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THE CYLINDER HOMOMORPHISM ASSOCIATED TO QUINTIC FOURFOLDS

James D. Lewis

§0. Introduction

Let X be a quintic fourfold (smooth hypersurface of degree 5 in \mathbb{P}^5), and Ω_X the variety of lines in X . According to [1], if X is generically chosen, then Ω_X is a smooth surface. Let $\Phi_*: H_2(\Omega_X, \mathbb{Q}) \rightarrow H_4(X, \mathbb{Q})$ be the “cylinder homomorphism” obtained by blowing up each point on $\gamma \in H_2(\Omega_X, \mathbb{Q})$ to a corresponding line in X (thus sweeping out a 4 cycle in X). This homomorphism was studied in [4], and in particular, viewing Φ_* on cohomology (viz Poincaré duality):

(0.1) THEOREM: ([4; (4.4)]). *Let X be generic, $\omega \in H^{1,1}(X, \mathbb{Q})$ the Kähler class dual to the hyperplane section of X . Then $\Phi_*: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})/\mathbb{Q}\omega \wedge \omega$ is an epimorphism.*

For relatively elementary reasons (see (5.5)), it is also true that $\Phi_: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})$ is an epimorphism for generic X . This paper is devoted to the answering of the following question:*

(0.2) *What is the kernel of Φ_* ?*

In order to satisfactorily answer (0.2), some terminology has to be introduced. The family of hypersurfaces $\{X_v\}_{v \in \mathbb{P}^N}$ of degree 5 in \mathbb{P}^5 is a projective space of dimension $N = 251$. Let $U \subset \mathbb{P}^N$ be the open set parameterizing the smooth X_v , $U_0 \subset U$ the open subset corresponding to those X for which Ω_X is a smooth, irreducible surface. Let $\Delta \subset U_0$ be a polydisk centered at $0 \in \Delta$, $X = X_0$, and for any $v \in \Delta$, define $j_v: \Omega_{X_v} \hookrightarrow \coprod_{v \in \Delta} \Omega_{X_v}$ to be the inclusion morphism. Now $\coprod_{v \in \Delta} \Omega_{X_v}$ is topologically equivalent to $\Delta \times \Omega_X$ (see [7]) for any given $v \in \Delta$, and therefore there is an isomorphism $j_v^* \circ (j_0^*)^{-1}: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^2(\Omega_{X_v}, \mathbb{Q})$.

(0.3) DEFINITION:

- (i) $H_A^{1,1}(\Omega_X, \mathbb{Q}) = \{\gamma \in H^2(\Omega_X, \mathbb{Q}) \mid j_v^* \circ (j_0^*)^{-1}(\gamma) \in H^{1,1}(\Omega_{X_v}, \mathbb{Q}) \text{ for all } v \in \Delta\}$.
- (ii) $H_P^2(\Omega_X, \mathbb{Q}) =$ orthogonal complement of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ in $H^2(\Omega_X, \mathbb{Q})$.

defined as follows (see (3.1) for a precise definition): (0.5) Let l_x be the line corresponding to $x \in \Omega_X$. Define $D(x) = \{y \in \Omega_X \mid y \neq x \ \& \ l_x \cap l_y \neq \emptyset\}$. It is proven (see (2.5)) that for generic X , $D(x)$ is a finite set for generic $x \in \Omega_X$.

Our theorem is: (X generic)

(0.6) THEOREM:

(i) i preserves the subspaces defined in (0.3)(i)&(ii); moreover i :

$$H_P^2(\Omega_X, \mathbb{Q}) \rightarrow H_P^2(\Omega_X, \mathbb{Q}) \text{ is an isomorphism.}$$

(ii) There is a s.e.s.:

$$0 \rightarrow (i + 119 \cdot I) H_P^2(\Omega_X, \mathbb{Q}) \xrightarrow{i} H_P^2(\Omega_X, \mathbb{Q}) \xrightarrow{\Phi_*} \text{Prim}^4(X, \mathbb{Q}) \rightarrow 0,$$

where i and I are respectively the inclusion and identity morphisms.

(iii) $\Phi_*(H_A^{1,1}(\Omega_X, \mathbb{Q})) = \mathbb{Q} \omega \wedge \omega.$

(0.7) COROLLARY:

$$\begin{array}{ccc} H_P^2(\Omega_X, \mathbb{Q}) & \xrightarrow{\Phi_*} & \text{Prim}^4(X, \mathbb{Q}) \\ i \downarrow & & \downarrow \times 119 \\ H_P^2(\Omega_X, \mathbb{Q}) & \xrightarrow{\Phi_*} & \text{Prim}^4(X, \mathbb{Q}) \end{array}$$

is sign commutative.

Much of the techniques of this paper are borrowed from an interesting paper by Tyurin ([6]).

§1. Notation

- (i) \mathbb{Z} = integers, \mathbb{Q} = rational numbers, \mathbb{C} = complex numbers
- (ii) X is a quintic fourfold, \mathbb{P}^M is complex, projective M -space.
- (iii) If Y is a projective, algebraic manifold, then $H^{p,p}(Y)$ is Dolbeault cohomology of type (p, p) and $H^{p,p}(Y, K) = H^{p,p}(Y) \cap H^{2p}(Y, K)$, where $K = \mathbb{Z}, \mathbb{Q}$.
- (iv) Prim stands for primitive cohomology.
- (v) There are 2 senses to the word “generic” in this paper. We say that X is generic if it is a member of a family $\{X_v\}_{v \in W}$ satisfying a given property, and where $W \subset \mathbb{P}^N$ is a Zariski open subset. The other use of the word “generic” is where X satisfies a given property that is transcendental in nature, and in this case the word generic will be prefixed by “transcendental”.

(vi) Let $Y \subset \mathbb{P}^M$ be given as in (iii) above, $G =$ Grassmannian of lines in \mathbb{P}^M . For $x \in G$, let l_x be the corresponding line in \mathbb{P}^M . The variety of lines in Y , denoted by Ω_Y is defined as follows: $\Omega_Y = \{x \in G \mid l_x \subset Y\}$.

(vii) Given Y as in (iii) and $S \subset Y$ an algebraic subset. Then $\dim S = \max\{\dim \text{ of irreducible components of } S\}$, and $\text{codim}_Y S = \dim Y - \dim S$.

§2. The variety of lines in X

Let $Y \subset \mathbb{P}^n$ be a generic hypersurface of degree d , and assume $2n - d - 5 \geq 0$. An immediate consequence of [1] is:

(2.1) THEOREM: Ω_Y is smooth and irreducible, of dimension $2n - d - 3$.

There are two noteworthy cases to consider:

(2.2) COROLLARY: Given X a generic quintic fourfold, and Z a generic fivefold of degree 5 in \mathbb{P}^6 , then:

(i) Ω_X is a smooth, irreducible surface and

(ii) Ω_Z is a smooth, irreducible fourfold.

An argument identical to one given in [6; p. 38] yields:

(2.3) PROPOSITION: Given Z as in (2.2). Then through a generic point of Z passes $5!$ lines.

Before stating the main result of this section, we introduce the following notation: Let $c \in \Omega_X$, $l_c \subset X$ the corresponding line.

(2.4) $\Omega_{X,c} = \{y \in \Omega_X \mid l_y \cap l_c \neq \emptyset\}$. We prove:

(2.5) THEOREM: Let X be generic.

(i) $\dim \Omega_{X,c} = 0$ for generic $c \in \Omega_X$.

(ii) Let $c \in \Omega_X$ be generic. Then for any $y \in l_c$, there is at most one line $l_0 \subset X$ other than l_c passing through y .

PROOF: We start by letting X be any degree 5 hypersurface in \mathbb{P}^5 , and $x \in X$. If we let $[X_0, X_1, X_2, X_3, X_4, X_5]$ be the homogeneous coordinates defining \mathbb{P}^5 , then X admits as its defining equation $F = 0$, $F \in \mathbb{C}[X_0, \dots, X_5]$ a homogeneous polynomial of degree 5. Now after applying a projective transformation, there is no loss of generality in assuming $x = [0, 0, 0, 0, 0, 1]$. In this case F takes the form: $F = X_5^4 F_1 + X_5^3 F_2 + X_5^2 F_3 + X_5 F_4 + F_5$, where $F_i \in \mathbb{C}[X_0, \dots, X_4]$ is homogeneous of degree i . We now convert to affine coordinates by setting $x_i = X_i/X_5$, $i = 0, \dots, 4$. Define $f_i = F_i/X_5^i$ and note that $f_i \in \mathbb{C}[x_0, \dots, x_4]$ is homogeneous of degree i . Likewise, set $f = F/X_5^5$, and note that $f = f_1 + f_2 + f_3 + f_4 + f_5$. In affine coordinates $x = (0, 0, 0, 0, 0)$, therefore any line l_a passing through x must be of the form $l_a = \{ta \mid t \in \mathbb{C}\}$, where $a \in \mathbb{C}^5$ is non-zero.

It follows that

$$\begin{aligned}
 l_a \subset X &\Leftrightarrow f_1(ta) + \dots + f_5(ta) = 0 \quad \text{for all } t \\
 \text{i.e.} &\Leftrightarrow tf_1(a) + \dots + t^5 f_5(a) = 0 \quad \text{for all } t \\
 &\Leftrightarrow f_1(a) = \dots = f_5(a) = 0.
 \end{aligned}$$

The upshot of this argument is that the lines in X passing through x correspond to the zeros of f_1, \dots, f_5 in \mathbb{P}^4 . Note that for generic $x \in X$, no such line exists. Let $V(i)$ be the vector space of homogeneous polynomials of degree i in $\mathbb{C}[x_0, \dots, x_4]$, and set $V = V(1) \oplus \dots \oplus V(5)$. It is clear from our construction that X determines a point $v \in \mathbb{P}(V)$, conversely, any $v \in \mathbb{P}(V)$ determines X so that $x \in X$.

(2.6) Every $v \in \mathbb{P}(V)$ determines an algebraic set $S(v)$ defined as the zeros of f_1, \dots, f_5 in \mathbb{P}^4 . Define $V_1 = \{v \in \mathbb{P}(V) \mid \dim S(v) \geq 0\}$. If $v \in \mathbb{P}(V)$ is given so that $\dim S(v) = 0$, then define $\#S(v)$ to be the cardinality of $S(v)$ as a set. For $i = 2, 3$ define $V_i = \{v \in V_1 \mid \dim S(v) \geq 1 \text{ or } \#S(v) \geq i\}$, and set $V_B = \{v \in V_1 \mid \dim S(v) \geq 1\}$. We need the following:

(2.7) LEMMA: $\text{codim}_{\mathbb{P}(V)} V_i = i$, for $i = 1, 2, 3$ & $\text{codim}_{\mathbb{P}(V)} V_B \geq 5$.

PROOF: Let $V' = V(j) \oplus \dots \oplus V(5) \subset V$, for $j = 1, \dots, 5$, and $\mathbb{P}(V') \subset \mathbb{P}(V)$ the corresponding projective subspaces. Note that for $v \in \mathbb{P}(V')$, $S(v) = \text{zeros of } \{f_j, \dots, f_5\} \text{ in } \mathbb{P}^4$. We will prove (2.7) case-by-case:

(a) $\text{codim}_{\mathbb{P}(V)} V_1 = 1$: It follows from general principles ([5; (3.30)]) that $v \in \mathbb{P}(V^2) \Rightarrow S(v) \neq \emptyset$, so for such v , choose any $y \in S(v)$. Clearly $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y) = 0\}$ cuts out a codimension 1 subspace of $\mathbb{P}(V(1))$, hence $\text{codim}_{\mathbb{P}(V)} V_1 = 1$.

(b) $\text{codim}_{\mathbb{P}(V)} V_2 = 2$: Let $v \in V^2$ be given so that $\dim S(v) = 0$ and $\#S(v) \geq 2$. Let $y_1, y_2 \in S(v)$ with $y_1 \neq y_2$. Then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y_1) = f_1(y_2) = 0\}$ cuts out a subspace of codimension 2 in $\mathbb{P}(V(1))$. Statement (b) now follows from:

(2.8) SUBLEMMA: $\{v \in \mathbb{P}(V^2) \mid \dim S(v) \geq 1\}$ has codimension ≥ 3 in $\mathbb{P}(V^2)$.

PROOF: If $v \in \mathbb{P}(V^3)$, then $\dim S(v) \geq 1$ and equal to 1 for generic v . Define $H = \{(y, v) \in \mathbb{P}^4 \times \mathbb{P}(V^3) \mid y \in S(v)\}$, and let q_1, q_2 be the canonical projections in the diagram below:



Note that the fibers of q_1 are projective spaces, all of which are projectively equivalent to each other; moreover q_1 (and q_2) are surjective, hence H is irreducible. In addition $q_2^{-1}(v) = S(v)$, and by our construction, the generic fiber of q_2 is a smooth, irreducible curve of degree 60 (Bezout's theorem). Let $K = \{v \in \mathbb{P}(V^3) \mid \dim S(v) \geq 2\}$. Then by considering the morphism q_2 , it follows that $\text{codim}_{\mathbb{P}(V^3)} K \geq 2$, (in fact $\text{codim}_{\mathbb{P}(V^3)} K \geq 3$). If $v \in \mathbb{P}(V^3)$ is given so that $\dim S(v) = 1$, then elementary reasoning implies $\{f_2 \in \mathbb{P}(V(2)) \mid f_2 \text{ vanishes on a component of } S(v) \text{ of dimension } 1\}$ is of codimension ≥ 3 in $\mathbb{P}(V(2))$. On the other hand if $v \in \mathbb{P}(V^3)$ is given so that $\dim S(v) \geq 2$, then one constructs a diagram analogous to (2.9), replacing $\mathbb{P}(V^3)$ by $\mathbb{P}(V^4)$, modifying H accordingly, and applying a similar reasoning as above to conclude $\text{codim}_{\mathbb{P}(V^3)} K \geq 3$, hence (2.8).

(c) $\text{codim}_{\mathbb{P}(V)} V_3 = 3$: If $v \in V^2$ is generically chosen, then $\#S(v) = 5!$ (bezout's theorem), moreover no 3 points in $S(v)$ are collinear. If $y_1, y_2, y_3 \in S(v)$ are distinct, then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y_1) = f_1(y_2) = f_1(y_3) = 0\}$ is a subspace of codimension 3 in $\mathbb{P}(V(1))$. The case that $v \in V^2$ is given so that $\dim S(v) \geq 1$ is taken care of by (2.8). There remains the possibility that $v \in V^2$ is given so that $\dim S(v) = 0$ and that some collinearity (of 3 points) exists. For this to happen, v would have to belong to a proper subvariety of V^2 , and one can easily argue that statement (c) still holds.

(d) $\text{codim}_{\mathbb{P}(V)} V_B \geq 5$: A construction similar to the proof of (2.8) implies $\{v \in \mathbb{P}(V^2) \mid \dim S(v) \geq 2\}$ is of codimension ≥ 5 in $\mathbb{P}(V^2)$. Now suppose $v \in \mathbb{P}(V^2)$ is given so that $\dim S(v) = 1$. Then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1 \text{ vanishes on a dimension } 1 \text{ component of } S(v)\}$ is of codimension ≥ 2 in $\mathbb{P}(V(1))$. We now apply (2.8) to conclude statement (d), and the proof of (2.7).

(2.10) Conclusion of the proof of (2.5)

Recall at the beginning of the proof a choice of $x \in \mathbb{P}^5$ which determines $\mathbb{P}(V)$, V_1, V_2, V_3, V_B , where $\mathbb{P}(V)$ corresponds to those $X \subset \mathbb{P}^5$ for which $x \in X$. To indicate that our choice of x determines $\mathbb{P}(V)$, we will relabel things with the obvious meaning as $\mathbb{P}(V_x), V_{1,x}, V_{2,x}, V_{3,x}, V_{B,x}$. Now define $W = \coprod_{x \in \mathbb{P}^5} \mathbb{P}(V_x)$, $W_i = \coprod_{x \in \mathbb{P}^5} V_{i,x}$ for $i = 1, 2, 3$, $W_B = \coprod_{x \in \mathbb{P}^5} V_{B,x}$. It is easy to verify that W, W_i 's, W_B all have the structure of an algebraic variety, moreover by (2.7):

(2.11) $\text{codim}_W W_i = i$ for $i = 1, 2, 3$ and $\text{codim}_W W_B \geq 5$.

Recall the statement just preceding (2.6), that for any X and $x \in X$, X determines a point $v_x \in \mathbb{P}(V_x)$. Therefore X determines a fourfold $X_W \subset W$ given by the formula $X_W = \coprod_{x \in X} v_x$. For generic $X \subset \mathbb{P}^5$, $\dim\{X_W \cap W_i\} = 4 - i$, and $X_W \cap W_B = \emptyset$. Translating this in terms of Ω_X , (2.5) clearly holds.

Q.E.D.

§3. The incidence and cylinder homomorphisms

Let $D_1 \subset \Omega_X \times \Omega_X$ be given by the formula: $D_1 = \{(x_1, x_2) \in \Omega_X \times \Omega_X \mid l_{x_1} \cap l_{x_2} \neq \emptyset \ \& \ x_1 \neq x_2\}$. It is clear from the definition that $\{x, D_1(x)\} = \Omega_{X,x}$. Throughout this section X will be assumed to be generic.

(3.1) DEFINITION: The incidence correspondance $D \subset \Omega_X \times \Omega_X$ is defined to be: $D = \bar{D}_1$.

Note that $\text{codim}_{\Omega_X \times \Omega_X} D = 2$, therefore the fundamental class of D defines a cocycle $[D] \in H^4(\Omega_X \times \Omega_X, \mathbb{Q})$. Now the component of $[D]$ in $H^2(\Omega_X, \mathbb{Q}) \otimes H^2(\Omega_X, \mathbb{Q})$, via the Künneth formula $H^4(\Omega_X \times \Omega_X, \mathbb{Q}) = \bigoplus_{p+q=4} H^p(\Omega_X, \mathbb{Q}) \otimes H^q(\Omega_X, \mathbb{Q})$, induces a morphism $i: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^2(\Omega_X, \mathbb{Q})$, where we use the fact $H^2(\Omega_X, \mathbb{Q})^* \cong H^2(\Omega_X, \mathbb{Q})$ (Poincaré duality).

(3.2) DEFINITION: The homomorphism $i: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^2(\Omega_X, \mathbb{Q})$ is called the incidence homomorphism.

The morphism i factors into a composite of 3 other morphisms given as follows:

(3.3) Let

- (i) $p: D \rightarrow \Omega_X$ be the projection onto the first factor,
- (ii) $j: \Omega_X \times \Omega_X \rightarrow \Omega_X \times \Omega_X$ the morphism which permutes the factors, i.e. $j(x_1, x_2) = (x_2, x_1)$. Note that $j(D) = D$.

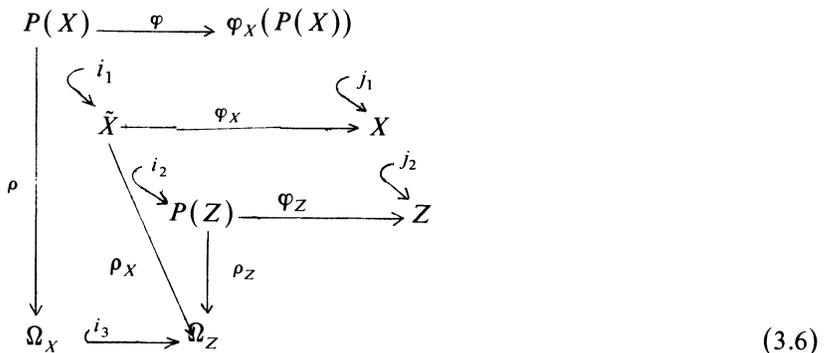
Then:

(3.4) PROPOSITION: $i = p_* \circ j \circ p^*$.

PROOF: Use the fact that the correspondence defined by $p_* \circ j \circ p^*$ in $\Omega_X \times \Omega_X$ is precisely D .

(3.5) The cylinder homomorphism

We will be constantly referring to the following diagram:



(3.6)

where, Z is a smooth degree 5 hypersurface in \mathbb{P}^6 , for which $X \subset Z$ is a (smooth) hyperplane section

$$P(X) = \{(c, x) \in \Omega_X \times X \mid x \in l_c\}$$

$$P(Z) = \{(c, z) \in \Omega_Z \times Z \mid z \in l_c\}$$

ρ (resp. ρ_Z) is the projection of $P(X)$ (resp. $P(Z)$) onto the first factor

φ (resp. φ_Z) is the projection of $P(X)$ (resp. $P(Z)$) onto the second factor

$$\tilde{X} = \varphi_Z^{-1}(X), \varphi_X = \varphi_Z|_{\tilde{X}}: \tilde{X} \rightarrow X, \rho_X = \rho_Z|_{\tilde{X}}: \tilde{X} \rightarrow \Omega_Z$$

i_1, i_2, i_3, j_1, j_2 are inclusion morphisms.

The same reasoning given in [2; p. 81] implies the following:

(3.7) PROPOSITION (see [4]):

- (i) $P(X), P(Z)$ are \mathbb{P}^1 bundles over Ω_X and Ω_Z respectively.
- (ii) $P(X), P(Z), \tilde{X}, \Omega_X, \Omega_Z$ are smooth and irreducible.
- (iii) All morphisms in (3.6), except for inclusions, are surjective.
- (iv) $\deg \varphi_Z = \deg \varphi_X = 5!$.
- (v) ρ_X is birational and induces: $\tilde{X} \cong$ blow up of Ω_Z along Ω_X .

(3.8) REMARKS:

- (i) (2.2) implies the smoothness and irreducibility for Ω_X and Ω_Z .
- (ii) (3.7) (iv) is a consequence of (2.3).

As will be discussed in §4, the threefold $\varphi(P(X))$ has a 2-dimensional singular set. Let S be a generic hyperplane section of $\varphi(P(X))$. One should expect S to be singular. The next result is a direct consequence of (2.5), together with the definitions of $P(X), \rho, \varphi$:

(3.9) PROPOSITION: φ is a birational morphism, moreover φ induces a birational map $\Omega_X \approx S$.

(3.10) DEFINITION: The cylinder homomorphism $\Phi_*: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})$ is given by: $\Phi_* = j_{1,*} \circ \varphi_* \circ \rho^*$.

Let $I: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^2(\Omega_X, \mathbb{Q})$ be the identity morphism, $\omega \in H^{1,1}(X, \mathbb{Z})$ the Kähler class defined in (0.1). The next result ties in a relationship between i and Φ_* .

(3.11) PROPOSITION: $\Phi_*((i + 119 \cdot I)H^2(\Omega_X, \mathbb{Q})) = 0$ in $H^4(X, \mathbb{Q}) / \mathbb{Q}\omega \wedge \omega$

PROOF: The proof of (3.11) is formally identical to the proof of lemma 6 in [6; p. 42] where

- (a) Z and $\deg \varphi_Z = 5!$ replace X_4 and $\deg \varphi$ in [6].
- (b) the cycles are even dimensional.
- (c) the weak Lefschetz theorem applied to the inclusions $Z \subset \mathbb{P}^6$ & $j_2: X \hookrightarrow Z$ implies $j_2^*(H^4(Z, \mathbb{Q})) = \mathbb{Q} \omega \wedge \omega$.
- (d) $119 = 5! - 1$.

§4. The numerical characteristic of the surface Ω_X

Let $\psi_1: D_1 \rightarrow X$ be the morphism defined by the formula: $\psi_1(x_1, x_2) = l_{x_1} \cap l_{x_2} \in X$. Then ψ_1 extends to a rational map $\psi_0: D \rightarrow X$, moreover $\deg \psi_0 = 2$ by (2.5)(ii). Let $\Gamma = D/\{j\}$ with quotient morphism $\psi: D \rightarrow \Gamma$. There is a factorization of ψ_0 :

$$\begin{array}{ccc}
 D & \xrightarrow{\psi_0} & X \\
 \psi \searrow & & \nearrow k \\
 & \Gamma &
 \end{array}
 \tag{4.1}$$

where k is a birational map onto its image, $\psi_0(D)$. This factorization will be useful in the next section where we consider an analogue to the fundamental computational lemma in [6; p. 45]. Note that the fibers of φ in (3.6) are discrete over every point in $\varphi(P(X))$, moreover $\#\varphi^{-1}(x) \geq 2$ over $\overline{\psi_1(D_1)}$ and $\#\varphi^{-1}(x) = 1$ over $\varphi(P(X)) - \overline{\psi_1(D_1)}$, where $\#$ includes multiplicity. By applying Zariski's Main theorem to φ , it is clear that $\text{sing}(\varphi(P(X))) = \overline{\psi_1(D_1)}$. On the other hand, $\overline{\psi_1(D_1)} = \psi_0(D)$, therefore, taking into account the result (2.5)(ii), we can summarize the above discussion in:

(4.2) PROPOSITION: $\text{sing}(\varphi(P(X))) = \psi_0(D)$, moreover through a generic point of $\text{sing}(\varphi(P(X)))$ passes exactly 2 lines in X .

So far we have only focused on the number of lines passing through a given point in $\varphi(P(X))$. We now turn our attention to the problem of determining the number of lines meeting a generic line in X . This number will be denoted by N_0 , and bears the title of this section, namely, recall the definition of p in (3.3)(i):

(4.3) DEFINITION: The numerical characteristic N_0 of Ω_X is given by: $N_0 = \deg p$.

(4.4) REMARK: This definition is borrowed in part from [6; p. 40].

There is another ingredient we want to introduce, but before doing so, we recall from the Lefschetz theorem applied to $X \subset \mathbb{P}^5$ that $H^2(X, \mathbb{Z}) = \mathbb{Z} \omega$. Let $[\varphi(P(X))]$ be the fundamental class of $\varphi(P(X))$ in $H^2(X, \mathbb{Z})$. Then there is a positive integer d for which $[\varphi(P(X))] = d\omega$. Geometri-

cally, d is the degree of the hypersurface in \mathbb{P}^5 cutting out $\varphi(P(X))$ in X . A partial generalization of Fano's work (see [6; p. 40]) implies d and N_0 are related by the simple:

(4.5) PROPOSITION: $d - N_0 \leq -2$.

PROOF: The proof is essentially borrowed from lemma 5 in [6; p. 40], but there are important differences accounting for the changes in statements between (4.5) and [6]. Let $l = \mathbb{P}^1, \mathbb{P}^3, X$ be generically chosen in \mathbb{P}^5 , so that $l \subset X \cap \mathbb{P}^3$, and that $S_0 = \mathbb{P}^3 \cap X$ is a smooth quintic surface. The adjunction formula for $S_0 \subset \mathbb{P}^3$ implies $\Omega_{S_0}^2 = \mathcal{O}_{S_0}(1)$, where $\Omega_{S_0}^2$ is the canonical sheaf of S_0 . Note that l is the only line in S_0 , since a generic hyperplane section of X contains only a finite number of lines ([1]), and S_0 is cut out by a generic \mathbb{P}^3 . If H is a generic hyperplane in \mathbb{P}^5 containing l , then $H \cap S_0 = l + C_0$, where C_0 is a smooth and irreducible curve. Note from the above expression for $\Omega_{S_0}^2$ that $\Omega_{S_0}^2 = \mathcal{O}_{S_0}(H \cap S_0) = \mathcal{O}_{S_0}(l + C_0)$. Now taking intersections: $1 = (l \cdot H)_{\mathbb{P}^5} = (l \cdot (H \cdot S_0))_{S_0} = (l \cdot (l + C_0))_{S_0}$, (where $\cdot = \cap$), consequently $(l \cdot C_0)_{S_0} = 1 - l^2$. On the other hand, the adjunction formula applied to $l \subset S_0$ implies: $-2 = (l \cdot (l + (H \cdot S_0)))_{S_0} = l^2 + 1$, hence $l^2 = -3$, afortiori $(l \cdot C_0)_{S_0} = 4$. Next $S_0 \cap \varphi(P(X)) = l + C_1 \sim d(H \cdot S_0) = dl + dC_0$, hence $C_1 \sim (d-1)l + dC_0$, therefore $(C_1 \cdot l)_{S_0} = (d-1)l^2 + d(l \cdot C_0)_{S_0} = d+3$. Now $\varphi^{-1}(\varphi(P(X)) \cap S_0) = l + \varphi^{-1}(C_1)$ where $\varphi^{-1}(C_1)$ is no longer regarded as a global section of the fibering $p: P(X) \rightarrow \Omega_X$ as in [6], but rather as a section of ρ over a curve in Ω_X , where we use the aforementioned fact that l is the only line in S . Then among the points of intersection in $C_1 \cdot l$ is a possible point of intersection of l with $\varphi^{-1}(C_1)$, and the remaining points are the intersections of l with at most the other lines in X meeting l . Therefore $(C_1 \cdot l)_{S_0} \leq N_0 + 1$, afortiori $d+3 \leq N_0 + 1$, which proves (4.5).

Let H_1 be the hypersurface of degree d which cuts out $\varphi(P(X)) \subset X$, and let $l \subset X$ be any line. Since $l \subset X$, we have $(H_1 \cdot l)_{\mathbb{P}^5} = ((H_1 \cdot X) \cdot l)_X$. Furthermore $d = (H_1 \cdot l)_{\mathbb{P}^5}$, moreover $H_1 \cap X = \varphi(P(X))$. In summary:

(4.6) PROPOSITION: $d = (\varphi(P(X)) \cdot l)_X$.

This concludes §4.

§5. The fundamental computational lemma (F.C.L.)

In this section we will arrive at a version of the F.C.L. in [6] for $\Phi_*: H_2(\Omega_X, \mathbb{Q}) \rightarrow H_4(X, \mathbb{Q})$ where Φ_* is studied on the homology level via Poincaré duality. As in §4, X will be a generic quintic. Now recalling the

diagram in (4.1) together with (4.2), there is a diagram:

$$\begin{array}{ccc}
 & \Omega_X \times \Omega_X & \\
 & \cup & \\
 & D & \\
 p \swarrow & & \searrow \psi \\
 \Omega_X & & \Gamma \xrightarrow{k} \text{sing}(\varphi(P(X))) \\
 & & \cong
 \end{array} \tag{5.1}$$

Define $\Gamma_0 = \{y \in \Gamma \mid k \text{ is regular at } y \text{ \& } k(y) \notin \text{sing}(\text{sing } \varphi(P(X)))\}$. Clearly Γ_0 is smooth and Zariski open in Γ . Next define $D_0 = \psi^{-1}(\Gamma_0)$, $\Sigma_0 = D - D_0$, and note that $j(D_0) = D_0$ and Σ_0 is closed in D . Note that $\Sigma = p(\Sigma_0) \subset \Omega_X$ is closed and of codimension ≥ 1 . Define $\Omega_{X,0} = \Omega_X - \Sigma$. We can desingularize the diagram in (5.1) to:

$$\begin{array}{ccc}
 & \tilde{D} & \\
 p \swarrow & & \searrow \sigma_1 \\
 \Omega_X & & \tilde{\Gamma},
 \end{array} \tag{5.2}$$

where all maps are morphisms, and $\tilde{D}, \tilde{\Gamma}$ are smooth. Diagrams (5.1) & (5.2) are analogous to the diagrams on p. 46 & 47 in [6], indeed we have even tried to retain similar notation. Let $i_0: \Omega_{X,0} \hookrightarrow \Omega_X$ be the inclusion, and set $H_2(\Omega_X, \mathbb{Q})_\Sigma = i_{0,*}(H_2(\Omega_{X,0}, \mathbb{Q})) \subset H_2(\Omega_X, \mathbb{Q})$. We can now state:

(5.3) THEOREM (F.C.L.): *Let $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})_\Sigma$. Then $(\Phi_*(\gamma_1) \cdot \Phi_*(\gamma_2))_X = (d - N_0)(\gamma_1 \cdot \gamma_2)_{\Omega_X} + (i\gamma_1 \cdot \gamma_2)_{\Omega_X}$.*

PROOF: Except for dimensions of cycles in question, the proof of (5.3) is formally identical to the proof of the F.C.L. in [6; p. 45], which begins on p. 46 of [6], and involves the integral invariants N_0 , and d of (4.6).

(5.4) For the remainder of this section, we will occupy ourselves with the problem of reformulating (5.3) so as to not involve the particular algebraic cycle $\Sigma \subset \Omega_X$.

We will now fulfill a promise made earlier:

(5.5) PROPOSITION: $\Phi_*: H^2(\Omega_X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})$ is surjective.

PROOF: We will use the notation following (0.2) where $\Delta \subset U_0$ is a polydisk centered at $0 \in \Delta$, $X = X_0 \in \coprod_{v \in \Delta} X_v$. Let $i_v: X_v \hookrightarrow \coprod_{v \in \Delta} X_v$ be the inclusion morphism. Let X be transcendentally generic. Now because Δ is uncountable, any $\gamma \in H^{2,2}(X, \mathbb{Q})$ will have a horizontal displace-

ment in $\coprod_{v \in \Delta} H^4(X_v, \mathbb{Q})$ which is also of Hodge type (2, 2), i.e. $i_v^* \circ (i_0^*)^{-1}(\gamma) \in H^{2,2}(X_v, \mathbb{Q})$ for all $v \in \Delta$. However it is a general fact (using Lefschetz pencils) that such $\gamma \in \mathbb{Q} \omega \wedge \omega$, hence X transcendently generic $\Rightarrow H^{2,2}(X, \mathbb{Q}) = \mathbb{Q} \omega \wedge \omega$. This means that the only algebraic cocycle in $H^4(X, \mathbb{Q})$ is a \mathbb{Q} multiple of $\omega \wedge \omega$. Since Φ_* preserves algebraicity, clearly Φ_* is surjective for transcendental X . Now it can be easily seen that the cylinder homomorphisms $\Phi_{v,*}: H^2(\Omega_{X_v}, \mathbb{C}) \rightarrow H^4(X_v, \mathbb{C})$ piece together to form a morphism $\bar{\Phi}: \coprod_{v \in \Delta} H^2(\Omega_{X_v}, \mathbb{C}) \rightarrow \coprod_{v \in \Delta} H^4(X_v, \mathbb{C})$ of (trivial) analytic vector bundles over Δ . From the above discussion $\bar{\Phi}$ is fiberwise surjective on a uncountable dense subset of Δ , hence by analytic considerations, must be surjective over Δ . Q.E.D.

Let $k_0: \Sigma \hookrightarrow \Omega_X$ be the inclusion. Our next result is:

(5.6) PROPOSITION:

$$H_2(\Omega_X, \mathbb{Q})_{\Sigma} = \left\{ \begin{array}{l} \gamma \in H_2(\Omega_X, \mathbb{Q}) \mid (\gamma \cdot k_{0,*}(\alpha))_{\Omega_X} = 0 \\ \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}) \end{array} \right\}.$$

PROOF: It follows from [3; ch. 27] that there is a commutative diagram: (for our purposes $H^2(\Omega_X, \mathbb{C})$ will be viewed as deRham cohomology)

$$\begin{array}{ccc} H^2(\Omega_X, \mathbb{Q}) & \xrightarrow{k_0^*} & H^2(\Sigma, \mathbb{Q}) \\ D_P \uparrow \mid \lrcorner & & D_A \uparrow \mid \lrcorner \\ H_2(\Omega_{X,0}, \mathbb{Q}) & \xrightarrow{i_{0,*}} & H_2(\Omega_X, \mathbb{Q}) & \xrightarrow{f_*} & H_2(\Omega_X, \Omega_{X,0}) \end{array} \quad (5.7)$$

where D_P and D_A are respectively Poincaré and Alexander duality. Now for

$$\begin{aligned} & \gamma \in H_2(\Omega_X, \mathbb{Q}), f_*(\gamma) \\ & = 0 \Leftrightarrow k_0^* \circ D_P(\gamma) = 0 \\ & \Leftrightarrow \int_{k_{0,*}(\alpha)} D_P(\gamma) = 0 \quad \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}) \\ & \Leftrightarrow (\gamma \cdot k_{0,*}(\alpha))_{\Omega_X} = 0 \quad \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}). \end{aligned}$$

Now recall the Lefschetz (1, 1) theorem which states that $H^{1,1}(\Omega_X, \mathbb{Z})$ is generated by the fundamental classes of algebraic curves in Ω_X . We introduce the following notation:

(5.8) DEFINITION:

- (i) The transcendental cohomology, $H_T^2(\Omega_X, \mathbb{Q})$, is given by:
 $H_T^2(\Omega_X, \mathbb{Q}) = \{ \gamma \in H^2(\Omega_X, \mathbb{Q}) \mid \gamma \wedge H^{1,1}(\Omega_X, \mathbb{Q}) = 0 \}$.
- (ii) $H_\Sigma^2(\Omega_X, \mathbb{Q}) = D_P(H_2(\Omega_X, \mathbb{Q})_\Sigma)$.

(5.9) COROLLARY: $H_T^2(\Omega_X, \mathbb{Q}) \subset H_\Sigma^2(\Omega_X, \mathbb{Q})$.

PROOF: Compare (5.6) to (5.8)(i).

According to (5.9), it is clear that one can formulate a version of (5.3) for cocycles in $H_T^2(\Omega_X, \mathbb{Q})$, however there is another subspace in $H_\Sigma^2(\Omega_X, \mathbb{Q})$ which contains $H_T^2(\Omega_X, \mathbb{Q})$ and best suits our purposes. Recall the definition of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ in (0.3). There is an equivalent definition of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ using the notation in the proof of (5.5) and the Lefschetz (1, 1) Theorem.

(5.10) DEFINITION:

- (i) $H_A^{1,1}(\Omega_X, \mathbb{Q}) = \left\{ \begin{array}{l} \text{algebraic cocycles} \\ \gamma \in H^2(\Omega_X, \mathbb{Q}) \end{array} \middle| \begin{array}{l} \text{a horizontal} \\ \text{deformation of } \gamma \text{ in} \\ \prod_{v \in \Delta} H^2(\Omega_{X_v}, \mathbb{Q}) \text{ is} \\ \text{algebraic} \end{array} \right\}$.
- (ii) $H_P^2(\Omega_X, \mathbb{Q}) = \{ \gamma \in H^2(\Omega_X, \mathbb{Q}) \mid \gamma \wedge H_A^{1,1}(\Omega_X, \mathbb{Q}) = 0 \}$.

(5.11) REMARKS: From the general theory of Hilbert schemes, $H_A^{1,1}(\Omega_X, \mathbb{Q})$ is independent of the choice of polydisk $\Delta \subset U_0$, $\dim H_A^{1,1}(\Omega_{X_v}, \mathbb{Q})$ is constant over $v \in U_0$, and $H_A^{1,1}(\Omega_X, \mathbb{Q}) = H^{1,1}(\Omega_X, \mathbb{Q})$ for transcendently generic X .

(5.12) PROPOSITION: $H_T^2(\Omega_X, \mathbb{Q}) \subset H_P^2(\Omega_X, \mathbb{Q}) \subset H_\Sigma^2(\Omega_X, \mathbb{Q})$.

PROOF: The inclusion $H_T^2(\Omega_X, \mathbb{Q}) \subset H_P^2(\Omega_X, \mathbb{Q})$ is obvious from the definitions, moreover is an equality for transcendently generic X ((5.11)). Next as X varies, i.e. $v \in U_0$ varies, Σ also varies algebraically, hence $[\Sigma] \in H_A^{1,1}(\Omega_X, \mathbb{Q})$, therefore the second inclusion follows from (5.6), (5.8)(ii)&(5.10)(ii).

(5.13) REMARKS:

- (i) The well known properties of the pairing $H^2(\Omega_X, \mathbb{C}) \times H^2(\Omega_X, \mathbb{C}) \rightarrow \mathbb{C}$ imply $H^2(\Omega_X, \mathbb{Q}) = H_P^2(\Omega_X, \mathbb{Q}) \oplus H_A^{1,1}(\Omega_X, \mathbb{Q})$ is an orthogonal decomposition under \wedge .
- (ii) As X varies, i.e. $v \in U_0$ varies, the incidence correspondence $D \subset \Omega_X \times \Omega_X$ also varies algebraically. Therefore $i(H_A^{1,1}(\Omega_X, \mathbb{Q})) \subset H_A^{1,1}(\Omega_X, \mathbb{Q})$.

We need the following:

(5.14) LEMMA: $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \subset \text{Prim}^4(X, \mathbb{Q})$.

PROOF: Let H_1, H_2 be generic hyperplanes in \mathbb{P}^5 , $X_s = H_1 \cap H_2 \cap X$, $Y_s = X_s \cap \varphi(P(X))$. Note that $[X_s] = \omega \wedge \omega \in H^{2,2}(X, \mathbb{Z})$, and Y_s is a curve in $S = H_1 \cap \varphi(P(X))$. By (3.9), Y_s induces a corresponding curve C_1 in Ω_X , given by the formula $C_1 = \rho_* \circ \varphi^*(Y_s)$. Since Y_s varies algebraically as X varies, clearly $[C_1] \in H_A^{1,1}(\Omega_X, \mathbb{Q})$. Now let $\gamma \in H_2(\Omega_X, \mathbb{Q})$ be given so that $D_P(\gamma) \in H_P^2(\Omega_X, \mathbb{Q})$. From the techniques of the proof of (5.6), it is clear that γ can be chosen to be supported on $\Omega_X\text{-supp}(C_1)$. Therefore, on the cycle level, $\Phi_*(\gamma) \cap Y_s = 0$, hence $(\Phi_*(\gamma) \cdot X_s)_X = 0$. By translating this in terms of cohomology, $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \wedge \omega \wedge \omega = 0$. But $\wedge \omega: H^6(X, \mathbb{Q}) \rightarrow H^8(X, \mathbb{Q})$ is an isomorphism, hence $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \wedge \omega = 0$, i.e. $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \subset \text{Prim}^4(X, \mathbb{Q})$. Q.E.D.

There is another needed result:

(5.15) LEMMA: Let $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})$. Then $(i\gamma_1 \cdot \gamma_2)_{\Omega_X} = (\gamma_1 \cdot i\gamma_2)_{\Omega_X}$.

PROOF: Using the notation of (5.2), together with the definition of j , there is a commutative diagram:

$$\begin{array}{ccc}
 \tilde{D} & \xrightarrow{\tilde{j}} & \tilde{D} \\
 g \downarrow & & g \downarrow \\
 D & \xrightarrow{j} & D
 \end{array} \tag{5.16}$$

where \tilde{j} is biregular, and g is a birational morphism. Define $\tilde{p} = p \circ g: \tilde{D} \rightarrow \Omega_X$. It is easy to verify that the correspondence defined by $\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*$ is the same as $p_* \circ j \circ p^* = D$, hence $\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^* = i$. Now by applying the projection formula 3 times we have: (Note $\tilde{j}^* = \tilde{j}_*$)

$$\begin{aligned}
 (i\gamma_1 \cdot \gamma_2)_{\Omega_X} &= (\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*(\gamma_1) \cdot \gamma_2)_{\Omega_X} \\
 &= (\gamma_1 \cdot \tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*(\gamma_2))_{\Omega_X} \\
 &= (\gamma_1 \cdot i\gamma_2)_{\Omega_X}.
 \end{aligned}$$

(5.17) COROLLARY: $i(H_P^2(\Omega_X, \mathbb{Q})) \subset H_P^2(\Omega_X, \mathbb{Q})$.

PROOF: Otherwise there exists $\gamma_1 \in H_P^2(\Omega_X, \mathbb{Q})$, $\gamma_2 \in H_A^{1,1}(\Omega_X, \mathbb{Q})$ such that $i(\gamma_1) \wedge \gamma_2 \neq 0$. But $i(\gamma_1) \wedge \gamma_2 = \gamma_1 \wedge i(\gamma_2)$ by (5.15) = 0 by (5.13), a contradiction.

We can formulate (5.3) for $H_P^2(\Omega_X, \mathbb{Q})$:

(5.18) PROPOSITION: Given $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})$ with $D_P(\gamma_1), D_P(\gamma_2)$ in $H_P^2(\Omega_X, \mathbb{Q})$. Then $(\Phi_*(\gamma_1) \cdot \Phi_*(\gamma_2))_X = (d - N_0)(\gamma_1 \cdot \gamma_2)_{\Omega_X} + (i\gamma_1 \cdot \gamma_2)_{\Omega_X}$.

PROOF: Use (5.3) & (5.12).

Combining everything together so far we arrive at the final result of this section:

(5.19) THEOREM. *The following subspaces are the same:*

- (i) $S_1 = \{\gamma \in H_P^2(\Omega_X, \mathbb{Q}) \mid \Phi_*(\gamma) = 0\}$
- (ii) $S_2 = \{\gamma \in H_P^2(\Omega_X, \mathbb{Q}) \mid (d - N_0)\gamma + i(\gamma) = 0\}$
- (iii) $S_3 = (i + 119 \cdot I)H_P^2(\Omega_X, \mathbb{Q})$.

PROOF: $S_1 = S_2$ follows immediately from (5.18) and (5.5). Next (3.11), (5.14), (5.17) imply $S_3 \subset S_1$. We first justify the claim: $\{\ker(i + 119 \cdot I)\} \cap S_1 = 0$. If $\gamma \in \ker(i + 119 \cdot I) \cap S_1$, then $i(\gamma) + 119\gamma = (d - N_0)\gamma + i(\gamma) = 0$, hence $(119 - (d - N_0))\gamma = 0$, $\Rightarrow \gamma = 0$ by (4.5), which proves the claim. Using the claim, it is clear that the homomorphism $(i + 119 \cdot I): S_1 \rightarrow S_3$ is injective, hence an isomorphism as $S_3 \subset S_1$. (5.19) now follows.

§6. A quadratic relation and the proof of the main theorem

We now attend to the proof of the main theorem ((0.6)). Let $r = d - N_0$, and set $Q(i) = (rI + i)(i + 119 \cdot I) = i^2 + (119 + r)i + r \cdot 119 \cdot I$. We prove:

(6.1) PROPOSITION:

- (i) $Q(i): H_P^2(\Omega_X, \mathbb{Q}) \rightarrow H_P^2(\Omega_X, \mathbb{Q})$ is the zero morphism.
- (ii) $i: H_P^2(\Omega_X, \mathbb{Q}) \rightarrow H_P^2(\Omega_X, \mathbb{Q})$ is an isomorphism.

PROOF: Part (i) is an immediate consequence of (5.19). For part (ii), note that $i(\gamma) = Q(i)(\gamma) = 0 \Rightarrow r \cdot 199\gamma = 0$, afortiori $\gamma = 0$. Q.E.D.

Note that for any $\gamma \in H_A^{1,1}(\Omega_X, \mathbb{Q})$, $\Phi_*(\gamma)$ has the property that under a horizontal displacement in $\coprod_{v \in \Delta} H^4(X_v, \mathbb{Q})$, $\Phi_*(\gamma)$ is still algebraic. One concludes from the proof of (5.5) that $\Phi_*(\gamma) \in \mathbb{Q}\omega \wedge \omega$. Therefore $\Phi_*(H_A^{1,1}(\Omega_X, \mathbb{Q})) = \mathbb{Q}\omega \wedge \omega$, hence:

(6.2) COROLLARY: $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) = \text{Prim}^4(X, \mathbb{Q})$.

PROOF: Use the above remark, (5.5)&(5.14).

Combining (6.2) with (5.19)&(6.1), we arrive at our main result.

(6.3) THEOREM:

- (i) i respects the decomposition $H^2(\Omega_X, \mathbb{Q}) = H_P^2(\Omega_X, \mathbb{Q}) \oplus H_A^{1,1}(\Omega_X, \mathbb{Q})$, moreover $i: H_P^2(\Omega_X, \mathbb{Q}) \rightarrow H_P^2(\Omega_X, \mathbb{Q})$ is an isomorphism.

