MAREK LASSAK

Relative extreme subsets

Compositio Mathematica, tome 56, n° 2 (1985), p. 233-236

<http://www.numdam.org/item?id=CM_1985__56_2_233_0>
RELATIVE EXTREME SUBSETS

Marek Lassak

Generalizing the notion of extreme point of a set in the real linear space $L$, Klee [2] introduced the following definition of relative extreme point. Let $B \subseteq L$ and $C \subseteq L$. If a point of $B$ does not belong to any open segment $(b, c) = \{(1 - \lambda)b + \lambda c; 0 < \lambda < 1\}$ determined by distinct points $b \in B$ and $c \in C$, then it is called an extreme point in $B$ relative to $C$.

Observe that the known notion of extreme subset can be generalized analogously:

**Definition:** Let $A \subseteq B \subseteq L$ and $C \subseteq L$. We say that $A$ is an extreme subset of $B$ relative to $C$ if, together with any point $a \in A$, the set $A$ contains every point $b \in B$ such that $a \in (b, c)$ for some $c \in C$.

Let us note that the definition can be expressed more geometrically using the notion of the penumbra ([5], p. 22) of $A$ with respect to $C$. Namely, a subset $A$ of $B \subseteq L$ is an extreme subset of $B$ relative to a non-empty set $C \subseteq L$ if and only if

$$P_C(A) \cap B = A.$$

Obviously in the case $A = \{a\}$ of our definition we get the notion of extreme point $a$ in $B$ relative to $C$ and in the case $B = C$ we obtain the usual notion of extreme subset $A$ of $B$. On the other hand, the above definition is a special case of the notion (presented as Remark in [3]) of $\Phi$-extreme subset, where $\Phi : \mathcal{D} \to 2^L$ is a function such that $\mathcal{D}$ consists of all one-point subsets of $L$ and $\Phi(\{b\}) = \bigcup_{c \in C}(b, c)$. Let us observe also a connection of our definition with the notion of semi-extreme subset.

Remember that a subset $A$ of a convex set $B \subseteq L$ is called a semi-extreme subset of $B$ if $B \setminus A$ is convex (comp. [1], p. 32). As in [6], pp. 186–187, this notion of semi-extreme subset can be extended to arbitrary (i.e. not necessary convex) set $B$: if $A \subseteq B$ and $A \cap \text{conv}(B \setminus A) = \emptyset$, then we call $A$ a semi-extreme subset of $B$. The above mentioned connection is expressed by the following easily provable:
PROPOSITION: If \( A \) is a semi-extreme subset of \( B \), then \( A \) is an extreme subset of \( B \) relative to \( B \setminus A \). When \( B \) is convex, the inverse implication also holds.

The reader can without difficulty verify six properties of relative extreme subsets presented in Theorem 1, the first five of which generalize well-known properties of extreme subsets in the usual sense.

**THEOREM 1:** Relative extreme subsets have the following properties

(a) Any intersection of extreme subsets of \( B \) relative to \( C \) is an extreme subset of \( B \) relative to \( C \).

(b) Any union of extreme subsets of \( B \) relative to \( C \) is an extreme subset of \( B \) relative to \( C \).

(c) If \( A \) is an extreme subset of \( B \) relative to \( C \) and if \( A_1 \) is an extreme subset of \( A \) relative to \( C \), then \( A_1 \) is an extreme subset of \( B \) relative to \( C \).

(d) If \( A \subset B_1 \subset B_2 \) and if \( A \) is an extreme subset of \( B_2 \) relative to \( C \), then \( A \) is an extreme subset of \( B_1 \) relative to \( C \).

(e) Sets \( B \) and \( \emptyset \) are extreme subsets of \( B \) relative to any set \( C \).

(f) If \( C_1 \subset C_2 \) and if \( A \) is an extreme subset of \( B \) relative to \( C_2 \), then \( A \) is an extreme subset of \( B \) relative to \( C_1 \). Any subset of \( B \) is extreme in \( B \) relative to empty set.

The notion of the usual extreme subset of a set \( B \) is considered mainly in the case when \( B \) is convex. Also the notion of extreme point of \( B \) relative to \( C \) plays an important part in the case when \( B \) is convex and \( C \subset B \) (comp. [2] and [4]). This is why we now consider extreme subsets of a convex set \( B \) relative to a subset of \( B \).

**THEOREM 2:** Let \( B \) be a convex set of a real linear space \( L \) and let \( A \subset B \), \( C \subset B \). The set \( A \) is an extreme subset of \( B \) relative to \( C \) if and only if \( A \) is an extreme subset of \( B \) relative to the convex hull \( \text{conv } C \).

**PROOF:** Suppose that \( A \) is an extreme subset of \( B \) relative to \( C \). To verify if \( A \) is an extreme subset of \( B \) relative to \( \text{conv } C \) we shall show that for any \( a \in A, b \in B \) and \( c \in \text{conv } C \) such that \( a \in (b, c) \) we have \( b \in A \).

As an element of \( \text{conv } C \), the point \( c \) belongs to the convex hull of a finite number of points of \( C \). Consequently, there exists a minimal finite collection of points \( c_1, \ldots, c_k \in C \) such that

\[ c \in \text{conv}\{b, c_1, \ldots, c_k\}. \]

In other words

\[ c = \alpha_0 b + \alpha_1 c_1 + \ldots + \alpha_k c_k, \]

where \( \alpha_0 \geq 0, \alpha_1 > 0, \ldots, \alpha_k > 0 \) and \( \alpha_0 + \alpha_1 + \ldots + \alpha_k = 1 \). Since \( a = \beta b + \gamma c \) for some \( \beta > 0 \) and \( \gamma > 0 \) such that \( \beta + \gamma = 1 \), we have

\[
a = (1 - \delta_1 - \ldots - \delta_k) b + \delta_1 c_1 + \ldots + \delta_k c_k,
\]

where \( \delta_1 = \gamma \alpha_1 > 0, \ldots, \delta_k = \gamma \alpha_k > 0 \) and \( 1 - \delta_1 - \ldots - \delta_k = 1 - \gamma(\alpha_1 + \ldots + \alpha_k) = 1 - \gamma(1 - \alpha_0) = \beta + \gamma \alpha_0 > 0 \).

Now, we recurrently define points \( b_k, b_{k-1}, \ldots, b_1 \) as follows

\[
b_k = b,
\]

\[
b_i = \frac{\delta_{i+1}}{1 - \delta_1 - \ldots - \delta_i} c_{i+1} + \frac{1 - \delta_1 - \ldots - \delta_{i+1}}{1 - \delta_1 - \ldots - \delta_i} b_{i+1}, \quad i = k - 1, \ldots, 1.
\]

Since the coefficients

\[
\delta_{i+1}/(1 - \delta_1 - \ldots - \delta_i), (1 - \delta_1 - \ldots - \delta_{i+1})/(1 - \delta_1 - \ldots - \delta_i)
\]

are positive and since the sum of them is equal to 1, the definition of \( b_i \) implies that

\[
b_i \in (c_{i+1}, b_{i+1}), \quad i = 1, \ldots, k - 1.
\]  

(1)

By the definition of \( b_i \), the equality

\[
\delta_{i+1} c_{i+1} + (1 - \delta_1 - \ldots - \delta_{i+1}) b_{i+1} = (1 - \delta_1 - \ldots - \delta_i) b_i
\]

holds for \( i = k - 1, \ldots, 1 \) and consequently

\[
a = \delta_1 c_1 + \ldots + \delta_k c_k + (1 - \delta_1 - \ldots - \delta_k) b_k
\]

\[
= \delta_1 c_1 + \ldots + \delta_{k-1} c_{k-1} + [\delta_k c_k + (1 - \delta_1 - \ldots - \delta_k) b_k]
\]

\[
= \delta_1 c_1 + \ldots + \delta_{k-1} c_{k-1} + (1 - \delta_1 - \ldots - \delta_{k-1}) b_{k-1}
\]

\[
= \ldots = \delta_1 c_1 + (1 - \delta_1) b_1.
\]

Thus in virtue of \( \delta_1 > 0 \) and \( 1 - \delta_1 > 0 \) we have

\[
a \in (c_1, b_1).
\]  

(2)

Since \( B \) is convex, from \( b_k \in B \) and \( c_k, \ldots, c_1 \in B \) and also from \( b_i \in (c_{i+1}, b_{i+1}) \) for \( i = k - 1, \ldots, 1 \) we get in turn that \( b_i \in B \) for \( i = k - 1, \ldots, 1 \).

Since \( A \) is an extreme subset of \( B \) relative to \( C \) and since \( a \in A \), \( b_i \in B \) and \( c_i \in C \) for \( i = 1, \ldots, k \), we first obtain from (2) that \( b_1 \in A \) and next (if \( k \geq 2 \)), applying \((k - 1)\)-times (1) we get in turn that
$b_2 \in A, \ldots, b_k \in A$. Thus $b = b_k \in A$. Hence $A$ is an extreme subset of $B$ relative to $C$.

The inverse implication of our theorem results immediately from the inclusion $C \subset \text{conv } C$ and from property $(f)$ of Theorem 1.

References


(Oblatum 28-X-1983)

Instytut Matematyki i Fizyki ATR
ul. Kaliskiego 7
85-790 Bydgoszcz
Poland