3(G_S, \mathbb{Q}/\mathbb{Z}_2) \cong \bigoplus_{\nu \text{ real}} H^3(G_v, \mathbb{Q}/\mathbb{Z}_2) \]

\[ = \bigoplus_{\nu \text{ real}} H^3(G_v, \mathbb{Z}/2) \cong H^3(G_S, \mathbb{Z}/2). \]

Considering the exact cohomology sequence

\[ H^2(G_S, \mathbb{Q}_2/\mathbb{Z}_2) \rightarrow H^2(G_S, \mathbb{Q}_2/\mathbb{Z}_2) \rightarrow H^3(G_S) \]

\[ \rightarrow H^3(G_S, \mathbb{Q}_2/\mathbb{Z}_2) \rightarrow 0, \]
Using Theorem 3.7 we now obtain the following result (see also [11, theorem 2]):

**THEOREM 5.3:** Suppose $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ (for instance, let $K_\infty | K$ be cyclotomic). Then the following are equivalent:

(i) $\mu_S = 0$.

(ii) $H_S = G(K_S(p)/K_\infty)$ is a free pro-$p$-group.

By [8, Prop. 22, Cor. 5], and Theorem 3.6 and Lemma 5.2 we get the next theorem:

**THEOREM 5.4:** Suppose $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$; then

(i) $\text{rg}_\Lambda X_S = r_2$, $r_2$ the number of complex primes of $K$

(ii) $X_S$ does not contain any finite non-trivial $\Lambda$-submodule.

**REMARK:** By Theorem 3.6 we see that the condition $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ is necessary for the validity of (i) and (ii). The above result is due to Iwasawa [9], if $K_\infty | K$ is the cyclotomic $\Gamma$-extension, and to Greenberg [6], if the Leopoldt-conjecture holds for $K$, that is if $H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ (which is, by Lemma 3.1, a stronger condition than ours).

Applying proposition 2.3 we obtain:

**PROPOSITION 5.5:** Let $K_\infty = \cup K_n$ be a $\Gamma$-extension of $K$; the following assertions are equivalent:

(i) The Leopoldt-conjecture is true for all intermediate fields $K_n$.

(ii) There exists a number $n_0 \geq \lambda_1$ such that the Leopoldt-conjecture holds for $K_{n_0}$ and the weak Leopoldt-conjecture $H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ is valid.

**PROOF:** The non-trivial implication is proved as follows: By Lemma 3.1 we have that $H^2_\Sigma$ is $n_0$-regular,

\[
(H^2_\Sigma)^{\Gamma_{n_0}} = H^1(\Gamma_{n_0}, H^1(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p))^* = H^2(G_\Sigma(K_{n_0}), \mathbb{Q}_p/\mathbb{Z}_p)^* = 0,
\]

and by Proposition 2.1 that $\text{hd}_\Lambda H^2_\Sigma$ is equal or less than 1. Therefore $H^2_\Sigma$ is a regular $\Lambda$-module by Proposition 2.3 and consequently $H^2(G_\Sigma(K_n), \mathbb{Q}_p/\mathbb{Z}_p)$ is zero for all $n \geq 0$.

In the following sections we will only consider the prime set $S = \Sigma$,
because we have:

**Proposition 5.6:** Let \( S \supseteq \Sigma \) be a finite set of primes, then
\[
\mu_S = \mu_1
\]

and, if the weak Leopoldt-conjecture is verified for \( K_\infty \mid K \),
\[
\lambda_S = \lambda_1 + \# \{ \wp \in S(K_\infty) \setminus \Sigma(K_\infty) : \mu_p \subset (K_\infty)_\wp \}
\]

**Proof:** The analogue of Riemann’s existence theorem [20, Theorem 2] tells us that the Galois group \( G(K_S(p)/K_\Sigma(p)) \) is a free pro-\( p \)-product
\[
G_{S \setminus \Sigma} = G(K_S(p)/K_\Sigma(p)) \cong \prod_{\wp \in S(K_\infty) \setminus \Sigma(K_\infty)} T_\wp,
\]
where \( T_\wp \) is the subgroup of \( G_{S \setminus \Sigma} \) generated by all inertia groups \( T_\wp' \), \( \wp' \) a fixed prime lying over \( \wp \) and \( \wp' \) varies over all primes of \( K_\Sigma(p) \) dividing \( \wp \). The inertia group \( T_\wp' \) is isomorphic to \( \mathbb{Z}_p \), if \( (K_\Sigma)_\wp \) contains the group \( \mu_p \), and otherwise equal to \( \{1\} \). The exact sequence
\[
0 \to G_{S \setminus \Sigma}[H_S, H_S]/[H_S, H_S] \to H^1_{\Sigma}(H_S) \to H^1_{\Sigma}(H_S) \to 0
\]
shows the equality \( \mu_S = \mu_\Sigma \), since the kernel is finitely generated. Now, if \( H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) \) is zero, we get
\[
G_{S \setminus \Sigma}[H_S, H_S]/[H_S, H_S] = G_{S \setminus \Sigma}/[G_{S \setminus \Sigma}, H_S].
\]
If \( \mu_p \) is contained in \( (K_\infty)_\wp \) we have \( \dim H^1(T_\wp)^{H_S} = 1 \) and, since \( H^1_{\Sigma}(H_S, \mathbb{Q}_p/\mathbb{Z}_p) \) contains no finite nontrivial \( \Delta \)-submodule, \( H^1(T_\wp, \mathbb{Q}_p/\mathbb{Z}_p)^{H_S} \cong \mathbb{Z}_p \). This proves the second part of the proposition. \( \square \)

**Corollary 5.7:** Let \( S \supseteq \Sigma \) be a finite set and \( H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \). Then \( H_S \) is a free pro-\( p \)-group if and only if \( H_\Sigma \) is free.

6. The \( \Delta[\Delta] \)-module structure of the local groups \( \mathcal{A} \) and \( U^1 \)

In this section \( p \) will be an odd prime, \( K_\infty/K \) the cyclotomic \( \Gamma \)-extension, \( \mathcal{A} = K(\mu_p) \), \( \mathcal{A}_\infty = K(\mu_{p^\infty}) \), \( d = [\mathcal{A} : K] \),
\[
\Delta = G(\mathcal{A}/K), \quad G_\infty = \Delta \times \Gamma = G(\mathcal{A}_\infty/K)
\]

and
\[
\psi : G_\infty \to \mathbb{Z}_p^\times
\]
the cyclotomic character given by the operation of \( G_\infty \) on \( \mu_{p^\infty} \).
restrictions of $\psi$ on $\Delta$ resp. $\Gamma$ are denoted by (resp.)

$$\theta: \Delta \to \mathbb{Z}/p-1, \quad \kappa: \Gamma \to (1+p\mathbb{Z}_p).$$

Furthermore $A(i)$, $i \in \mathbb{Z}$, is the usual $i$-fold Tate twist of a $p$-torsion $G_\infty$-module $A$: $A(i) = A \otimes \mathbb{Z}_p(i)$ with

$$
\begin{align*}
\mathbb{Z}_p(0) &= \mathbb{Z}_p, \\
\mathbb{Z}_p(i) &= \mathbb{Z}_p(i-1) \otimes \mathbb{Z}_p(1), \quad i \geq 1, \\
\mathbb{Z}_p(i) &= \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(-i), \mathbb{Z}_p), \quad i < 0.
\end{align*}
$$

There are canonical $\Gamma$-isomorphism

$$A(i)^\Delta \cong A(i)_\Delta \cong (e_{-i}A)(i), \quad i \in \mathbb{Z},$$

where $e_i A$ is the maximal submodule (or factor module) of $A$ on which $\sigma \in \Delta$ acts as multiplication by $\theta(\sigma)^i$. Finally

$$G_{\infty \mathfrak{B}} = \Delta_{\mathfrak{B}} \times \Gamma_{\mathfrak{B}} \leq G_{\infty}$$

denotes the local galois group of the extension $K_{\infty \mathfrak{B}} | K_{\mathfrak{B}}$, $\mathfrak{B} \in S_p(K)$ and $A | p$ an extension on $\mathcal{B}_{\infty \mathfrak{B}}$.

Let us analyse now the $G_{\infty \mathfrak{B}}$-module structure of

$$\mathcal{A}_{\mathfrak{B}} = \text{proj lim } \mathcal{X}_{n, m}^{\infty \mathfrak{B}} / \mathcal{X}_{n, m}^{\infty \mathfrak{B}, p^m}.$$ 

If the group $\Delta_{\mathfrak{B}}$ is trivial, this is completely done in [9]. More generally, in [26], $\mathcal{Z}_p^r \times \Delta_{\mathfrak{B}}$-extensions, $r \geq 1$, are considered but no explicit description of the module structure of $\mathcal{A}_{\mathfrak{B}}$ is given. The reason for the difficulties to do this for $\Delta_{\mathfrak{B}} \neq 1$ lies in the fact that certain results of the representation theory are only valid for $p$-groups. Using the cohomological methods established (for $p$-groups) in [15] and especially their generalisation to arbitrary finite groups by Jannsen in [13], we get the wanted description of $\mathcal{A}_{\mathfrak{B}}$. The result below is easily obtained by the methods developed in [13], but for the convenience of the reader we shall prove it here. We need two lemma’s:

**Lemma 6.1:** Let $G$ be a finite group and $A$ and $C$ finitely generated $\mathbb{Z}_p[G]$-modules. if $A$ is cohomological trivial and $C$ $\mathbb{Z}_p$-free, then $\text{Ext}(C, A) = 0$.

**Proof:** Let

$$0 \to P_1 \to P_2 \to A \to 0$$
be a presentation of \( A \) by projective finitely generated \( \mathbb{Z}_p[G] \)-modules \( P_2 \). This is possible by [7; 10.7, Theorem 3], since the kernel \( P_1 \) of a surjection of a projective module \( P_2 \) onto a cohomological trivial module \( A \) has to be cohomological trivial. Because of the \( \mathbb{Z}_p \)-freeness of \( C \) we obtain an exact sequence

\[
0 \to \text{Hom}_{\mathbb{Z}_p}(C, P_1) \to \text{Hom}_{\mathbb{Z}_p}(C, P_2) \to \text{Hom}_{\mathbb{Z}_p}(C, A) \to 0,
\]

in which the middle and left module are projective [7; 10.1, Ex. 2]; hence \( \text{Hom}_{\mathbb{Z}_p}(C, A) \) is cohomological trivial and therefore \( \text{Ext}(C, A) = H^1(G, \text{Hom}_{\mathbb{Z}_p}(C, A)) = 0 \) [7; 10.1, Prop. 2].

**Lemma 6.2:** Let \( G \) be a finite group with a minimal number of generators \( d = d(G) \) and \( \mathcal{M}_G \) be the class of all finitely generated \( \mathbb{Z}_p[G] \)-modules \( M \) with

(i) \( H^1(G_p, M) = 0 \),
(ii) \( H^2(G_p, M) \cong \mathbb{Z}/(G_p : 1) \),

for a \( p \)-Sylow group \( G_p \) of \( G \). Furthermore, let

\[
1 \to R_d \to F_d \to G \to 1
\]

be a minimal presentation of \( G \) by a free pro-finite group \( F_d \) of rank \( d \) and a closed normal subgroup \( R_d \) of \( F_d \). Then the following assertions are valid for \( M \in \mathcal{M}_G \) with a cohomological trivial \( \mathbb{Z}_p \)-torsion module \( T_{\mathbb{Z}_p}(M) \):

(a) There exists an \( m \geq d \) and a \( \mathbb{Z}_p[G] \)-projective module \( N \) such that

\[
M \otimes N \cong \left( R_d^{ab} \otimes \mathbb{Z}_p \right) \oplus \mathbb{Z}_p[G]^{m-d} \oplus T_{\mathbb{Z}_p}(M).
\]

(b) If there is an integer \( n \geq 0 \) and a \( \mathbb{Q}_p[G] \)-isomorphism

\[
M \otimes \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p[G]^n,
\]

then

\[
M \cong \left( R_d^{ab} \otimes \mathbb{Z}_p \right) \oplus \mathbb{Z}_p[G]^{n+1-d} \oplus T_{\mathbb{Z}_p}(M).
\]

**Proof:** Let \( \chi_E \) be a generator of the cyclic group \( H^2(G_p, M/T_{\mathbb{Z}_p}(M)) = H^2(G_p, M) \) corresponding to the group extension

\[
1 \to M/T_{\mathbb{Z}_p}(M) \to E \to G_p \to 1
\]

and let

\[
1 \to R_m \to F_m \to G_p \to 1
\]

be a presentation by a free pro-finite group \( F_m \) of rank \( m \geq d \), for which
there exists a surjection \( \varphi : F_m \to E \), and a closed normal subgroup \( R_m \) of \( F_m \). We get a commutative exact diagram:

\[
\begin{array}{cccccc}
1 & \to & R_{ab}^m & \to & F_m/[R_m, R_m] & \to & G_p & \to 1 \\
\downarrow \varphi & & \downarrow & & \downarrow & & \|
\end{array}
\]

\[
1 \to M/T_{\mathbb{Z}_p}(M) \to E \to G_p \to 1
\]

Because \( \text{scdp}(F_m) = 2 \) there exists a \( \chi_F \in H^2(G_p, R_{ab}^m) \equiv \mathbb{Z}/(G_p : 1) \) with \( \overline{\varphi}^* \chi_F = \chi_E \).

Let \( N \) be defined by the exact sequence

\[
0 \to N \to R_{ab}^m \otimes \mathbb{Z}_p \to M/T_{\mathbb{Z}_p}(M) \to 0.
\]

Since \( H^1(G_p, M/T_{\mathbb{Z}_p}(M)) = 0 = H^3(G_p, R_{ab}^m \otimes \mathbb{Z}_p) \) the exact cohomology sequence

\[
0 \to H^2(G_p, N) \to H^2(G_p, R_{ab}^m \otimes \mathbb{Z}_p)
\]

\[
\xrightarrow{\overline{\varphi}^*} H^2(G_p, M/T_{\mathbb{Z}_p}(M)) \to H^3(G_p, N) \to 0
\]

shows that

\[
H^2(G_p, N) = H^3(G_p, N) = 0
\]

for all the \( \ell \)-Sylow groups \( G_\ell \) of \( G \). Now, a theorem of Nakayama [19] says that \( N \) is a cohomological trivial \( \mathbb{Z}_p[G] \)-module. Hence, by Lemma 6.1 the sequence

\[
0 \to N \to R_{ab}^m \otimes \mathbb{Z}_p \to M/T_{\mathbb{Z}_p}(M) \to 0
\]

splits. By the lemma of Shanuel [7;8.10, Lemma 11] and the semi-local cancellation theorem [7;10.6, Theorem 1], we then get the \( \mathbb{Z}_p[G] \)-isomorphism

\[
M/T_{\mathbb{Z}_p}(M) \oplus N \equiv R_{ab}^m \otimes \mathbb{Z}_p \oplus \mathbb{Z}_p[G]^{m-d}.
\]

Again by Lemma 6.1 there is a splitting

\[
M \equiv M/T_{\mathbb{Z}_p}(M) \oplus T_{\mathbb{Z}_p}(M)
\]

and therefore the assertion (a) is proved. Now, let \( M \otimes \mathbb{Q}_p \) be isomorphic to \( \mathbb{Q}_p \oplus \mathbb{Q}_p[G]^n \), then by the Krull–Schmidt theorem and since

\[
R_{ab}^m \otimes \mathbb{Q}_p \equiv \mathbb{Q}_p \oplus \mathbb{Q}_p[G]^{d-1}
\]
we get
\[ N \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[G]^{m-n-1}. \]

A theorem of Swan [24, Corr. 6.4] shows, that the projective module \( N \) is \( \mathbb{Z}_p[G] \)-free. Hence the cancellation theorem proves the assertion (b). \( \square \)

**Theorem 6.3:** Let \( K \mid k \) be a finite cyclotomic extension of \( p \)-adic number fields, \( G = G(K \mid k) \), \( n = [k : \mathbb{Q}_p] \), \( p \neq 2 \), and \( \mid \mid p \) the valuation of \( K \). Then there is a commutative exact diagram of \( \mathbb{Z}_p[G] \)-modules:

\[
\begin{array}{ccccccccc}
0 & \to & J \oplus \mathbb{Z}_p[G]^{n-1} & \oplus & \mu_K & \to & \mathbb{Z}_p[G]^n & \oplus & \mu_K & \oplus \Gamma_K & \to & \mathbb{Z}_p & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & U^1(K) & \longrightarrow & \mathcal{A}(K) & \longrightarrow & \mathbb{Z}_p & \rightarrow & 0
\end{array}
\]

where \( \Gamma_K \) is the galois group \( G(k(\mu_p^n) \mid K) \cong \mathbb{Z}_p \) and \( J \) is a \( \mathbb{Z}_p[G] \)-module with an exact sequence

\[
0 \to J \to \mathbb{Z}_p[G] \oplus \Gamma_K \to \mathbb{Z}_p \to 0.
\]

(One can show: If \( K \mid k \) is unramified (resp. totally ramified) then \( J \) is isomorphic to \( \mathbb{Z}_p[G] \) (resp. \( \mu_1 \oplus \Gamma_K \)), where \( \mu_1 \) is the augmentation ideal of \( \mathbb{Z}_p[G] \); furthermore, the above theorem is also valid for \( p = 2 \) and \( \mu_4 \subseteq k \).)

**Proof:** We have the following facts: by local class field theory \( \mathcal{A}(K) \) is an element of \( \mathcal{M}_G \); \( \mu_K \) is cohomological trivial for the Galois group of the cyclotomic extension \( K \mid k \); the group of principal units \( U^1(K) \) contains a free \( \mathbb{Z}_p[G] \)-module of rank \( n \) with finite index (by the well-known argument using the \( p \)-adic exponential map and the normal basis theorem). Consequently we have \( \mathbb{Q}_p[G] \)-isomorphisms

\[
\mathcal{A}(K) \otimes \mathbb{Q}_p \cong \left( U^1(K) \otimes \mathbb{Q}_p \right) \oplus \mathbb{Q}_p[G]^n \oplus \mathbb{Q}_p,
\]

and therefore by Lemma 6.2(b) the \( \mathbb{Z}_p[G] \)-isomorphism

\[
\mathcal{A}(K) \cong \left( R_1^{ob} \otimes \mathbb{Z}_p \right) \oplus \mathbb{Z}_p[G]^n \oplus \mu_K.
\]

The free presentation

\[
0 \to R_1 \to \hat{\mathbb{Z}} \to G(K \mid k) \to 0,
\]

\[
R_1 \cong \Gamma_K \times \left( G(k(\mu_p) \mid k) : 1 \right) \cdot \prod_{\ell \neq p} \mathbb{Z}_\ell \cong \prod_{\ell} \mathbb{Z}_\ell = \hat{\mathbb{Z}},
\]

shows that \( R_1 \times \mathbb{Z}_p \cong \Gamma_K \). Finally, let \( \varphi \) be the homomorphism \( \mathbb{Z}_p[G]^n \oplus \mathbb{Z}_p[G] \).
Corollary 6.4: There is a commutative exact diagram of $G\mathcal{O}_K$-modules:

$$
0 \rightarrow \mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q})^n \oplus \mathbb{Z}_p(1) \rightarrow \mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q})^n \oplus \mathbb{Z}_p(1) \rightarrow \mathbb{Z}_p \rightarrow 0
$$

where $n_\nu$ is the degree $[K_\mathcal{Q}:\mathbb{Q}_p]$.

Proof: This follows by Theorem 6.3 passing to the inverse limit; indeed, the exact sequence

$$
0 \rightarrow J_\infty = \text{proj \ lim}_n J(\mathcal{A}_{nQ}) \rightarrow \mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q}) \rightarrow \mathbb{Z}_p \rightarrow 0
$$

shows, on the one hand, that $J_\infty$ is a free $\mathbb{Z}_p[[\Gamma_\mathcal{Q}]]$-module of rank $\#\Delta_\mathcal{Q}$ (see (1.2)) and on the other hand, because $(p, \#\Delta_\mathcal{Q}) = 1$, that $J_\infty$ is a cohomological trivial $\Delta_\mathcal{Q}$-module and hence $\mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q})$-projective [7; 10.7, Theorem 3]. Since $\mathbb{Z}_p$ is a $\mathbb{Z}_p[[\Gamma_\mathcal{Q}]]$-torsion module we have, by tensoring with quotient field of $\mathbb{Z}_p[[\Gamma_\mathcal{Q}]]$,

$$\text{quot}(\mathbb{Z}_p[[\Gamma_\mathcal{Q}]] \otimes J_\infty \cong \text{quot}(\mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q}))$$

and therefore by [24, Cor. 6.4]

$$J_\infty \cong \mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q}).$$

This proves the corollary. \[\square\]

Theorem 6.5: We have the following commutative exact diagram for the $\Gamma[\Delta] = \bigoplus_{\mathcal{Q} \mid \nu} \mathbb{Z}_p[[\Gamma_\mathcal{Q}]](\Delta_\mathcal{Q})$-modules $U^1$ and $\mathcal{A}$:

$$
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
\bigoplus_{\nu \in S_p(K)} (\Lambda[\Delta]^n \oplus \Lambda/(\omega_\nu)(\Delta/\Delta_\nu)(1)) & \bigoplus_{\nu \in S_p(K)} (\Lambda[\Delta]^n \oplus \Lambda/(\omega_\nu)(\Delta/\Delta_\nu)(1)) \\
\downarrow & \downarrow \\
\bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_\nu)[\Delta/\Delta_\nu] & \bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_\nu)[\Delta/\Delta_\nu] \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

where $r_\nu = (\Gamma: \Gamma_\mathcal{Q})$. 

7. $\lambda$-invariants for cyclotomic $\Gamma$-extensions; a duality theorem.

With the notations and conventions of section 6 we consider the $\Lambda[\Delta]$ modules $\mathcal{X}_r = X_r(\mathcal{X})$ and their decomposition by the action of $\Delta$,

$$\mathcal{X}_r = \bigoplus_{i \mod d} e_i \mathcal{X}_r, \quad r = 1, 2, 3.$$  

We put

$$\mu_r^{(\cdot)} = \mu(e, \mathcal{X}_r) \quad \text{and} \quad \lambda_r^{(\cdot)} = \lambda(e, \mathcal{X}_r);$$

if $K$ is totally real, in which case $\mathcal{X} = K(\mu_p)$ is a field of CM-type with maximal totally real subfield $K^+$, then we set

$$\mathcal{X}_r^+ = \bigoplus_{i \mod d \atop i \text{ even}} e_i \mathcal{X}_r \quad \text{and} \quad \mathcal{X}_r^- = \bigoplus_{i \mod d \atop i \text{ odd}} e_i \mathcal{X}_r,$$

$$\lambda_r^+ = \sum_{i \text{ even}} \lambda_r^{(i)} \quad \text{and} \quad \lambda_r^- = \sum_{i \text{ odd}} \lambda_r^{(i)}, \quad r = 1, 2, 3.$$  

Let $i, j \in \mathbb{Z}$ and $\nu \in S_p(K)$; we define

$$\delta_{i, \nu} = \begin{cases} 1, & \theta'_i | \Delta_{\nu} = 1 \\ 0, & \text{otherwise} \end{cases}, \quad \delta_{i, j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Obviously we have

$$\sum_{i=0}^{d-1} \delta_{i, \nu} = (\Delta : \Delta_{\nu})$$

and

$$s_{\mathcal{X}_\infty} = \sum_{\nu \in S_p(K_\infty)} (\Delta : \Delta_{\nu}) = \sum_{i=0}^{d-1} \sum_{\nu \in S_p(K_\infty)} \delta_{i, \nu},$$

where $s_F$ is the number of primes dividing $p$ in an algebraic number field $F$.

Our first objective is to study the connections between the $\mu_1$-invariants for finite $p$-extension of $K$. For the $\lambda_1$-invariant we obtain an analogue of the classical Riemann–Hurwitz formula for compact connected Riemann surfaces.

**Proposition 7.1** ([10, Theorem 3]): Let $E \subset K(p)$ be a finite extension of $K$ and $E_\infty = K_\infty E$, then

$$\mu_1(E) = 0 \text{ if and only if } \mu_1(K) = 0.$$
PROOF: Let $S \supseteq \Sigma$ be finite with $E \subset K_S(p)$. Since $H_S(E)$ is an open subgroup of $H_S(K)$ and $\text{cd}_p(H_S(K)) < \infty$ we obtain by [23; I, 3.3, Prop. 14]
$$\text{cd}_p(H_S(K)) = \text{cd}_p(H_S(E)).$$

The assertion now follows from Propositions 5.1 and 5.6, and Theorem 5.3. \qed

**Theorem 7.2:** Let $E | K$ be a finite extension of totally real number fields, $E \subset K(p)$, $E_\infty = K_\infty E$ and $\mu_1(K) = 0$. Then we have the equality

$$\lambda_1(E) - 1 = (\lambda_1(K) - 1)[E_\infty : K_\infty] + \sum_{\nu \mid p} (e_\nu - 1),$$

where $e_\nu$ is the ramification index of a prime $\nu$ of $E_\infty$ for the extension $E_\infty | K_\infty$.

**PROOF:** Let $S(K_\infty) \supseteq S_p(K_\infty)$ be the set of primes which ramify in $K_\infty$ or divide $p$, and $S(E_\infty)$ the set of primes lying above $S(K_\infty)$. Propositions 5.6 and 5.1 yields the equalities

$$\lambda_S(E) = \lambda_1(E) + \#S(E_\infty) \setminus S_p(E_\infty),$$
$$\lambda_S(K) = \lambda_1(K) + \#S(K_\infty) \setminus S_p(K_\infty),$$

and by Propositions 7.1 and 5.6 we have

$$\mu_S(E) = \mu_S(K) = 0.$$ 

By Theorem 5.4 the $\mathbb{Z}_p$-modules $H_S(E)^{ab}$ and $H_S(K)^{ab}$ are free of rank $\lambda_S(E)$ (resp. $\lambda_S(K)$). For the Euler–Poincaré characteristics $\chi(E)$ and $\chi(K)$ of the free pro-$p$-groups $H_S(E)$ (resp. $H_S(K)$) we obtain (Theorem 5.3)

$$\lambda_S(E) - 1 = -\chi(E) = -\chi(K)[E_\infty : K_\infty]$$
$$= (\lambda_S(K) - 1)[E_\infty : K_\infty].$$

Since we have

$$[E_\infty : K_\infty] \cdot \#S(K_\infty) \setminus S_p(K_\infty) - \#S(E_\infty) \setminus S_p(E_\infty)$$
$$= \sum_{\nu \mid p} (e_\nu - 1),$$

the theorem is proved. \qed
In [25] we proved the above theorem for $K = \mathbb{Q}$; it is possible in this case to give a more explicit description of $H_S(E)^{ab}$ even for $p = 2$.

Next we established a duality theorem, which gives us relations between the invariants $\lambda^{1-i}$ and $\lambda^i$. We need the following lemma’s.

**Lemma 7.3:** Let $i \in \mathbb{Z}$, then

$$\text{inj lim}_{n, \text{res}} \text{inj lim}_{m} \ker^1\left(G_{S_p}(K_n), \mathbb{Z}/p^m(i)\right) \cong (e_i, \mathcal{A}_3)(-i)^*.$$  

**Proof:** Passing in (4.1) to the direct limit we obtain the exact sequence

$$0 \to \text{inj lim}_{n, m} \ker^1\left(G_{S_p}(K_n), \mathbb{Z}/p^m(i)\right) \to H^1\left(H_{S_p}(K), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)$$

$$\to \prod_{\nu \in S_p(K_\infty)} H^1\left(G_{\nu}(K_{\infty}), \mathbb{Q}_p/\mathbb{Z}_p(i)\right).$$

Since $d | p - 1$ and by [8; Prop. 22, Cor. 1] there are isomorphisms

$$H^1\left(H_{S_p}(K), \mathbb{Q}_p/\mathbb{Z}_p(i)\right) \cong H^1\left(H_{S_p}(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)^\Delta$$

$$\cong H^1\left(H_{S_p}(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)^\Delta$$

(and similarly for the cohomology groups of the local Galois groups).

Hence

$$\text{inj lim}_{n, m} \ker^1\left(G_{S_p}(K_n), \mathbb{Z}/p^m(i)\right) \cong (e_i, \mathcal{A}_3)(-i)^* \cong (e_i, \mathcal{A}_3)(-i)^*.$$

**Lemma 7.4:** The following diagram commutes and is exact:

$$H^2\left(G_{S_p}(\mathcal{X}_n), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)^\text{res} \oplus_{\nu \in S_p(\mathcal{X}_n)} H^2\left(G_{\nu}(\mathcal{X}_n), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)$$

$$\xrightarrow{T_A(\mathcal{X}_1)(-i)\Gamma^n} (T_A(\mathcal{X})(-i)\Gamma^n)^*$$

$$\to \text{inj lim}_{m} H^0\left(G_{S_p}(\mathcal{X}_n), \mathbb{Z}/p^m(1-i)^*\right) \to 0$$

$$\xrightarrow{T_A(\mathcal{E}'(-i))\Gamma^*} 0$$

in particular,

$$T_A(\mathcal{E}') \cong \mathbb{Z}_p(1).$$
PROOF: By (4.6) and (4.2) (passing to the direct limit) the lines are exact. The vertical morphisms are defined as follows: By the spectral sequence, Lemma 3.1, we have a commutative diagram

\[
\begin{align*}
(\mathbf{G}_S^p(\mathcal{X}_\infty)^{\text{ab}}(\mathcal{X}_\infty))^{e_{\mathcal{X}_1}} &\rightarrow (\mathbf{G}_p(\mathcal{X}_\infty)^{\text{ab}}(\mathcal{X}_\infty))^{e_{\mathcal{X}_1}} \\
\downarrow &\downarrow \\
H^2(\mathbf{G}_S^p(\mathcal{X}_n), \mathbb{Z}/p^m(i)) &\rightarrow H^2(\mathbf{G}_p(\mathcal{X}_n), \mathbb{Z}/p^m(i))
\end{align*}
\]

where the right map is an isomorphism because of \( \text{cd}_p(\mathbf{G}_p(\mathcal{X}_\infty)) = 1 \). Since we have

\[
\text{inj lim}_m \left( \mathbf{G}_S^p(\mathcal{X}_\infty)^{\text{ab}}(\mathcal{X}_\infty) \right)^{e_{\mathcal{X}_1}} = \left( \mathcal{A}_1^{e_{\mathcal{X}_1}} \right)^* = \left( T_\Lambda(\mathcal{A}_1)(-i)^{e_{\mathcal{X}_1}} \right)^*,
\]

\[
\text{inj lim}_m \bigoplus_{\nu \in S_p(\mathcal{X}_n)} \left( \mathbf{G}_p(\mathcal{X}_\infty)^{\text{ab}}(\mathcal{X}_\infty) \right)^{e_{\mathcal{X}_1}} = \left( \mathcal{A}(\mathcal{X}_\infty)^{e_{\mathcal{X}_1}} \right)^* = \left( T_\Lambda(\mathcal{A})(-i)^{e_{\mathcal{X}_1}} \right)^*
\]

and

\[
\text{inj lim}_m H^2(H_S^p, \mathbb{Z}/p^m(i)) = H^2(H_S^p, \mathbb{Q}_p/\mathbb{Z}_p)(i) = 0,
\]

by Proposition 5.1, we obtain the left square and hence the right. Now, because of

\[
T_\Lambda(\overline{E}) \subset T_\Lambda(\mathcal{A}) \cong \bigoplus_{\nu \in S_p(K)} \Lambda^/\omega_{\nu}[\Delta/\Delta_{\nu}](1).
\]

Theorem 6.5 and \( T_\Lambda(\overline{E})^{e_{\mathcal{X}_n}} \cong \mu_{\mathcal{X}_n} \) we obtain the last assertion. \( \square \)

**Lemma 7.5:** Let \( i \in \mathbb{Z} \), then there is an exact sequence

\[
0 \rightarrow \alpha(T_\Lambda(e,\mathcal{X}_1)(-i)/T_{\mathbb{Z}_p}(e,\mathcal{X}_1)(-i)) \rightarrow \text{proj lim}_{n, \text{cor}} \text{proj lim}_m H^2(S_p(\mathcal{X}_n), \mathbb{Z}/p^m(i)) \rightarrow \alpha(T_{\mathbb{Z}_p}(e,\mathcal{X}_1)(-i)) \rightarrow 0;
\]

furthermore, there exist a quasi-isomorphism \( \phi_1 \) and two isomorphisms \( \phi_2 \) and \( \phi_3 \), such that the following diagram with quasiexact upper line and
exact bottom line commutes:

\[
\begin{array}{c}
\alpha(T_A(e, \mathcal{X})(-i)) \to \operatorname{proj \ lim} \ H^2_{\phi_1} (\mathcal{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)) \\
\downarrow \\
\alpha(T_A(e, \mathcal{A})(-i)) \to \operatorname{proj \ lim} \ H^2_{\phi_2} (\mathcal{G}_{\mathcal{V}}(K_n), \mathbb{Z}/p^m(i)) \\
\downarrow \\
\alpha(T_A(e, \mathcal{E}')(i)) \to \operatorname{proj \ lim} \ H^0_{\phi_1} (\mathcal{G}_{S_p}(K_n), \mathbb{Z}/p^m(1-i))^* \\
\downarrow \\
0 \\
\end{array}
\]

PROOF: By [8; Prop. 22, Cor. 1] and because of Lemma 3.1 and \((p, d) = 1\) we obtain the exact sequence

\[
0 \to H^1(\Gamma^n, H^1(S_p(\mathcal{X}), \mathbb{Z}/p^m(i)))^\Delta \to H^2(S_p(K_n), \mathbb{Z}/p^m(i)) \\
\to H^2(S_p(\mathcal{X}), \mathbb{Z}/p^m(i))^\Gamma \times \Delta \to 0.
\]

Since

\[
H^1(\Gamma^n, H^1(S_p(\mathcal{X}), \mathbb{Z}/p^m(i)))^\Delta \cong \left(\mathcal{X}_{1,m}(-i)^{\Gamma^n}\right)^\Delta^* \\
\cong (e_i, \mathcal{X}_1)(-i)^\Gamma^* \cong \mathcal{X}_1(-i)^\Gamma^*
\]

and because of proposition 5.1

\[
H^2(S_p(\mathcal{X}), \mathbb{Z}/p^m(i))^\Gamma \times \Delta \cong \left(\rho^m S_p(\mathcal{X})^{ab}(-i)^*\right)^\Gamma \times \Delta \\
\cong \left(\rho^m(e_i, \mathcal{X}_1)(-i)\right)^\Gamma^*
\]

Lemma 1.9 proves the first assertion.

The lines of the diagram are quasi-exact resp. exact by (4.6) resp. (4.2). With the diagram in the proof of Lemma 7.4 (now passing to the inverse limit) and the structure theorem (Theorem 6.5) we obtain the commuta-
tive diagram

\[
\alpha\left(e_i(T_{\Lambda}(\mathcal{H}_1/T_{\mathbb{Z}}(\mathcal{H}_1))(-i)) \rightarrow \alpha(e_iT_{\Lambda}(\mathcal{H}')(\mathcal{H}')(i)) \right) \cong \bigoplus_{\phi \in \mathbb{S}(\mathbb{K})} \Lambda^\phi(\omega_{\eta})^\phi \cdot \cdot \cdot (i-1)
\]

\[
\downarrow \quad \downarrow \quad \phi_2
\]

\[
\text{proj} \lim_{n, m} H^2(\mathcal{G}_p(\mathbb{K}_n), \mathbb{Z}/p^m(i)) \rightarrow \text{proj} \lim_{n, m} \bigoplus_{\phi \in \mathbb{S}(\mathbb{K}_n)} H^2(\mathcal{G}_p(\mathbb{K}_n), \mathbb{Z}/p^m(i))
\]

\[
\downarrow \quad \alpha(e_iT_{\mathbb{Z}}(\mathcal{H}_1))(-i)
\]

Since the vertical exact sequence is quasi-splitting and \(\alpha(T_{\Lambda}(\mathcal{H}'))\) is \(\mathbb{Z}_p\)-torsion free, we can define the quasi-isomorphism \(\phi_1\), such that the left square commutes. By Lemma 7.4 we have

\[
\alpha(T_{\Lambda}(\mathcal{H}')(\mathcal{H}')(i-2)) = \text{proj} \lim_{n} \alpha(T_{\Lambda}(\mathcal{H}')(\mathcal{H}')(i-2)^* \cong \text{proj} \lim_{n, m} H^0(\mathcal{G}_p(\mathcal{H}_n), \mathbb{Z}/p^m(i-1)).
\]

Thus there is an isomorphism

\[
\phi_2: \alpha(e_iT_{\Lambda}(\mathcal{H}')(\mathcal{H}')(i)) \rightarrow \text{proj} \lim_{n, m} H^0(\mathcal{G}_p(\mathcal{H}_n), \mathbb{Z}/p^m(1-i))^*,
\]

which because of Lemma 7.4 commutes with \(\phi_2\). \(\square\)

**Lemma 7.6:** Let \(M\) and \(N\) be noetherian \(\Lambda[\Delta]\)-torsion modules, \(i \in \mathbb{Z}\), and let \(\hat{\psi}\) be a \((\Gamma \times \Delta)\)-invariant pairing with finite kernels; let \(\hat{\psi}_m\) th pairing induced by \(\hat{\psi}\).

\[
\hat{\psi}: \quad M \times N^* \rightarrow \mathbb{Q}_p/\mathbb{Z}_p(i)
\]

\[
\uparrow \quad \uparrow \quad \uparrow
\]

\[
\hat{\psi}_m: \quad M_{p^m} \times p^m(N^*) \rightarrow p^{-m}\mathbb{Z}_p/\mathbb{Z}_p(i)
\]

Then there exists a \((\Gamma \times \Delta)\)-invariant pairing \(\psi\),

\[
\psi: \quad M \times \alpha(N/T_{\mathbb{Z}_p}(N)) \rightarrow \mathbb{Z}_p(i)
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\psi_m: \quad M_{p^m} \times \text{Hom}(N_{p^m}, \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathbb{Z}/p^m(i),
\]

such that the pairing \(\psi_m, m \geq 0\), induced by \(\psi\), are equal to \(\hat{\psi}_m\) and the pairing

\[
\psi \otimes \mathbb{Q}_p: \quad (M \otimes \mathbb{Q}_p) \times (\alpha(N) \otimes \mathbb{Q}_p) \rightarrow \mathbb{Q}_p(i)
\]
is non-degenerated.

PROOF: It is easy to see, that the diagram

\[
\begin{array}{c}
M_p^m \times p^m (N^*) \xrightarrow{\varphi_m} \mathbb{Z} / p^m(i) \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(i) \\
\downarrow \downarrow \quad \downarrow p^{n-m} \quad \downarrow \text{can} \quad \downarrow p^{n-m} \\
M_p^m \times p^m (N^*) \xrightarrow{\varphi_m} \mathbb{Z} / p^m(i) \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(i)
\end{array}
\]

commutes. Since we can choose \( \pi_m = p^m \) to obtain the adjoint of \( N / T_{Z_p}(N) \), we get

\[
\text{proj lim}_{m} p^m (N^*) = \text{proj lim}_{m} (N / T_{Z_p}(N))^* = \alpha \left( N / T_{Z_p}(N) \right)
\]

and hence a pairing \( \psi := \text{proj lim} \tilde{\psi}_m \) with the required properties. \( \square \)

Now we define the \( \Lambda[\Delta] \)-torsion module \( \mathcal{X} \) by the following exact sequence, which is induced by (4.6):

\[
0 \rightarrow T_{\Lambda}(\mathcal{E}') \rightarrow T_{\Lambda}(\mathcal{A}) \rightarrow T_{\Lambda}(\mathcal{X}_1) \rightarrow \mathcal{X} \rightarrow 0.
\] (7.7)

Lemma 7.5 implies a quasi-isomorphism

\[
((e_i \mathcal{X})(-i))^o \sim \text{proj lim}_{n,m} \ker^2 \left( G_{S_p}(K_n), \mathbb{Z} / p^m(i) \right),
\] (7.8)

and therefore we get by the global duality theorem, that is (4.3), (4.4), (4.5) and Lemma 7.6 the following duality assertion:

**THEOREM 7.9:** There exists a pairing induced by the cupproduct

\[
(e_{1-i} \mathcal{X}_3)^* \times (e_i \mathcal{X})^o \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(-1)
\]

with finite kernels and hence a quasi-isomorphism

\[
((e_i \mathcal{X})(-i))^o \rightarrow (e_{1-i} \mathcal{X}_3)(i-1).
\]

This bilinear form induces a non-degenerated pairing of \( \mathbb{Q}_p \)-vector spaces

\[
\psi_0^{(1-i)} : (e_{1-i} \mathcal{X}_3 \otimes \mathbb{Q}_p) \times (e_i \mathcal{X} \otimes \mathbb{Q}_p) \rightarrow \mathbb{Q}_p(1),
\]

for which we have

\[
\psi_0^{(1-i)}(\gamma x, \gamma z) = \kappa(\gamma) \psi_0^{(1-i)}(x, z),
\]
\[ T_{\mathbb{Z}_p}(e_{1-i}, \mathcal{X}_3) \times T_{\mathbb{Z}_p}(e_i, \mathcal{X}) \to \mathbb{Z}/p^\mu(1), \]

with \( \mu = \mu_1^{(i)} = \mu_3^{(1-i)} \).

**Corollary 7.10:** Let \( i \in \mathbb{Z} \); then the following sequence is quasi-exact:

\[ 0 \to ((e_{1-i}, \mathcal{X}_3)(i - 1))^\circ \to T_\Lambda(e_i, \mathcal{X}_1)(-i) \to \bigoplus_{\nu \in S_\nu(K)} \Lambda/(\nu_\nu)_{\delta_{1-i,\nu}(1-i)} \]
\[ \to \mathbb{Z}_p(1 - i)_{\delta_{1-i,0}} \to 0; \]

As a consequence, we have

\[ \mu_1^{(i)} = \mu_3^{(1-i)} \quad \text{and} \quad \lambda_1^{(i)} = \lambda_3^{(1-i)} + n_{1-i} - \delta_{1-i,0}, \]

where

\[ n_j = \sum_{\nu \in S_\nu(K_\infty)} \delta_{j,\nu}, \quad j \in \mathbb{Z}, \]

and in the case \( \mu_p \subset K = \mathcal{X} \)

\[ \mu_1 = \mu_3 \quad \text{and} \quad \lambda_1 = \lambda_3 + s_{\mathcal{X}_\infty} - 1. \]

**Corollary 7.11:** Let \( \mathcal{X} \) be a number field of CM-type with maximal totally real subfield \( K^+ \), then

\[ \lambda_1^- = \lambda_3^+ + s_{K_\infty} - 1, \quad \lambda_1^+ = \lambda_3^- + s_{\mathcal{X}_\infty} - s_{K_\infty}. \]

**Remark:** The assertions of Corollary 7.10 and Theorem 5.4(i) in the case \( \mu_p \subset K \) (without the exact description of the factors \( \omega_\nu \) of the characteristic polynomial of \( \mathcal{X}_1 \)) is due to Iwasawa [9, Theorems 15 and 16] (if \( \mathcal{T} := \kappa(y_\nu)(1 + T)^{-1} - 1 \) denotes the “Iwasawa-involution”, we have for a compact \( \Lambda \)-module \( M \) the equality \( \mathcal{T} = (M(-1))^\circ = M(1) \)).

**Corollary 7.12:** Let \( E|K \) be a finite extension of totally real number fields, \( E \subset K(p) \), \( \mu_1(K) = 0, \delta = E(\mu_p) \) and \( \mathcal{X} = K(\mu_p) \), then

\[ \lambda_3^{(1)}(\delta) - 1 = (\lambda_3^{(1)}(\mathcal{X}) - 1)[\mathcal{X}_\infty \cdot \mathcal{X}_\infty^\circ] + \sum_{\nu \in S(E_\infty) \atop \mu_p \subset E_\nu}(n_\nu - 1), \]
where $S(E_\infty)$ is a finite set of primes containing $S_p(E_\infty)$ and all prime divisors, which ramify in the extension $E_\infty | K_\infty$, and $n_\wp = [E_\infty : K_\infty]$ (= ramification index if $\wp \nmid p$).

**PROOF:** We have

$$n_1(K) = \sum_{\wp \in S_p(K)} \delta_1(\wp) = \# \{ \wp \in S_p(K) : \mu_p \subset K_\infty \},$$

hence

$$n_1(K)[E_\infty : K_\infty] - n_1(E) = \sum_{\wp \in S_p(E_\infty) \subset E_\infty \wp \subset E_\infty} (n_\wp - 1).$$

Since the condition $\mu_p \subset E_\infty$ is necessary for $n_\wp > 1$, $\wp \in S(E_\infty) \setminus S_p(E_\infty)$, and $\lambda_1(\wp) = \lambda_1(E)$, $\lambda_1(\wp) = \lambda_1(K)$ the corollary follows from Theorem 7.2 and Corollary 7.10. □

Finally, we give a description of the fixed part of $\mathcal{X}_1(-1)$ used in the next section. For this we define

$$B := T_A(\mathcal{A})/T_A(\overline{E}'),$$

and obtain the following:

**PROPOSITION 7.13:** Let $n \geq 0$ be sufficiently large, then

$$B(-1) = \mathcal{X}_1(-1)^{\Gamma^n}.$$

**PROOF:** The assertion follows from the commutative exact diagram in Lemma 7.4, since we have, for large $n$ (Theorem 6.5),

$$T_A(\mathcal{A})(-1)^{\Gamma^n} = T_A(\mathcal{A})(-1),$$

$$T_A(\overline{E}')(1)^{\Gamma^n} = T_A(\overline{E}')(1)$$

for all $n \geq 0$ (Lemma 7.4) and by (4.3)

$$\operatorname{inj}\lim_m \ker^2(\mathcal{G}_{S_p}(\mathcal{X}_n), \mathbb{Z}/p^m(1))$$

$$\cong \left( \operatorname{proj}\lim_m \ker^1(\mathcal{G}_{S_p}(\mathcal{X}_n), \mathbb{Z}/p^m) \right)^*$$

$$= \operatorname{inj}\lim_m \operatorname{Cl}_{S_p}(\mathcal{X}_n)_{p^n} = 0,$$

where $\operatorname{Cl}_{S_p}(\mathcal{X}_n)$ denotes the finite $S_p$-ideal class group of $\mathcal{X}_n$. 
8. The relations between $\lambda$-invariants for fields of CM-type

With the notations and conventions of section 7 we assume further that $K$ is a totally real number field and hence $\mathcal{X} = K(\mu_p)$ is a CM-field with maximal totally real subfield $K^+$. In this case the subgroup generated by the units $E(K^+_n)$ and the roots of unity of $\mathcal{X}_n$, $n \geq 0$, has index 1 or 2 in the group $E(\mathcal{X}_n)$. Since $p \neq 2$ we have

$$e_i\left( E_{\infty} \otimes \mathbb{Q}_p / \mathbb{Z}_p \right) = 0 \quad \text{for } i \text{ odd.} \quad (8.1)$$

To get the relations between the invariants $\lambda_2$ and $\lambda_3$ we use Kummer-theory.

We summarize some results of [9, §7]: Let $I_{\infty}$ be the free abelian group on the non-archimedian primes of $\mathcal{X}_{\infty}$ which do not lie above $p$. Define $\mathfrak{M}$ by the exactness of the sequence

$$0 \to \mathfrak{M} \to \mathcal{X}_{\infty} \otimes \mathbb{Q}_p / \mathbb{Z}_p \to I_{\infty} \otimes \mathbb{Q}_p / \mathbb{Z}_p,$$

where $\varphi(a \otimes \alpha) = (a)' \otimes \alpha$, $(a)' = \sum v_p(a)p$ with the $p$-adic valuation $v_p$. Then there exists the perfect Kummer-pairing

$$\left( , \right) : \mathfrak{M}_1 \times \mathfrak{M} \to \mathbb{Q}_p / \mathbb{Z}_p(1) \quad (8.2)$$

satisfying $(\sigma x, \sigma m) = \sigma (x, m)$ for all $\sigma \in \Gamma \times \Delta$. For the fields

$$N = \mathcal{X}_{\infty}\left( E_{\infty}^{1/p^\infty} \right) \quad \text{and} \quad N' = \mathcal{X}_{\infty}\left( E_{\infty}^{\lambda_1/p^\infty} \right)$$

exist quasi-exact sequences induced by the above pairing [9, Lemma 10 with the remark following it and Theorem 11]:

$$0 \to \mathfrak{M}_1(1) \to \mathfrak{M} \to G\left( N/\mathcal{X}_{\infty} \right) \to 0, \quad (8.3)$$

$$0 \to \mathfrak{M}_1(1) \to \mathfrak{M} \to G\left( N'/\mathcal{X}_{\infty} \right) \to 0. \quad (8.4)$$

Since the duals of $E_{\infty} \otimes \mathbb{Q}_p / \mathbb{Z}_p$ and $E_{\infty}' \otimes \mathbb{Q}_p / \mathbb{Z}_p$ are free $\mathbb{Z}_p$-modules we see by Proposition 2.1 that

$$\text{hd}_\Lambda G\left( N/\mathcal{X}_{\infty} \right) \leq 1 \quad \text{and} \quad \text{hd}_\Lambda G\left( N'/\mathcal{X}_{\infty} \right) \leq 1. \quad (8.5)$$

Now, from the exact sequence (4.6) we obtain by the structure theorem (Theorem 6.5)

$$\text{rg}_\Lambda e_i\mathfrak{M}_1 \leq \sum_{v \in S_p(K)} n_v = n = [K : \mathbb{Q}].$$
Since $\mathcal{X}_1^+$ is a $\Lambda$-torsion module and

$$\text{rg}_\Lambda \mathcal{X}_1 = \text{rg}_\Lambda \mathcal{X}_1^- = \sum_{i \text{ odd}} \text{rg}_\Lambda e_i \mathcal{X}_1 = n \cdot d/2$$

(Theorems 5.4) the following proposition is proved:

**PROPOSITION 8.6:**

$$\text{rg}_\Lambda e_i \mathcal{X}_1 = \begin{cases} n, & i \text{ odd,} \\ 0, & i \text{ even.} \end{cases}$$

Since a noetherian $\Lambda$-module $M$ is quasi-isomorphic to $T_\Lambda(M) \oplus F_\Lambda(M)$, the quasi-exact sequence in Corollary 7.10 induces a quasi-exact sequence

$$0 \to G(N'/\mathcal{X}_\infty) \to (\Lambda[\Delta]^n)^{\sim} \oplus \bigoplus_{\wp \in S_p(K)} \Lambda/(\wp)^{\Delta/\Delta_v}(1) \to \mathbb{Z}_p(1) \to 0. \quad (8.7)$$

Especially we see that the $\mu$-invariant of the $\Lambda$-torsion module $G(N'/N)$ is zero, hence

$$\mu_1 = \mu_2 = \mu_3. \quad (8.8)$$

Since $e_i(\mathcal{X}_\infty \otimes \mathbb{Q}_p / \mathbb{Z}_p) = 0$ for an odd number $i$ we get via Kummer pairing $e_{1-i}G(N/\mathcal{X}_\infty) = 0$ ($i$ odd). The existence of a quasi-isomorphism

$$(e_i \mathcal{X}_2)(-i) \sim (e_{1-i} \mathcal{X}_1)(i-1) = T_\Lambda(e_{1-i} \mathcal{X}_1)(i-1), \quad i \text{ odd}, \quad (8.9)$$

follows from (8.3).

From this last assertion and Theorem 7.2 we obtain now a Riemann–Hurwitz formula for $\lambda^{(i)}_2$ which for $K=K^+$ is the result obtained by Kida [16] and Iwasawa [11].

**COROLLARY 8.10:** Let $E | K$ be a finite extension of totally real number fields, $E \subset K(\mathfrak{p})$, $\mu(K) = 0$, $\mathcal{X} = E(\mu_p)$ and $K' = K(\mu_p)$, then

$$\lambda^{(i)}_2(\mathcal{X}) - 1 = (\lambda^{(i)}_2(\mathcal{X}') - 1)[\mathcal{X}_\infty : \mathcal{X}_\infty] + \sum_{\wp \neq p} (e_\wp - 1),$$

where $e_\wp$ denotes the ramification index of $\wp$ for the extension $\mathcal{X}_\infty / \mathcal{X}_\infty$. 
**Proposition 8.11**: Let $i \in \mathbb{Z}$ be odd, then the sequence

$$0 \to \bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_{\nu})^{\delta_{\nu}} \to e_i\mathfrak{A}_2 \to e_i\mathfrak{A}_3 \to 0$$

is exact. Consequently

$$\lambda_{1(i)}^{(-i)} = \lambda_2^{(i)} = \lambda_3^{(i)} + n_i$$

and

$$s_{\mathfrak{A}_\infty} - 1 \geq \lambda_2 - \lambda_3 \geq \lambda_2^+ - \lambda_3^+ = s_{\mathfrak{A}_\infty} - s_{K_\infty^+}.$$

**Proof**: Since we have the equalities

$$\sum_{i \text{ odd}} n_i = s_{\mathfrak{A}_\infty} - \sum_{\nu \in S_p(\mathfrak{A}_\infty)} \sum_{i \text{ even}} \delta_{i, \nu} = s_{\mathfrak{A}_\infty} - s_{K_\infty^+}$$

and

$$\lambda_{1(i)}^{(-i)} = \lambda_2^{(i)} = \lambda_3^{(i)} + n_i, \quad i \text{ odd},$$

(by (8.7) and Corollary 7.10) we get $\lambda_2^+ - \lambda_3^+ = s_{\mathfrak{A}_\infty} - s_{K_\infty^+}$. The quasi-exact sequence (8.7) shows the inequality

$$\lambda_2 - \lambda_3 = \lambda\left(G\left(N'/N\right)\right) \leq \lambda\left(G\left(N'/\mathfrak{A}_\infty\right)\right) = s_{\mathfrak{A}_\infty} - 1.$$

By the snake lemma and the exact sequence (Theorem 6.5)

$$0 \to e_iU^1 \to e_i\mathfrak{A} \to \bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_{\nu})^{\delta_{\nu}} \to 0,$$

we obtain from the commutative exact diagram (4.6) the exact sequence

$$\bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_{\nu})^{\delta_{\nu}} \to e_i\mathfrak{A}_2 \to e_i\mathfrak{A}_3 \to 0.$$ If $i$ is odd, the equality $\lambda_2^{(i)} = \lambda_3^{(i)} + n_i$ yields the injectivity of the left morphism. □

Let

$$\nabla = T_\Lambda(\mathfrak{A}_1)$$

be the orthogonal complement of $T_\Lambda(\mathfrak{A}_1)$ in the Kummer pairing (8.2). Since the morphism

$$\text{Hom}\left(E_{\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p(1)\right) = G\left(N/\mathfrak{A}_\infty\right) \to F_\Lambda(\mathfrak{A}_1)$$

$$= \text{Hom}\left(\nabla, \mathbb{Q}_p/\mathbb{Z}_p(1)\right)$$
is surjective, we have the inclusion

\[ \nabla \subseteq E_{\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p. \]

In [2, Theorem 5], Coates showed that these groups are equal, if there is only one prime of \( \mathcal{X}_\infty \) lying above \( p \). But in general this is not true; we prove that this assertion is still valid under the weaker assumption \( s_{K,\infty} = 1 \) and, if the Leopoldt-conjecture is true for all fields \( K_n, \ n \geq 0 \), that this condition is also necessary for the equality \( \nabla = E_{\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p \).

**Proposition 8.12:** Let \( s_{K,\infty} = 1 \), then

(a) \[ \lambda^{(i-1)}_1 = \lambda_2^{(i)} + \left( s_{\mathcal{X}_\infty} - 1 \right) \cdot \delta_{i, d/2}, \quad i = 0, 1, \ldots, d - 1; \]

in particular \( \lambda^{+}_2 = \lambda^{+}_3 \).

(b) \[ G(N'/\mathcal{X}_\infty) \sim \Lambda^{\nabla} \oplus \mathbb{Z}_p(1)^{s_{\mathcal{X}_\infty} - 1}. \]

(c) \[ G(N/\mathcal{X}_\infty) = F_{\Lambda}(\mathcal{X}_1) \sim \Lambda^{\nabla} \text{ and } \nabla = E_{\infty} \otimes \mathbb{Q}_p/\mathbb{Z}. \]

**Proof:** With \( S_p(K_\infty) = \{ p \} \) and \( \Delta^+ = G(\mathcal{X}_\infty/K_\infty^+) \) we have \( \Delta/\Delta^+ = (\Delta/\Delta^+)_p = \Delta_p \Delta^+ / \Delta^+ \), hence \( \Delta = \Delta_p \Delta^+ \);

\[ s_{\mathcal{X}_\infty} = 1 \Rightarrow \Delta = \Delta_p \quad \text{and} \quad n_i = \delta_{i, p} = \delta_{i, 0}, \]

\[ s_{\mathcal{X}_\infty} = 2 \Rightarrow \Delta = \Delta_p \oplus \Delta^+ \quad \text{and} \quad n_i = \delta_{i, p} = \delta_{i, 0} + \delta_{i, d/2}. \]

By Proposition 8.11 we obtain the assertion (a) for an odd number \( i \); furthermore we see from the equality

\[ s_{\mathcal{X}_\infty} - 1 = \lambda_2 - \lambda_3 = \lambda^{+}_2 - \lambda^{+}_3 \]

that

\[ \lambda^{(i)}_2 = \lambda^{(i)}_3 \quad \text{for } i \text{ even}, \]

and that \( d/2 \) is necessarily odd, if \( s_{\mathcal{X}_\infty} \) is equal to 2. Hence, by Corollary 7.10 the assertion (a) is proved for all \( i \). The quasi-exact sequence (8.7) proves (b). Because of the equality \( \lambda(G(N'/N)) = \lambda_2 - \lambda_3 = s_{\mathcal{X}_\infty} - 1 \) and the fact, that \( G(N/\mathcal{X}_\infty) \) does not contain any nontrivial submodule (8.5), the kernel of the surjection

\[ G(N/\mathcal{X}_\infty) \rightarrow F_{\Lambda}(\mathcal{X}_1) \]

has to be trivial. \( \square \)
PROPOSITION 8.13: The following assertions are equivalent:

(i) \( \lambda_2^+ = \lambda_3^+ \).

(ii) \( \lambda(G(N/\mathcal{K}_\infty)) = s_{K_\infty}^- - 1 \).

PROOF: This is trivial because of

\[
\lambda_2^+ - \lambda_3^- = \lambda_2 - \lambda_3 - (\lambda_2^- - \lambda_3^-) = \lambda(G(N'/N)) - s_{\mathcal{K}_\infty} + s_{K_\infty}^+
\]

\[= s_{K_\infty}^- - 1 - \lambda(G(N/\mathcal{K}_\infty)).\]

REMARK 8.14: We may conjecture that \( \lambda_2^+ = \lambda_3^+ \) is true in general and not only for \( s_{K_\infty}^- = 1 \) (Proposition 8.12) (Greenberg’s conjecture asserts even more: \( \lambda_2^+ = 0 \) and hence \( \lambda_3^+ = 0 \)). Proposition 8.13 shows that for fields \( K \) with \( \lambda_2^+ = \lambda_3^+ \) the equality \( \nabla = E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p \) holds if and only if \( s_{K_\infty}^- = 1 \). In analogy to the quasi-exact sequence (8.7) we may further expect the existence of a quasi-exact sequence

\[
0 \to G(N/\mathcal{K}_\infty) \to (\Lambda[\Delta]^\nu) \oplus \bigoplus_{v \in S_p(K)} \Lambda/(\omega_v)[\Delta/\Delta_v\Delta^+] \to \mathbb{Z}_p(1) \to 0.
\]

We shall now prove the conjecture \( \lambda_2^+ = \lambda_3^+ \) assuming the Leopoldt-conjecture. For that purpose we consider the following fundamental exact sequence obtained by the snake lemma form the commutative exact diagram (4.6) and the exact sequence

\[
0 \to e_iU^1 \to e_i\mathcal{A} \to \bigoplus_{v \in S_p(K)} \Lambda/(\omega_v)^{\delta_{i,v}}e_i \to 0;
\]

\[
0 \to e_i\overline{E} \to e_i\overline{E'} \to \bigoplus_{v \in S_p(K)} \Lambda/(\omega_v)^{\delta_{i,v}}e_i \to e_i\mathcal{X}_2 \to e_i\mathcal{X}_3 \to 0.
\]

\[(8.15)\]

We show now:

PROPOSITION 8.16: Let \( n_0 \geq \lambda_1^+ \) and let the Leopoldt-conjecture be true for the maximal totally real subfield \( K_{n_0}^+ \) of \( \mathcal{K}_{n_0} \) (for example, if \( K/\mathbb{Q} \) is an abelian extension). Then the equality

\[
\lambda_1^{(i-1)} - n_0 + \delta_{i,0} = \lambda_2^{(i)} = \lambda_3^{(i)}, \quad i \text{ even,}
\]

holds, and in particular

\[
\lambda_2^+ = \lambda_3^+.
\]
PROOF: By Propositions 5.5 and 5.1 the Leopoldt-conjecture is true for all fields \( K_n^+ \), \( n \geq 0 \), and hence by Lemma 3.1

\[
(e, \mathcal{X}_1)^{\Gamma_n} = 0 \quad \text{for } i \text{ even and all } n.
\]

Consequently, the characteristic polynomials \( f_1^{(i)}(T) \) of \( e, \mathcal{X}_1 \), \( i \) even, are prime to \( \omega_n \) for all \( n \). Since \( \mathcal{X}_1^+ \) is a \( \Lambda \)-torsion module, this also holds for the character polynomials \( f_2^{(i)}(T) \) of \( e, \mathcal{X}_2 \). Hence, because of the exact sequence (8.15) the kernel of \( e, \mathcal{X}_2 \to e, \mathcal{X}_3 \) is finite and therefore we obtain \( \lambda_2^{(i)} = \lambda_3^{(i)} \), \( i \) even. \( \square \)

In the following we describe the \( \Lambda[\Delta] \)-module structure of \( \tilde{E} \) and \( \tilde{E}' \).

**Theorem 8.17:** Let \( n \) be the degree \([K:Q]\), then there are \( \Lambda[\Delta] \)-isomorphisms

\[
\bar{E}(\mathcal{X}_\infty) \cong \tilde{E}(K_\infty^+) \oplus \tilde{E}(\mathcal{X}_\infty)^- \cong \Lambda[\Delta^+]^n \oplus \mathcal{Z}_p(1),
\]

\[
\bar{E}'(\mathcal{X}_\infty) \cong \tilde{E}'(K_\infty^+) \oplus \tilde{E}'(\mathcal{X}_\infty)^- \cong \Lambda[\Delta^+]^n \oplus \mathcal{Z}_p(1).
\]

**Proof:** Since the operation \( ^- \) commutes with the \( \Delta \)-action, we get

\[
\bar{E}(\mathcal{X}_n) = \bar{E}(K_\infty^+) \oplus \mu_{\mathcal{X}_n},
\]

hence

\[
\bar{E}(\mathcal{X}_\infty) = \bar{E}(K_\infty^+) \oplus \limproj_n \mu_{\mathcal{X}_n} = \bar{E}(\mathcal{X}_\infty)^+ \oplus \bar{E}(\mathcal{X}_\infty)^-.
\]

The exact sequences (8.15) and (8.11) show, for an odd \( i \), that

\[ e_i \bar{E} = e_i \bar{E}', \]

hence by Lemma 7.4

\[ \mathcal{Z}_p(1) = (\bar{E})^- = (\bar{E}')^- = T_\Lambda(\bar{E}'). \]

Consequently the module \( (\bar{E}')^+ \) is \( \Lambda \)-torsion free. Let \( C_1 \) and \( C_2 \) be defined by the commutative exact diagram

\[
0 \to (\bar{E}')^+ \to F_\Lambda(\mathcal{A}) \to C_2 \to 0
\]

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[
0 \to \bar{E}' \to \mathcal{A} \to C_1 \to 0.
\]

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[
0 \to T_\Lambda(\bar{E}') \to T_\Lambda(\mathcal{A}) \to B \to 0
\]
Since $C_1^+ \subseteq \mathcal{X}_1^+$ is a $\Lambda$-torsion module, it follows from Theorem 6.5 for the $\Lambda$-torsion free module $(\overline{E'})^+$ the quasi-isomorphism
\[ e_i \overline{E'} \sim \Lambda^n, \quad i \text{ even}. \]

We show now that $(\overline{E'})^+_1$ is $\mathbb{Z}_p$-free. Indeed, by Proposition 7.13 we get, for a large $n$,
\[ B(-1) \leq C_1(-1)^{\Gamma^n} \leq \mathcal{X}_1(-1)^{\Gamma^n} = B(-1), \]
and hence the first map in the exact sequence
\[ 0 \to B(-1) \to C_1(-1)^{\Gamma^n} \to C_2(-1)^{\Gamma^n} \to B(-1) \]
is an isomorphism. Therefore $C_2(-1)^{\Gamma^n}$ is $\mathbb{Z}_p$-free ($B(-1) = \mathcal{X}_1(-1)^{\Gamma^n}$ is $\mathbb{Z}_p$-free, by Theorem 5.4 (ii)) and consequently $C_2(-1)$ (resp. $C_2$) does not contain any non-trivial finite $\Lambda$-submodule. Now, the exact sequence
\[ 0 \to C_2^\Gamma \to (\overline{E'})^+_\Gamma \to F_\Lambda(\mathcal{A})^\Gamma \cong \mathbb{Z}_p^{n \cdot \#\Delta} \]
proves that $(\overline{E'})^+_\Gamma$ is $\mathbb{Z}_p$-free. By (1.2) we see
\[ e_i \overline{E'} \sim \Lambda^n, \quad i \text{ even}, \]
and then the exact sequence (8.15) yields our assertion for $e_i \overline{E}$. □

For the homological dimensions of the $\Lambda$-modules $\mathcal{X}_2^-$ and $\mathcal{X}_3^-$ we obtain the following:

**Proposition 8.18:**

\[ \text{hd}_\Lambda \mathcal{X}_2^- \leq 1 \quad \text{and} \quad \text{hd}_\Lambda \mathcal{X}_3^- \leq 1. \]

**Proof:** Because $(\overline{E'})^- = T_\Lambda(\overline{E'})$ the diagram in the proof of Theorem 8.17 implies $T_\Lambda(C_1)^- = B^-$ and hence induces the following commutative exact diagram:
\[ \begin{array}{ccc}
F_\Lambda(\mathcal{A})^- & \to & F_\Lambda(C_1) \\
\uparrow & & \uparrow \\
0 \to (\overline{E'})^- & \to & \mathcal{A}^- \to C_1^- \to 0.
\end{array} \]
\[ \begin{array}{ccc}
0 \to T_\Lambda(\overline{E'})^- & \to & T_\Lambda(\mathcal{A})^- \to B^- \to 0
\end{array} \]

By Proposition 7.13 we obtain for large $n$
\[ T_\Lambda(C_1)^-(-1) = B^-(-1) = B(-1)^{\Delta^\ast} = \left(\mathcal{X}_1(-1)^{\Gamma^n}\right)^{\Delta^\ast} \]
\[ = \left(\mathcal{X}_1(-1)^{\Delta^\ast}\right)^{\Gamma^n} = \mathcal{X}_1(-1)^{\Gamma^n}. \]
Since $B(-1)$ is $\mathbb{Z}_p$-free, the exact sequence
\[ 0 \to T_\Lambda(C_1)(-1) \to C_1(-1)^{\Gamma_n} \to F_\Lambda(C_1)^{\Gamma_n} \to 0 \]
shows $C_1(-1)^{\Gamma_n}$ to be $\mathbb{Z}_p$-free too. Furthermore, we have
\[ C_1(-1)^{\Gamma_n} = T_\Lambda(C_1)(-1) = \mathfrak{X}_1(-1)^{\Gamma_n} \]
and hence an inclusion induced by the exact sequence $0 \to C_1^- \to \mathfrak{X}_1^- \to \mathfrak{X}_3^- \to 0$,
\[ \mathfrak{X}_3^-(-1)^{\Gamma_n} \to C_1^-(1)^{\Gamma_n}. \]

This proves the proposition for $\mathfrak{X}_3^-$ and the exact sequence (8.11) gives the assertion for $\mathfrak{X}_2^-$. □

REMARK: There are examples by Greenberg [5] showing that the module $\mathfrak{X}_2^+$ is finite but not trivial, hence $\text{hd}_N\mathfrak{X}_2^+ > 1$.

Since $\mathfrak{X}_3^-$ does not contain any non-trivial finite $\Lambda$-submodule we have as a consequence of the exact sequence (8.11) the following divisibility assertion which is connected with the famous Lichtenbaum-conjecture:

**Proposition 8.19:** Let $i \in \mathbb{Z}$ be odd and the order of the group $(e,\mathfrak{X}_3)(-i)^\Gamma$ be finite. Then the order of $e,\mathfrak{X}_2(-i)^\Gamma$ is finite and divisible by
\[ \prod_{\nu \in S_p(K)} w_i(K_\nu), \]

where $w_i(K_\nu) = \max\{ p^j : [K_\nu(\mu_{p^j}) : K_\nu]|i \}.$

REMARK: (a) For $i \geq 1$ this is the well-known result of Lichtenbaum [3, Theorem 9; 4].

(b) For negative $i$ the group $(e,\mathfrak{X}_3)(-i)^\Gamma$ is finite. This is proved in [22; §7, Satz 12] using results obtained by algebraic $K$-theory.

PROOF: The exact sequence (8.11) induces the exactness of the following sequence
\[ (e,\mathfrak{X}_3)(-i)^\Gamma \to \bigoplus_{\nu \in S_p(K)} \Lambda/(\omega_\nu)^{\delta_{i,-}}(-i)^\Gamma \]
\[ \to (e,\mathfrak{X}_2)(-i)^\Gamma \to (e,\mathfrak{X}_3)(-i)^\Gamma \to 0, \]
i odd. Because of our assumption \((e, A_3)(-i)\) and consequently \((e, A_3)(-i)^\Gamma\) are finite groups; hence, \((e, A_3)(-i)^\Gamma\) is zero (Proposition 8.18). Now with the equalities

\[
\quad = \prod_{\nu \in S_p(K)} H^0\left(\mathcal{G}_p(K_\infty), \mathbb{Q}_p / \mathbb{Z}_p(i)^*\right)
\]

we obtain

\[
\# (e, A_3)(-i)^\Gamma = \prod_{\nu \in S_p(K)} w_i(K_\nu) \cdot \# (e, A_3)(-i)^\Gamma.
\]

Or expressed in another way: if \(|\cdot|_p\) is the normalized valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-1}\) and if \(f^{(i)}(T)\) denotes the characteristic polynomials of \(e, A_r, r = 2, 3,\) we have the equality

\[
|f_2^{(i)}(\kappa^{-1} - 1)|_p^{-1} = \prod_{\nu \in S_p(K)} w_i(K_\nu) \cdot |f_3^{(i)}(\kappa_0^{-1} - 1)|_p^{-1}.
\]

9. Symplectic pairings for CM-fields

We keep the notations of section 8. In the following we shall define two \(\Lambda\)-modules \(L_+\) and \(L_-\) with skew symmetric, \(\Gamma\)-invariant pairings \(\psi_+\) resp. \(\psi_-\), which are non degenerated after tensoring with \(\mathbb{Q}_p\). Thus we obtain functional equations for their characteristic polynomials. In the function field case this will be the well-known functional equation for the zeta function of \(F\) (in this case we get \(L_3 = L = A_3\)).

Because \(T_\Lambda(E^-) = (E^-)^\Lambda, T_\Lambda(E^+) = 0\) and \(T_\Lambda(A_1^+) = A_1^+\) the exact sequence \(0 \to E^- \to A \to A_1 \to A_3 \to 0\) gives us the following commutative exact diagrams:

\[
0 \to F_\Lambda(A^-) \to F_\Lambda(A_1) \to A^- \to 0
\]

\[
0 \to (E^-) \to A^- \to A_1^- \to A_3^- \to 0
\]

\[
0 \to (E^-) \to T_\Lambda(A^-) \to T_\Lambda(A_1^-) \to A^- \to 0
\]

9.1
where $\mathcal{V}^- = F_\Lambda(\mathcal{A}_1)/F_\Lambda(\mathcal{A})^-$ and $\mathcal{V}^+ = F_\Lambda(\mathcal{A})^+/(E')^+$. The $\lambda$-invariant of the $\Lambda$-torsion modules $\mathcal{V}$ and $\mathcal{X}$ are (Theorem 7.9)

$$
\begin{align*}
\lambda(\mathcal{V}^-) &= \lambda_3^-, \\
\lambda(\mathcal{V}^+) &= \lambda_3^+, \\
\lambda(\mathcal{X}^-) &= \lambda_3^+ - \lambda_3^+, \\
\lambda(\mathcal{X}^+) &= \lambda_3^- - \lambda_3^+;
\end{align*}
$$

(9.3)

REMARK: If $\mathcal{K}$ is not necessarily a number field of CM-type, instead of (9.4) we get the exact sequence

$$
0 \rightarrow \mathcal{V}^+ \rightarrow \mathcal{X} \rightarrow \mathcal{X}_3 \rightarrow \mathcal{V}^- \rightarrow 0.
$$

(9.4)

As before, let $B = T_\Lambda(\mathcal{A})/T_\Lambda(\overline{E'})$; we define the $\Lambda$-torsion modules $\mathcal{L}_+$ and $\mathcal{L}_-$ by the commutative exact diagrams

$$
\begin{align*}
0 &\rightarrow F_\Lambda(\mathcal{A})^- \rightarrow F_\Lambda(\mathcal{A}_1) \rightarrow \mathcal{V}^- \rightarrow 0 \\
0 &\rightarrow \overline{E'} \rightarrow \mathcal{A} \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_3 \rightarrow 0 \\
0 &\rightarrow \overline{E'} \rightarrow T_\Lambda(\mathcal{A}) \rightarrow T_\Lambda(\mathcal{X}_1) \rightarrow \mathcal{L}_+ \rightarrow 0
\end{align*}
$$

(9.6)
and

\[
\begin{array}{cccccc}
0 & \rightarrow & (E')^+ & \rightarrow & F_\Lambda(A)^+ & \rightarrow & \mathcal{L} & \rightarrow & X_3 & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & E' & \rightarrow & A & \rightarrow & X_1 & \rightarrow & X_3 & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & T_\Lambda(E') & \rightarrow & T_\Lambda(A) \oplus F_\Lambda(A)^- & \rightarrow & B \oplus F_\Lambda(A)^- & \rightarrow & 0
\end{array}
\]

(9.7)

Obviously, we have

\[
\begin{align*}
\mathcal{L}_+ &= G\left(\frac{L'}{L' \cap T}\right), & T &= M(\mathcal{X})^{T_\Lambda(X_1)} \\
\mathcal{L} &= G\left(\frac{M'}{\mathcal{X}_\infty}\right), & M' &= M(\mathcal{X})^{B \oplus F_\Lambda(A)^-} \subseteq M(K^+) \cdot L'.
\end{align*}
\]

(9.8)

By the exact sequence (9.4) we obtain

\[
\mathcal{L}_+ = X_3^+ \oplus X^- & \quad \text{and} \quad \mathcal{L}_- = X_3^- \oplus X^+,
\]

(9.9)

hence

\[
\lambda(\mathcal{L}_+) = 2\lambda_3^+ \quad \text{and} \quad \lambda(\mathcal{L}_-) = 2\lambda_3^-.
\]

(9.10)

In the case when none of the prime divisors of \( p \) splits in the extension \( \mathcal{X}/K^+ \), we have \( T_\Lambda(A)^+ = 0 \), hence by (9.2) and (8.11)

\[
\mathcal{L} = X_2^- \oplus X_1^+ = G\left(\frac{M(K^+) L(X')}{\mathcal{X}_\infty}\right).
\]

Further, we have the following properties of the \( \Lambda \)-module \( \mathcal{L}_- \): Let \( E^+/K^+ \) be a finite extension with \( E^+ \subseteq K^+(p) \) and \( \mathfrak{e} = E^+(\mu_p) \). If we assume \( \mu(k^+) = 0 \), then the \( \lambda \)-invariants of \( \mathcal{L}_-(\mathfrak{e}) \) and \( \mathcal{L}_-(\mathcal{X}) \) are connected by a Riemann–Hurwitz formula; indeed by Corollary 7.12 we get

\[
\begin{align*}
(\lambda(\mathcal{L}_-(\mathfrak{e})) - 2) &= (\lambda(\mathcal{L}_-(\mathcal{X})) - 2)[\mathfrak{e}_\infty : \mathcal{X}_\infty] \\
&\quad + \sum_{\mathfrak{e}_\infty \subseteq (E^*_\infty)_{\wp}} (n_{\wp} - 1).
\end{align*}
\]

(9.11)

Secondly we see that \( \mathcal{L}_- \) does not contain any finite \( \Lambda \)-submodule:
**Proposition 9.12:** We have $\text{hd}_{\Lambda}(L_-) \leq 1$.

**Proof:** In the exact sequence induced by $0 \to B \oplus F_\Lambda(\mathcal{A})^- \to F_\Lambda(B(A))^- \to 0$ (9.7),

$$0 \to B(-1) \to F_\Lambda(B(A))^- \to L_-(-1)^{\Gamma^n} \to B(-1) \oplus F_\Lambda(\mathcal{A})^-,$$

the first map is an isomorphism for sufficiently large $n$ (Corollary 7.13); consequently, the $\mathbb{Z}_p$-module $L_-(1)^{\Gamma^n}$ is free and by Proposition 2.1 the proposition is proved. \[\Box\]

In order to prove the existence of a functional equation for the characteristic polynomial of $L_-$ (resp. $L_+$) we define two pairings,

$$\psi_{\pm} : (L_\pm \otimes \mathbb{Q}_p) \times (L_\pm \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1);$$

$$\psi_{+}(x, y) = \sum_{i \text{ even}} \psi_0^{(i)}(e_i x, e_{1-i} y) - \psi_0^{(i)}(e_i y, e_{1-i} x)$$

and

$$\psi_{-}(x, y) = \sum_{i \text{ odd}} \psi_0^{(i)}(e_i x, e_{1-i} y) - \psi_0^{(i)}(e_i y, e_{1-i} x),$$

where $\psi_0^{(i)}$ is the pairing introduced in Theorem 7.9.

Recall the following definitions: Let $R$ be a commutative ring with 1, $A$ an associative algebra over $R$ and $a \to a^*$ an involutory antiautomorphism of $A$. A symplectic $A$-space is a pair $(M, \phi)$ consisting of a $A$-module $M$ and a nonsingular skew-symmetric $A$-invariant (that is, $\phi(xa, y) = \phi(x, ya^*), x, y \in M, a \in A$) $R$-bilinear form $\phi$ on $M$. We say $\phi$ is hyperbolic, if there exists a direct decomposition of $M$ in totally isotropic $A$-submodules. Two forms $\phi_1$ and $\phi_2$ on $M$ are said to be equivalent if there exists an $A$-isometry, that is an $A$-isomorphism $\rho$ such that $\phi_1(x, y) = \phi_2(\rho(x), \rho(y))$ for any $x, y \in M$.

By Theorem 7.9 and the decomposition (9.9) we obtain:

**Theorem 9.13:** The pairings

$$\psi_{\pm} : (L_\pm \otimes \mathbb{Q}_p) \times (L_\pm \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1)$$

are $\Gamma$-invariant, that is,

$$\psi_{\pm}(\gamma x, \gamma y) = \kappa(\gamma) \psi_{\pm}(x, y), \quad \gamma \in \Gamma, \ x, y \in L_\pm \otimes \mathbb{Q}_p,$$

nonsingular, skew-symmetric and hyperbolic.
REMARKS: (i) If the divisors of the $\Lambda$-module $G(L'/L' \cap N')$ are disjoint from all principal divisors $(\omega_n), n \geq 0$ (for instance, if the Leopoldt-conjecture holds for all intermediate fields $\mathcal{N}_n, n \geq 0$ [9, Lemma 21]), then there is a quasi-isomorphism $G(L'/L' \cap N') \sim G(L'/L' \cap T) = \mathcal{L}_+$ (see (9.10) and [9, Theorem 24]) and Iwasawa defined in [9, Theorem 23] a pairing $\psi_I$ on the $\Lambda$-module $\mathcal{L}_+$ with the same properties as $\psi_+$. Now, a theorem of Jakovlev concerning symplectic hyperbolic forms [12, Theorem 1], tells that the forms $\psi_I$ and $\psi_+$ are equivalent (the involutory antiautomorphism on $\mathbb{Q}_p[\mathcal{T}]$ is given by $T \rightarrow T = (T + 1)^{-1} \kappa(\gamma_0) - 1$). Remember that if the conjecture of Greenberg ($\lambda_+^2 = 0$) is true, the pairing $\psi_+$ is trivial.

(ii) In [17] Kuz'min obtained a pairing on $\mathcal{L} \otimes \mathbb{Q}_p$ with the properties of Theorem 9.13. The same argument as in (i) shows that this pairing is equivalent to $\psi_-$.

COROLLARY 9.14: The characteristic polynomials $h_+(T)$ and $h_-(T)$ of $\mathcal{L}_+$ resp. $\mathcal{L}_-$ satisfy the following functional equations

\[
\kappa(\gamma_0)^{2\lambda_+} \cdot h_+(T - 1) = T^{2\lambda_+} \cdot h_+(\kappa(\gamma_0)T^{-1} - 1),
\]

\[
\kappa(\gamma_0)^{2\lambda_-} \cdot h_-(T - 1) = T^{2\lambda_-} \cdot h_-(\kappa(\gamma_0)T^{-1} - 1).
\]

PROOF: Since the $p^n$-factor is the same on both sides of the equations the corollary follows by a standard argument of linear algebra (see for instance [Hartshorne, Algebraic geometry, p. 456]).

Considering the exact sequence

$$0 \rightarrow \mathcal{Y}^+ \rightarrow \mathcal{L}_- \rightarrow \mathcal{X}_3 \rightarrow 0$$

we may say, that $h_-(T)$ is the product of the “global” characteristic polynomial $f_3(T)$ of $\mathcal{X}_3$ and the “local” polynomial of $\mathcal{Y}^+$. In order to justify this terminology we consider the commutative exact diagram induced by (9.7),

\[
\begin{array}{ccccccccc}
0 & \rightarrow & G(M'/L') & = & \mathcal{Y}^+ & \rightarrow & \mathcal{X}^+ & \rightarrow & \mathcal{X}_3^+ & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & G(M(K')L'/L') & \rightarrow & \mathcal{X}_1^+ & \rightarrow & \mathcal{X}_3^+ & \rightarrow & 0 \\
\uparrow & & & & & & & & & \\
T_\Lambda(\mathcal{A})^+ & = & T_\Lambda(\mathcal{A})^+ & = & T_\Lambda(\mathcal{A})^+
\end{array}
\]

Hence, the group $\mathcal{Y}^+$ is generated by the local galois groups

$$G \left( M_{p/v}/\mathcal{X}_{\infty v} \right) = G \left( M_{p/v}/L_{p/v} \right), \quad p | p.$$
If \( s_{K_{\infty}} = 1 \) we obtain the following exact sequence:

\[
0 \to T_{\lambda}(A)^+ \to (A_1^+)_v \to \mathcal{G}^+ = \mathcal{G}^+ \to 0;
\]

and consequently an isomorphism

\[
\mathcal{G}^+ \cong G(M(K^+)_v | K_{\infty}^+), \quad \text{if } \mu_p \not\in K_v^+.
\]

Thus, by Corollary 9.14 (and Proposition 8.12(a)) we get:

**Theorem 9.15:** If there exists only one prime \( v \) of \( K_{\infty}^+ \) dividing \( p \) and if \( f_v(T) \in \mathbb{Z}_p[T] \) denotes the characteristic polynomial of the local galois group

\[
G(M(K^+)_v | K_{\infty}^+)^{ab},
\]

then the polynomial \( f_v \cdot f_2, f_2 = f_{x,2}, \) satisfies the functional equation

\[
k(\gamma_0)^{\lambda_2} \cdot f_v(T - 1)f_2(T - 1) = T^2 \cdot f_v(k(\gamma_0)T^{-1} - 1)f_2(k(\gamma_0)T^{-1} - 1).
\]

**Appendix: The analogy with the function field case**

Let \( \mathcal{K} \) be an algebraic function field of one variable over the finite constant field \( F \) containing the \( p \)th roots of unity (hence \( p \neq \text{char } F \)). Let \( A \) denote the subgroup of all points of \( p \)-power order in the jacobian variety associated with \( \mathcal{K} \) and \( F_{\infty} \) the field generated over \( F \) by \( A \). Replacing \( F \) by a finite extension we may assume that every point in \( pA \) is rational over \( F \). Thus \( F_{\infty} = F(\mu_{p^\infty}) \) and \( \mathcal{K}_{\infty} = \mathcal{K}, F_{\infty} \) are \( \Gamma \)-extensions of \( F \) (resp. \( \mathcal{K} \)). We fix a prime \( v \) of \( \mathcal{K} \); if \( \mathfrak{B}_1, \ldots, \mathfrak{B}_2 \) denote the finitely many primes of \( \mathcal{K}_{\infty} \) dividing \( v \) and \( \Sigma = \{ v \} \), then with the notations of Chapter II (if it makes sense) we have \( L = L' \) and \( \mathcal{A}_2 = \mathcal{A}_3 \) and by global class field theory we obtain an exact sequence of \( \Lambda \)-torsion modules

\[
0 \to \mathcal{E}_{\Sigma} \to \mathcal{A} \to \mathcal{A}_1 \to \mathcal{A}_3 \to 0;
\]

\[
\mathbb{Z}_p(1) \quad \mathbb{Z}_p[\Gamma/\Gamma_p](1)
\]

(see [9; 12.3]).
hence
\[ \lambda_1 = \lambda_3 + s - 1. \]

It is well known that as \( \mathbb{Z}_p \)-modules
\[ \mathcal{X}_3 \cong \mathbb{Z}_p^{2g}, \]
where \( g \) denotes the genus of the function field \( \mathcal{X} \). Consequently we get
\[ \mu = 0, \quad \lambda_3 = 2g. \]

Since \( F_\lambda(\mathcal{A}) = 0 \) we obtain by the analog construction (9.6), (9.7) for \( \mathcal{L}_\pm \), the module \( \mathcal{X}_3 \) and, since the global duality theorem is also valid in the function field case, a symplectic pairing \( \psi = \psi_\pm \),
\[ \psi : (\mathcal{X}_3 \otimes \mathbb{Q}_p) \times (\mathcal{X}_3 \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1). \]

Let \( q = \kappa(\gamma_0) = \#F \) then the functional equation
\[ q^g \cdot f_3(T - 1) = T^{2g} \cdot f_3(qT^{-1} - 1) \]
induces the functional equation for the zeta-function of the curve \( X \) associated with \( \mathcal{X} \),
\[ Z(X, 1/qT) = q^{1-g} \cdot T^{2-2g} \cdot Z(X, T), \]
where
\[ Z(X, T) = \frac{\det(1 - \gamma_0^{-1}T, H^1_{et}(X, \mathbb{Q}_p))}{(1 - T)(1 - qT)} = \frac{T^{2g}f_3(T^{-1} - 1)}{(1 - T)(1 - qT)}, \]
\[ \overline{X} = X \times_F F_\infty. \]

**Bibliography**


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