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ON TORSION IN THE SLOPE SPECTRAL SEQUENCE

Richard Crew *

Introduction

The purpose of this paper is to prove, and give some simple applications of, a formula relating the Hodge numbers of a variety X, smooth and proper over a perfect field k, and certain numerical invariants that can be extracted from the slope spectral sequence of X. These invariants are of two kinds; the first, $m^{i,j}(X)$, only depend on the Newton polygon of the crystalline cohomology $H^{i+j}_{\text{cirs}}(X)$, while the second, $T^{i,j}(X)$, describe the torsion in the E_1 terms of the slope spectral sequence of X. We will make extensive use of the Illusie-Raynaud structure theory [4] of this spectral sequence. After proving this formula (Theorem 4 below) we give some simple applications. These all concern a *surface* X; we give a criterion, in terms of the Hodge and crystalline cohomology of X, for the slope spectra sequence to degenerate, and prove a semicontinuity theorem for $T^{0,2}$ which generalizes a result of Nygaard [5]. Applications of theorem 4 to higher dimensional varieties figure in recent work of Ekedahl [2], who gives, notably, a general criterion for degeneration of the slope spectra sequence in terms of the Hodge and crystalline cohomology.

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Notation

All unexplained notation and terminology will be as in Illusie-Raynaud [4]. In what follows k is a perfect field, W its ring of Witt vectors, and K the fraction field of W. If $M^{\bullet}$ is a graded W-module, then we will take $\text{length}_W(M^{\bullet})$ to be the function which to the integer i assigns the number $\text{length}_W(M^i)$.

We will never make explicit use of hypercohomology, so that an expression such as $H^i(\Omega^{\bullet}_X)$ denotes the cohomology of a graded sheaf, not a hypercohomology group.

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Let X be a smooth proper scheme over a perfect field k of characteristic $p > 0$. The *slope spectra sequence*

$$E_1^{p,q} = H^q(X, W\Omega_X^p) \Rightarrow H_{\text{cris}}^{p+q}(X/W) \tag{0.1}$$

is defined and studied in [3], [4]. We recall from [4] that the rows $H^q(X, W\Omega_X^{\cdot})$ in the matrix of E_1 terms can be viewed as graded modules over a graded ring $R^{\cdot}$, the *Raynaud algebra* (c.f. [4] for the definition). One central result is a finiteness theorem for the $R^{\cdot}$-modules $H^q(X, W\Omega_X^{\cdot})$, namely the assertion that they are *coherent* $R^{\cdot}$-modules; we recall that

1. Definition: A graded $R^{\cdot}$-module $M^{\cdot}$ is *coherent* if it possesses a finite filtration with quotients of the following types:

Type I_a: A W-module of finite length concentrated in a single degree

Type I_b: A finite free W-module concentrated in a single degree. Via the action of F, V, such a module is an F-crystal with slopes contained in the interval [0,1).

Type II_i: The module U_i defined in [4] *I* 2.14.3. It is concentrated in two consecutive degrees and is killed by p.

(In [4], the notation of a coherent $R^{\cdot}$-module is actually defined by an equivalent condition.)

Recall that the right $R^{\cdot}$-module $R_n^{\cdot}$ is defined by $R_n^{\cdot} = (V^n, dV^n)\backslash R^{\cdot}$. If $M^{\cdot}$ is coherent, then any graded component of $\operatorname{Tor}_i^R(R_n^{\cdot}, M^{\cdot})^{\cdot}$ has finite W-length. We denote by $\ell(M^{\cdot})(j)$ the $\mathbb{Z}$-valued function on $\mathbb{Z}$ defined by

$$\ell(M^{\cdot})(j) = \sum_i (-1)^i \text{length}_W \operatorname{tor}_i^R(R_1^{\cdot}, M^{\cdot})^j \tag{1.1}$$

It is clear that ℓ is additive for exact sequences of coherent $R^{\cdot}$-modules. Its value on the various modules of Types I_a, I_b, II_i is as follows:

2. Lemma:

(*i*) *If* $M^{\cdot}$ *is of Type* I_a, *then* $\ell(M^{\cdot}) = 0$

(*ii*) *If* $M^{\cdot}$ *is of Type* I_b, *concentrated in degree* 0, *then*

$$\begin{aligned} \ell(0) &= \mathit{length}_W(M/VM) \\ \ell(1) &= -\mathit{length}_W(M/FM) \end{aligned} \qquad (M = M^0)$$

and $\ell(M^{\cdot})(j) = 0$ *for* $j \neq 0, 1$.

(*iii*) *If* $M^{\cdot}$ *is of Type* II_i, *concentrated in degrees* 0, 1, *then*

$$\ell(0) = \ell(2) = 1, \quad l(1) = 2$$

$$\ell(j) = 0 \qquad \textit{for} \quad j \neq 0, 1, 2.$$

PROOF: This is a consequence of the explicit calculation of the Tors made in [4]. For $M^{\cdot}$ of Type I_a or I_b concentrated in degree 0, we have

$$\mathrm{Tor}_0(R_1^{\cdot}, M^{\cdot})^0 = M/VM \quad \mathrm{Tor}_1(R_1^{\cdot}, M)^0 = {}_VM$$

$$\mathrm{Tor}_1(R_1^{\cdot}, M^{\cdot})^1 = M/FM \quad \mathrm{Tor}_2(R_1^{\cdot}, M)^1 = {}_FM$$

Then (ii) follows from M being p-torsion free (hence F- and V-torsion free) and (i) follows from the exactness of

$$0 \to {}_VM \to M \xrightarrow{V} M \to M/VM \to 0$$

$$0 \to {}_FM \to M \xrightarrow{F} M \to M/FM \to 0.$$

The proof of (iii) is similar.

We want to define the numerical invariants alluded to in the introduction. If $M^{\cdot}$ is coherent, we recall that $M^i \otimes \mathbb{Q}$ is an F-isocrystal with slopes in [0, 1) for each i, and set

$$m_\lambda^i(M^{\cdot}) = \text{the multiplicity of the slope } \lambda \text{ in } \quad M^i \otimes \mathbb{Q}$$

Now choose a filtration for $M^{\cdot}$ as in Definition 1, and let

$$T^i(M^{\cdot}) = \text{the number of times any module of the form } \quad U_j(-i)$$

$$\text{appears as a quotient in the given filtration.}$$

In this situation we have

3. LEMMA: *The number* $T^i(M^{\cdot})$ *is independent of the filtration chosen for* $M^{\cdot}$, *and we have*

$$\ell(M^{\cdot})(i) = \sum_{\lambda \in [0,1)} (1-\lambda) m_\lambda^i(M^{\cdot}) - \sum_{\lambda \in [0,1)} \lambda m_\lambda^{i-1}(M^{\cdot})$$

$$+ T^i(M^{\cdot}) + 2T^{i-1}(M^{\cdot}) + T^{i-2}(M^{\cdot})$$

PROOF: Given formula 3.1, the independence is obvious. It is enough, given the additivity of l, to check 3.1 when $M^{\cdot}$ is of Type I_a, I_b or II_i, and for these it is a consequence of lemma 2, once we show that for $M^{\cdot}$ of Type I_b in degree 0,

$$\sum_{\lambda\in[0,1)} (1-\lambda)m_\lambda^0(M^{\cdot}) = \text{length}_W(M/VM) \quad M=M^0$$

$$\sum_{\lambda\in[0,1)} \lambda m_\lambda^0(M^{\cdot}) = \text{length}_W(M/FM) \quad M=M^0.$$

In fact, the right hand sides of the above equations are isogeny invariants, as are the left hand sides; it is therefore enough to check them for M of "standard type," i.e. of the form $D/(F^r - V^s)$, where D is the Dieudonne ring. For these latter modules the above equalities are clear.

Turning again to X smooth and proper over k, we can now define

$$m^{i,j}(X) = \sum_{\lambda\in[i-1,i)} (\lambda - i + 1)\dim_K H_{\text{cris}}^{i-j}(X/W)_\lambda + \sum_{\lambda\in[i,i+1)} (i+1-\lambda)\dim_K H_{\text{cris}}^{i+j}(X/W)_\lambda \tag{3.2}$$

where $H_{\text{cris}}^{i+j}(X/W)_\lambda$ is the part of $H_{\text{cris}}^{i+j}(X/W)\otimes\mathbb{Q}$ with slope λ, and

$$T^{i,j}(X) = T^i(H^j(X, W\Omega_X^{\cdot})) \tag{3.3}$$

The $m^{i,j}(X)$ could be called the Hodge-Newton numbers of X, for it is easily checked that they have the following interpretation: for each n, we form the polygon HN(n) whose break-points are at the points (0, 0) and

$$\left(\sum_{0\leqslant \ell\leqslant i} m^{\ell,n-\ell}, \sum_{0\leqslant \ell\leqslant i} \ell m^{\ell,n-\ell}\right)$$

for $0\leqslant i\leqslant n$. Then HN(n) is the uppermost convex polygon with integer slopes and integral breakpoints lying below the Newton polygon of $H_{\text{cris}}^n(X/W)$. To interpret the $m^{i,j}(X)$ in terms of the slopes spectral sequence 0.1 of X, we recall that $H^j(X, W\Omega_X^i)\otimes\mathbb{Q}$ is canonically isomorphic to the part of $H_{\text{cris}}^{i+j}(X/W)\otimes\mathbb{Q}$ where the geometric Frobenius acts with slopes in the interval $[i, i+1)$ (cf [3] II3.2), and that, via the isomorphism, the corresponding action on $H^j(X, W\Omega_X^i)\otimes\mathbb{Q}$ is p^iF. This means that the formula for $m^{i,j}(X)$ can be rewritten

$$m^{i,m}(X) = \sum_{\lambda\in[0,1)} \lambda\dim_K H^{j-1}(W\Omega^{i-1})_\lambda\otimes K + \sum_{\lambda\in[0,1)} (1-\lambda)\dim_K H^j(W\Omega^i)_\lambda\otimes K \tag{3.4}$$

Now since $R\Gamma(X, W\Omega^{\cdot}_X)$ is an element of $D^b(R^{\cdot})$ whose homology is coherent we may apply ℓ: the result, by 3.1, 3.3, and 3.4, is

$$\ell(R\Gamma(X, W\Omega^{\cdot}_X))(i) = \sum_j (-1)^j \ell(H^j(X, W\Omega^{\cdot}_X))(i)$$
$$= \sum_j (-1)^j m^{i,j} + \sum_j (-1)^j T^{i,j}$$
$$+ 2\sum_j (-1) T^{i-1,j} + \sum_j (-1)^j T^{i-2,j}$$

Let us write simply

$$m^i(X) = \sum_j m^{i,j}(X)(-1)^j$$

$$T^i(X) = \sum_j (-1)^j T^{i,j}(X)$$

Then we have

4. THEOREM: *If X/k is a proper smooth variety over a perfect field, then for all i,*

$$m^i(X) + T^i(X) + 2T^{i-1}(X) + T^{i-2}(X) + \chi(\Omega^i_X)$$

PROOF: By II. Theorem 1.2 of [4] we have

$$R_1 \overset{L}{\otimes} W\Omega^{\cdot}_X \simeq R_1 \otimes W\Omega^{\cdot}_X \simeq \Omega^{\cdot}_X$$

whence

$$R_1 \overset{L}{\otimes} R\Gamma(W\Omega^{\cdot}_X) \simeq R\Gamma(\Omega^{\cdot}_X)$$

This gives a spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p(R_1, H^q(W\Omega^{\cdot}_X)) \Rightarrow H^{q-p}(\Omega^{\cdot})$$

which implies

$$\sum_{p,q} (-1)^{p+q} \mathrm{length}_W \mathrm{Tor}_p(R_1, H^q(W\Omega^{\cdot}_X))$$
$$= \sum_q (-1)^q \mathrm{length}_W H^q(\Omega^{\cdot}_X) \tag{4.2}$$

The left hand side is just

$$\sum_q (-1)^q \ell(H^q(W\Omega^{\cdot}_X)),$$

so that 4.1 follows from 4.2 and 3.5.

REMARK: If we introduce, following Ekedahl [2], the "Hodge-Witt" numbers

$$h_W^{p,q} \overset{\text{def}}{=} m^{p,q} + T^{p,q} - 2T^{p-1,q+1} + T^{p-2,q+2}$$

then the formula 4.1 takes the form

$$\sum_q (-1)^q h_W^{p,q} = \sum_q (-1)^q h^{p,q}$$

used by Ekedahl.

In order to illustrate 4.1 we shall consider the case of a *surface X/k;* then in 4.1 the only relation of interest is the one given by $i = 0$, the other being linearly dependent on this one. After some rearrangement 4.1 gives, for $i = 0$ and dim $X = 2$,

$$m^{0,2} + T^{0,2} + \delta = h^{0,2} \tag{4.3}$$

where δ is the "defect of smoothness"

$$\begin{aligned} \delta &= h^{0,1} - m^{0,1} \\ &= \dim \operatorname{Pic} X/\operatorname{Pic}^{\mathrm{red}} X \end{aligned} \tag{4.4}$$

To get the second line of 4.4, we recall that $H^1(W\mathcal{O}_X)$ is the covariant Dieudonne module of $\operatorname{Pic}^{\mathrm{red}} X$, so that $m^{0,1} = \dim_k H^1(W\mathcal{O})/VH^1(W\mathcal{O})$ $= \dim \operatorname{Pic}^{\mathrm{red}} X$. We should also recall that for a surface X, $T^{0,2}$ is the only one of the $T^{i,j}$ that can possibly be nonzero (since the only differential in 0.1 that can be nonzero is $d_1^{0,2}$). Since degeneration of the slope spectral sequence at E_1 is equivalent to the vanishing of *all* the $T^{i,j}$, we obtain

5. COROLLARY: *If X/k is a surface, then the slope spectral sequence*

$$E_1^{pq} = H^q(X, W\Omega_X^p) =)H^{p+q}_{cris}(X/W)$$

degenerates at E_1 if and only if

$$m^{0,2} + \delta = h^{0,2}$$

6. Corollary: *If X/k is a surface and $m^{0,2} + \delta = h^{0,2}$, then all 1-forms on X are closed.*

Proof: The hypothesis implies that the slope spectral sequence degenerates at E_1, and it is known (e.g. [3]) that this implies that the 1-forms are closed.

The next theorem and its corollary were inspired by a result of Nygaard ([5], 3.1 and 3.2):

7. Theorem: *If S is a k-scheme and X/S is proper and smooth of relative dimension 2, then the function on geometric points $\bar{s} \to S$ of S*

$$\bar{s} \to T^{0,2}(X_{\bar{s}})$$

is upper semicontinuous on S.

Proof: For $i = 0$, 4.1 reads

$$m^{0,2}(X_{\bar{s}}) - m^{0,1}(X_{\bar{s}}) + T^{0,2}(X_{\bar{s}}) = h^{0,2}(X_{\bar{s}}) - h^{0,1}(X_s)$$

Now it is well known that $h^{0,2}(X_{\bar{s}}) - h^{0,1}(X_{\bar{s}}) = p_a(X_{\bar{s}})$ and $m^{0,1}(X_{\bar{s}}) = \frac{1}{2} b_1(X_{\bar{s}})$ are constant on S. In order, then, to show that $T^{0,2}(X_{\bar{s}})$ is upper semicontinuous in $\bar{s}$, it is enough to show that $m^{0,2}(X_{\bar{s}})$ is lower semicontinuous in $\bar{s}$, it is enough to show that $m^{0,2}(X_{\bar{s}})$ is lower semicontinuous. Now $m^{0,2}$ is just the length of the slope zero segment of the polygon HN(2) associated to $H^2_{\text{cris}}(X_{\bar{s}})$. By [1], Theorem 2.6, we known that the Newton polygon of $H^2_{\text{cris}}(X_{\bar{s}})$ and hence the polygon HN(2), rises under specialization; and this implies that $m^{0,2}$ is lower semicontinuous.

8. Corollary: *With X/S as before, suppose in addition that S is connected. If there is a geometric point $\bar{s} \to S$ such that $m^{0,2}(X_{\bar{s}}) + \delta(X_{\bar{s}}) = h^{0,2}(X_{\bar{s}})$, then the differential d: $f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S}$ is zero.*

Proof: Since $f^* \Omega^2_{X/S}$ is locally free, the subset of S on which $d = 0$ is closed. To show that it contains the generic point of S, we need only remark that $T^0(X_{\bar{s}}) = 0$ by 4.5, and that the condition $T^0 = 0$ is open, by Theorem 7. The result follows then from Corollary 6.

We conclude with a discussion of algebraic surfaces X satisfying $q = -p_a$, where $q = \dim \operatorname{Alb} X$.

9. Proposition: ([6], *Prop.* 4) *Let X/k be a smooth, complete surface. Then $q(X) = -p_a(X)$ if and only if $H^2(W\mathcal{O}_X)$ is V-torsion.*

Proof: Since $H^2(W\Omega^{\cdot}_X)$ is coherent, one has that $H^2(W\mathcal{O})$ is V-torsion if and only if $T^{0,2} = m^{0,2} = 0$. By 4.3 and 4.4, this last condition is equivalent to $q(X) = -p_a(X)$, *since* $q(X) = m^{0,1}$.

In particular, $H^2(W\mathcal{O}_X)$ *is of finite length for such a surface.*

10. COROLLARY: *If* X *is a smooth proper surface with* $q(X) = -p_a(X)$, *then* X *is Hodge-Witt and* $H^2_{cris}(X/W)$ *is purely of slope* 1. *In particular, all global* 1-*forms on* X *are closed.*

PROOF: This follows from 9,5, and 6.

From Proposition 9 and its corollary, one may go on to compute the Hodge numbers of X in terms of the structure of the group Pic $X/\mathrm{Pic}^{\mathrm{red}}\, X$ and dim Alb (X). We refer the reader to [6] for the result and the details.

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