COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 56, nº 3 (1985), p. 315-329
<http://www.numdam.org/item?id=CM_1985__56_3_315_0>

Compositio Mathematica 56 (1985) 315–329.

THE CYLINDER HOMOMORPHISM ASSOCIATED TO QUINTIC FOURFOLDS

James D. Lewis

§0. Introduction

Let X be a quintic fourfold (smooth hypersurface of degree 5 in $\mathbb{P}^5$), and Ω_X the variety of lines in X. According to [1], if X is generically chosen, then Ω_X is a smooth surface. Let Φ_*: $H_2(\Omega_X, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ be the "cylinder homorphism" obtained by blowing up each point on $\gamma \in \mathrm{H}_2(\Omega_X, \mathbb{Q})$ to a corresponding line in X (thus sweeping out a 4 cycle in X). This homomorphism was studied in [4], and in particular, viewing Φ_* on cohomology (viz Poincaré duality):

(0.1) Theorem: ([4; (4.4)]). *Let X be generic, $\omega \in H^{1,1}(X, \mathbb{Q})$ the Kähler class dual to the hyperplane section of X. Then Φ_*: $H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q})/\mathbb{Q}\omega \wedge \omega$ is an epimorphism.*

For relatively elementary reasons (see (5.5)), it is also true that Φ_*: $H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$ is an epimorphism for generic X. This paper is devoted to the answering of the following question:

(0.2) What is the kernel of Φ_*?

In order to satisfactorily answer (0.2), some terminology has to be introduced. The family of hypersurfaces $\{X_v\}_{v \in \mathbb{P}^N}$ of degree 5 in $\mathbb{P}^5$ is a projective space of dimension $N = 251$. Let $U \subset \mathbb{P}^N$ be the open set parameterizing the smooth X_v, $U_0 \subset U$ the open subset corresponding to those X for which Ω_X is a smooth, irreducible surface. Let $\Delta \subset U_0$ be a polydisk centered at $0 \in \Delta$, $X = X_0$, and for any $v \in \Delta$, define j_v: $\Omega_{X_t} \hookrightarrow \coprod_{v \in \Delta} \Omega_{X_v}$ to be the inclusion morphism. Now $\coprod_{v \in \Delta} \Omega_{X_v}$ is topologically equivalent to $\Delta \times \Omega_{X_t}$ (see [7]) for any given $v \in \Delta$, and therefore there is an isomorphism $j_v^* \circ (j_0^*)^{-1}$: $H^2(\Omega_X, \mathbb{Q}) \to H^2(\Omega_{X_v}, \mathbb{Q})$.

(0.3) Definition:

(i) $H_A^{1,1}(\Omega_X, \mathbb{Q}) = \{\gamma \in H^2(\Omega_X, \mathbb{Q}) \mid j_v^* \circ (j_0^*)^{-1}(\gamma) \in H^{1,1}(\Omega_{X_t}, \mathbb{Q})$ for all $v \in \Delta\}$.

(ii) $H_P^2(\Omega_X, \mathbb{Q})$ = orthogonal complement of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ in $H^2(\Omega_X, \mathbb{Q})$.

defined as follows (see (3.1) for a precise definition): (0.5) Let l_x be the line corresponding to $x \in \Omega_X$. Define $D(x) = \{ y \in \Omega_X \mid y \neq x \ \& \ l_x \cap l_y \neq \emptyset\}$. It is proven (see (2.5)) that for generic X, $D(x)$ is a finite set for generic $x \in \Omega_X$.

Our theorem is: (X generic)

(0.6) THEOREM:

(*i*) *i preserves the subspaces defined in* (0.3)(i)&(ii); *moreover* i: $H_P^2(\Omega_X, \mathbb{Q}) \to H_P^2(\Omega_X, \mathbb{Q})$ *is an isomorphism.*

(*ii*) *There is a s.e.s.:*

$$0 \to (i + 119 \cdot I) H_P^2(\Omega_X, \mathbb{Q}) \xrightarrow{i} H_P^2(\Omega_X, \mathbb{Q}) \xrightarrow{\Phi_*} \mathrm{Prim}^4(X, \mathbb{Q}) \to 0,$$

where i and I are respectively the inclusion and identity morphisms.

(*iii*) $\Phi_*(H_A^{1,1}(\Omega_X, \mathbb{Q})) = \mathbb{Q}\omega \wedge \omega$.

(0.7) COROLLARY:

$$\begin{array}{ccc} H_P^2(\Omega_X, \mathbb{Q}) & \xrightarrow{\Phi_*} & \mathrm{Prim}^4(X, \mathbb{Q}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \times 119} \\ H_P^2(\Omega_X, \mathbb{Q}) & \xrightarrow{\Phi_*} & \mathrm{Prim}^4(X, \mathbb{Q}) \end{array}$$

is sign commutative.

Much of the techniques of this paper are borrowed from an interesting paper by Tyurin ([6]).

§1. Notation

(i) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{C}$ = complex numbers

(ii) X is a quintic fourfold, $\mathbb{P}^M$ is complex, projective M-space.

(iii) If Y is a projective, algebraic manifold, then $H^{p,p}(Y)$ is Dolbeault cohomology of type (p, p) and $H^{p,p}(Y, K) = H^{p,p}(Y) \cap H^{2p}(Y, K)$, where $K = \mathbb{Z}, \mathbb{Q}$.

(iv) Prim stands for primitive cohomology.

(v) There are 2 senses to the word "generic" in this paper. We say that X is generic if it is a member of a family $\{X_v\}_{v \in W}$ satisfying a given property, and where $W \subset \mathbb{P}^N$ is a Zariski open subset. The other use of the word "generic" is where X satisfies a given property that is transcendental in nature, and in this case the word generic will be prefixed by "transcendental".

(vi) Let $Y \subset \mathbb{P}^M$ be given as in (iii) above, G = Grassmannian of lines in $\mathbb{P}^M$. For $x \in G$, let l_x be the corresponding line in $\mathbb{P}^M$. The variety of lines in Y, denoted by Ω_Y is defined as follows: $\Omega_Y = \{x \in G \mid l_x \subset Y\}$.

(vii) Given Y as in (iii) and $S \subset Y$ an algebraic subset. Then dim S = max{dim of irreducible components of S}, and $\operatorname{codim}_Y S = \dim Y - \dim S$.

§2. The variety of lines in X

Let $Y \subset \mathbb{P}^n$ be a generic hypersurface of degree d, and assume $2n - d - 5 \geqslant 0$. An immediate consequence of [1] is:

(2.1) THEOREM: *Ω_Y is smooth and irreducible, of dimension $2n - d - 3$.*
There are two noteworthy cases to consider:

(2.2) COROLLARY: *Given X a generic quintic fourfold, and Z a generic fivefold of degree* 5 *in $\mathbb{P}^6$, then*:

(*i*) *Ω_X is a smooth, irreducible surface and*

(*ii*) *Ω_Z is a smooth, irreducible fourfold.*

An argument identical to one given in [6; *p*. 38] *yields*:

(2.3) PROPOSITION: *Given Z as in* (2.2). *Then through a generic point of Z passes* 5! *lines.*

Before stating the main result of this section, we introduce the following notation: *Let $c \in \Omega_X$, $l_c \subset X$ the corresponding line.*

(2.4) $\Omega_{X,c} = \{y \in \Omega_X \mid l_y \cap l_c \neq \emptyset\}$.We prove:

(2.5) THEOREM: *Let X be generic.*

(*i*) dim $\Omega_{X,c} = 0$ *for generic $c \in \Omega_X$.*

(*ii*) *Let $c \in \Omega_X$ be generic. Then for any $y \in l_c$, there is at most one line $l_0 \subset X$ other than l_c passing through y.*

PROOF: We start by letting X be any degree 5 hypersurface in $\mathbb{P}^5$, and $x \in X$. If we let $[X_0, X_1, X_2, X_3, X_4, X_5]$ be the homogeneous coordinates defining $\mathbb{P}^5$, then X admits as its defining equation $F = 0$, $F \in \mathbb{C}[X_0, \ldots, X_5]$ *a homogeneous polynomial of degree* 5. *Now after applying a projective transformation, there is no loss of generality in assuming* $x = [0, 0, 0, 0, 0, 1]$. *In this case F takes the form*: $F = X_5^4 F_1 + X_5^3 F_2 + X_5^2 F_3 + X_5 F_4 + F_5$, where $F_i \in \mathbb{C}[X_0, \ldots, X_4]$ is homogeneous of degree i. We now convert to affine coordinates by setting $x_i = X_i/X_5$, $i = 0, \ldots, 4$. Define $f_i = F_i/X_5^i$. and note that $f_i \in \mathbb{C}[x_0, \ldots, x_4]$ is homogeneous of degree i. Likewise, set $f = F/X_5^5$, and note that $f = f_1 + f_2 + f_3 + f_4 + f_5$. In affine coordinates $x = (0, 0, 0, 0, 0)$, therefore any line l_a passing through x must be of the form $l_a = \{ta \mid t \in \mathbb{C}\}$, where $a \in \mathbb{C}^5$ is non-zero.

It follows that

$$\begin{array}{llll} l_a \subset X & \Leftrightarrow & f_1(ta) + \cdots + f_5(ta) = 0 & \text{for all } t \\ \text{i.e.} & \Leftrightarrow & tf_1(a) + \cdots + t^5 f_5(a) = 0 & \text{for all } t \\ & \Leftrightarrow & f_1(a) = \cdots = f_5(a) = 0. & \end{array}$$

The upshot of this argument is that the lines in X passing through x correspond to the zeros of $f_1, \ldots, f_5$ in $\mathbb{P}^4$. Note that for generic $x \in X$, no such line exists. Let $V(i)$ be the vector space of homogeneous polynomials of degree i in $\mathbb{C}[x_0, \ldots, x_4]$, and set $V = V(1) \oplus \ldots \oplus V(5)$. It is clear from our construction that X determines a point $v \in \mathbb{P}(V)$, conversely, any $v \in \mathbb{P}(V)$ determines X so that $x \in X$.

(2.6) Every $v \in \mathbb{P}(V)$ determines an algebraic set $S(v)$ defined as the zeros of $f_1, \ldots, f_5$ in $\mathbb{P}^4$. Define $V_1 = \{v \in \mathbb{P}(V) \mid \dim S(v) \geqslant 0\}$.If $v \in \mathbb{P}(V)$ is given so that $\dim S(v) = 0$, then define $\#S(v)$ to be the cardinality of $S(v)$ as a set. For $i = 2, 3$ define $V_i = \{v \in V_1 \mid \dim S(v) \geqslant 1$ or $\#S(v) \geqslant i\}$, and set $V_B = \{v \in V_1 \mid \dim S(v) \geqslant 1\}$. We need the following:

(2.7) Lemma: $\operatorname{codim}_{\mathbb{P}(V)} V_i = i$, *for* $i = 1, 2, 3$ & $\operatorname{codim}_{\mathbb{P}(V)} V_B \geqslant 5$.

Proof: Let $V^j = V(j) \oplus \cdots \oplus V(5) \subset V$, for $j = 1, \ldots, 5$, and $\mathbb{P}(V^j) \subset \mathbb{P}(V)$ the corresponding projective subspaces. Note that for $v \in \mathbb{P}(V^j)$, $S(v) =$ zeros of $\{f_j, \ldots, f_5\}$ in $\mathbb{P}^4$. We will prove (2.7) case-by-case:
(*a*) $\operatorname{codim}_{\mathbb{P}(V)} V_1 = 1$: It follows from general principles ([5; (3.30)]) that $v \in \mathbb{P}(V^2) \Rightarrow S(v) \neq \emptyset$, so for such v, choose any $y \in S(v)$. Clearly $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y) = 0\}$ cuts out a codimension 1 subspace of $\mathbb{P}(V(1))$, hence $\operatorname{codim}_{\mathbb{P}(V)} V_1 = 1$.
(*b*) $\operatorname{codim}_{\mathbb{P}(V)} V_2 = 2$: Let $v \in V^2$ be given so that $\dim S(v) = 0$ and $\#S(v) \geqslant 2$. Let $y_1, y_2 \in S(v)$ with $y_1 \neq y_2$. Then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y_1) = f_1(y_2) = 0\}$ cuts out a subspace of codimension 2 in $\mathbb{P}(V(1))$. Statement (b) now follows from:

(2.8) Sublemma: $\{v \in \mathbb{P}(V^2) \mid \dim S(v) \geqslant 1\}$ *has codimension* $\geqslant 3$ *in* $\mathbb{P}(V^2)$.

Proof: If $v \in \mathbb{P}(V^3)$, then $\dim S(v) \geqslant 1$ and equal to 1 for generic v. Define $H = \{(y, v) \in \mathbb{P}^4 \times \mathbb{P}(V^3) \mid y \in S(v)\}$, and let q_1, q_2 be the canonical projections in the diagram below:

$$\begin{array}{ccc} & H & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ \mathbb{P}^4 & & \mathbb{P}(V^3) \end{array} \qquad (2.9)$$

Note that the fibers of q_1 are projective spaces, all of which are projectively equivalent to each other; moreover q_1 (and q_2) are surjective, hence H is irreducible. In addition $q_2^{-1}(v) = S(v)$, and by our construction, the generic fiber of q_2 is a smooth, irreducible curve of degree 60 (Bezout's theorem). Let $K = \{v \in \mathbb{P}(V^3) \mid \dim S(v) \geqslant 2\}$. Then by considering the morphism q_2, it follows that $\operatorname{codim}_{\mathbb{P}(V^3)} K \geqslant 2$, (in fact $\operatorname{codim}_{\mathbb{P}(V^3)} K \geqslant 3$). If $v \in \mathbb{P}(V^3)$ is given so that $\dim S(v) = 1$, then elementary reasoning implies $\{f_2 \in \mathbb{P}(V(2)) \mid f_2$ vanishes on a component of $S(v)$ of dimension $1\}$ is of codimension $\geqslant 3$ in $\mathbb{P}(V(2))$. On the other hand if $v \in \mathbb{P}(V^3)$ is given so that $\dim S(v) \geqslant 2$, then one constructs a diagram analogous to (2.9), replacing $\mathbb{P}(V^3)$ by $\mathbb{P}(V^4)$, modifying H accordingly, and applying a similar reasoning as above to conclude $\operatorname{codim}_{\mathbb{P}(V^3)} K \geqslant 3$, hence (2.8).
(c) $\operatorname{codim}_{\mathbb{P}(V)} V_3 = 3$: If $v \in V^2$ is generically chosen, then $\# S(v) = 5!$ (bezout's theorem), moreover no 3 points in $S(v)$ are collinear. If y_1, y_2, $y_3 \in S(v)$ are distinct, then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1(y_1) = f_1(y_2) = f_1(y_3) = 0\}$ is a subspace of codimension 3in $\mathbb{P}(V(1))$. The case that $v \in V^2$ is given so that $\dim S(v) \geqslant 1$ is taken care of by (2.8). There remains the possibility that $v \in V^2$ is given so that $\dim S(v) = 0$ and that some collinearity (of 3 points) exists. For this to happen, v would have to belong to a proper subvariety of V^2, and one can easily argue that statement (c) still holds.
(d) $\operatorname{codim}_{\mathbb{P}(V)} V_B \geqslant 5$: A construction similar to the proof of (2.8) implies $\{v \in \mathbb{P}(V^2) \mid \dim S(v) \geqslant 2\}$ is of codimension $\geqslant 5$ in $\mathbb{P}(V^2)$. Now suppose $v \in \mathbb{P}(V^2)$ is given so that $\dim S(v) = 1$. Then $\{f_1 \in \mathbb{P}(V(1)) \mid f_1$ vanishes on a dimension 1 component of $S(v)\}$ is of codimension $\geqslant 2$ in $\mathbb{P}(V(1))$. We now apply (2.8) to conclude statement (d), and the proof of (2.7).

(2.10) Conclusion of the proof of (2.5)

Recall at the beginning of the proof a choice of $x \in \mathbb{P}^5$ which determines $\mathbb{P}(V)$, V_1, V_2, V_3, V_B, where $\mathbb{P}(V)$ corresponds to those $X \subset \mathbb{P}^5$ for which $x \in X$. To indicate that our choice of x determines $\mathbb{P}(V)$, we will relabel things with the obvious meaning as $\mathbb{P}(V_x)$, $V_{1,x}$, $V_{2,x}$, $V_{3,x}$, $V_{B,x}$. *Now define* $W = \coprod_{x \in \mathbb{P}^5} \mathbb{P}(V_x)$, $W_i = \coprod_{x \in \mathbb{P}^5} V_{i,x}$ for $i = 1, 2, 3$, $W_B = \coprod_{x \in \mathbb{P}^5} V_{B,x}$. It is easy to verify that W, W_i's, W_B all have the structure of an algebraic variety, moreover by (2.7):

(2.11) $\operatorname{codim}_W W_i = i$ for $i = 1, 2, 3$ and $\operatorname{codim}_W W_B \geqslant 5$.
Recall the statement just preceeding (2.6), that for any X and $x \in X$, X determines a point $v_x \in \mathbb{P}(V_x)$. Therefore X determines a fourfold $X_W \subset W$ given by the formula $X_W = \coprod_{x \in X} v_x$. For generic $X \subset \mathbb{P}^5$, $\dim\{X_W \cap W_i\} = 4 - i$, and $X_W \cap W_B = \emptyset$. Translating this in terms of Ω_X, (2.5) clearly holds.

Q.E.D.

§3. The incidence and cylinder homomorphisms

Let $D_1 \subset \Omega_X \times \Omega_X$ be given by the formula: $D_1 = \{(x_1, x_2) \in \Omega_X \times \Omega_X \mid l_{x_1} \cap l_{x_2} \neq \emptyset \ \& \ x_1 \neq x_2\}$. It is clear from the definition that $\{x, D_1(x)\} = \Omega_{X,x}$. Throughout this section X will be assumed to be generic.

(3.1) Definition: The incidence correspondance $D \subset \Omega_X \times \Omega_X$ is defined to be: $D = \overline{D}_1$.

Note that $\operatorname{codim}_{\Omega_X \times \Omega_X} D = 2$, therefore the fundamental class of D defines a cocycle $[D] \in H^4(\Omega_X \times \Omega_X, \mathbb{Q})$. Now the component of $[D]$ in $H^2(\Omega_X, \mathbb{Q}) \otimes H^2(\Omega_X, \mathbb{Q})$, via the Künneth formula $H^4(\Omega_X \times \Omega_X, \mathbb{Q}) = \oplus_{p+q=4} H^p(\Omega_X, \mathbb{Q}) \otimes H^q(\Omega_X, \mathbb{Q})$, induces a morphism i: $H^2(\Omega_X, \mathbb{Q}) \to H^2(\Omega_X, \mathbb{Q})$, *where we use the fact* $H^2(\Omega_X, \mathbb{Q})^* \cong H^2(\Omega_X, \mathbb{Q})$ (Poincaré duality).

(3.2) Definition: The homomorphism i: $H^2(\Omega_X, \mathbb{Q}) \to H^2(\Omega_X, \mathbb{Q})$ is called the incidence homomorphism.

The morphism i factors into a composite of 3 other morphisms given as follows:

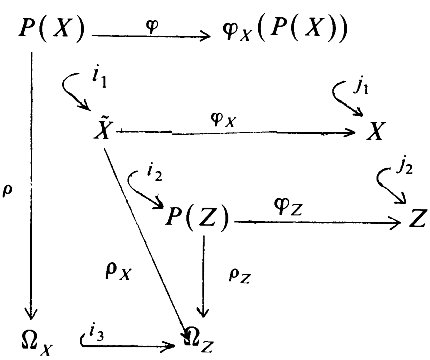
(3.3) Let
(i) p: $D \to \Omega_X$ be the projection onto the first factor,
(ii) j: $\Omega_X \times \Omega_X \to \Omega_X \times \Omega_X$ the morphism which permutes the factors, i.e. $j(x_1, x_2) = (x_2, x_1)$. Note that $j(D) = D$.
Then:

(3.4) Proposition: $i = p_* \circ j \circ p^*$.

Proof: Use the fact that the correspondence defined by $p_* \circ j \circ p^*$ in $\Omega_X \times \Omega_X$ is precisely D.

(3.5) The cylinder homomorphism

We will be constantly referring to the following diagram:



(3.6)

where, Z is a smooth degree 5 hypersurface in $\mathbb{P}^6$, for which $X \subset Z$ is a (smooth) hyperplane section

$$P(X) = \{(c, x) \in \Omega_X \times X \mid x \in l_c\}$$

$$P(Z) = \{(c, z) \in \Omega_Z \times Z \mid z \in l_c\}$$

ρ (resp. ρ_Z) is the projection of $P(X)$ (resp. $P(Z)$) onto the first factor

φ (resp. φ_Z) is the projection of $P(X)$ (resp. $P(Z)$) onto the second factor

$$\tilde{X} = \varphi_Z^{-1}(X),\ \varphi_X = \varphi_{Z|_{\tilde{X}}}\colon \tilde{X} \to X,\ \rho_X = \rho_{Z|_{\tilde{X}}}\colon \tilde{X} \to \Omega_Z$$

i_1, i_2, i_3, j_1, j_2 are inclusion morphisms.

The same reasoning given in [2; p. 81] implies the following:

(3.7) PROPOSITION (see [4]):

(*i*) *$P(X)$, $P(Z)$ are $\mathbb{P}^1$ bundles over Ω_X and Ω_Z respectively.*

(*ii*) *$P(X)$, $P(Z)$, $\tilde{X}$, Ω_X, Ω_Z are smooth and irreducible.*

(*iii*) *All morphisms in* (3.6), *except for inclusions, are surjective.*

(*iv*) $\deg \varphi_Z = \deg \varphi_X = 5!$.

(*v*) *ρ_X is birational and induces: $\tilde{X} \cong$ blow up of Ω_Z along Ω_X.*

(3.8) REMARKS:

(i) (2.2 implies the smoothness and irreducibility for Ω_X and Ω_Z.

(ii) (3.7) (iv) is a consequence of (2.3).

As will be discussed in §4, the threefold $\varphi(P(X))$ has a 2-dimensional singular set. Let S be a generic hyperplane section of $\varphi(P(X))$. One should expect S to be singular. The next result is a direct consequence of (2.5), together with the definitions of $P(X)$, ρ, φ:

(3.9) PROPOSITION: *φ is a birational morphism, moreover φ induces a birational map $\Omega_X \approx S$.*

(3.10) DEFINITION: The cylinder homomorphism $\Phi_*\colon H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$ is given by: $\Phi_* = j_{1,*} \circ \varphi_* \circ \rho^*$.

Let I: $H^2(\Omega_X, \mathbb{Q}) \to H^2(\Omega_X, \mathbb{Q})$ be the identity morphism, $\omega \in H^{1,1}(X, \mathbb{Z})$ *the Kähler class defined in* (0.1). *The next result ties in a relationship between i and* Φ_*.

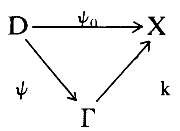
(3.11) PROPOSITION: $\Phi_*(\{(i + 119 \cdot I)H^2(\Omega_X, \mathbb{Q})\}) = 0$ *in* $H^4(X, \mathbb{Q})/\mathbb{Q}\omega \wedge \omega$

PROOF: The proof of (3.11) is formally identical to the proof of lemma 6 in [6; p. 42] where

(a) Z and deg $\varphi_Z = 5!$ replace X_4 and deg φ in [6].
(b) the cycles are even dimensional.
(c) the weak Lefschetz theorem applied to the inclusions $Z \subset \mathbb{P}^6 \& j_2$: $X \hookrightarrow Z$ implies $j_2^*(H^4(Z, \mathbb{Q})) = \mathbb{Q}\omega \wedge \omega$.
(d) $119 = 5! - 1$.

§4. The numerical characteristic of the surface Ω_X

Let ψ_1: $D_1 \to X$ be the morphism defined by the formula: $\psi_1(x_1, x_2) = l_{x_1} \cap l_{x_2} \in X$. Then ψ_1 extends to a rational map ψ_0: $D \to X$, moreover deg $\psi_0 = 2$ by (2.5)(ii). Let $\Gamma = D/\{j\}$ with quotient morphism ψ: $D \to \Gamma$. There is a factorization of ψ_0:



(4.1)

where k is a birational map onto its image, ψ_0(D). This factorization will be useful in the next section where we consider an analogue to the fundamental computational lemma in [6; p. 45]. Note that the fibers of φ in (3.6) are a discrete over every point in $\varphi(P(X))$, moreover $\#\varphi^{-1}(x) \geqslant 2$ over $\overline{\psi_1(D_1)}$ and $\#\varphi^{-1}(x) = 1$ over $\varphi(P(X)) - \overline{\psi_1(D_1)}$, where # includes multiplicity. By applying Zariski's Main theorem to φ, it is clear that sing($\varphi(P(X))$) $= \overline{\psi_1(D_1)}$. On the other hand, $\overline{\psi_1(D_1)} = \psi_0(D)$, therefore, taking into account the result (2.5)(ii), we can summarize the above discussion in:

(4.2) Proposition: sing($\varphi(P(X))$) $= \psi_0(D)$, *moreover through a generic point of* sing($\varphi(P(X))$) *passes exactly* 2 *lines in X.*

So far we have only focused on the number of lines passing through a given point in $\varphi(P(X))$. *We now turn our attention to the problem of determining the number of lines meeting a generic line in X. This number will be denoted by* N_0, *and bears the title of this section, namely, recall the definition of p in* (3.3)(*i*):

(4.3) Definition: The numerical characteristic N_0 of Ω_X is given by: $N_0 = \deg p$.

(4.4) Remark: This definition is borrowed in part from [6; p. 40].

There is another ingredient we want to introduce, but before doing so, we recall from the Lefschetz theorem applied to $X \subset \mathbb{P}^5$ that $H^2(X, \mathbb{Z}) = \mathbb{Z}\omega$. Let $[\varphi(P(X))]$ be the fundamental class of $\varphi(P(X))$ in $H^2(X, \mathbb{Z})$. Then there is a positive integer d for which $[\varphi(P(X))] = d\omega$. Geometrically, d is the degree of the hypersurface in $\mathbb{P}^5$ cutting out $\varphi(P(X))$ in X. A partial generalization of Fano's work (see [6; p. 40]) implies d and N_0 are related by the simple:

(4.5) PROPOSITION: $d - N_0 \leqslant -2$.

PROOF: The proof is essentially borrowed from lemma 5 in [6; p. 40], but there are important differences accounting for the changes in statements between (4.5) and [6]. Let $l = \mathbb{P}^1$, $\mathbb{P}^3$, X be generically chosen in $\mathbb{P}^5$, so that $l \subset X \cap \mathbb{P}^3$, and that $S_0 = \mathbb{P}^3 \cap X$ is a smooth quintic surface. The adjunction formula for $S_0 \subset \mathbb{P}^3$ implies $\Omega^2_{S_0} = \mathcal{O}_{S_0}(1)$, where $\Omega^2_{S_0}$ is the canonical sheaf of S_0. Note that l is the only line in S_0, since a generic hyperplane section of X contains only a finite number of lines ([1]), and S_0 is cut out by a generic $\mathbb{P}^3$. If H is a generic hyperplane in $\mathbb{P}^5$ containing l, then $H \cap S_0 = l + C_0$, where C_0 is a smooth and irreducible curve. Note from the above expression for $\Omega^2_{S_0}$ that $\Omega^2_{S_0} = \mathcal{O}_{S_0}(H \cap S_0) = \mathcal{O}_{S_0}(l + C_0)$. Now taking intersections: $1 = (l \cdot H)_{\mathbb{P}^5} = (l \cdot (H \cdot S_0))_{S_0} = (l \cdot (l + C_0))_{S_0}$, (where $\cdot = \cap$), consequently $(l \cdot C_0)_{S_0} = 1 - l^2$. On the other hand, the adjunction formula applied to $l \subset S_0$ implies: $-2 = (l \cdot (l + (H \cdot S_0))_{S_0} = l^2 + 1$, hence $l^2 = -3$, afortiori $(l \cdot C_0)_{S_0} = 4$. Next $S_0 \cap \varphi(P(X)) = l + C_1 \sim d(H \cdot S_0) = dl + dC_0$, hence $C_1 \sim (d-1)l + dC_0$, therefore $(C_1 \cdot l)_{S_0} = (d-1)l^2 + d(l \cdot C_0)_{S_0} = d + 3$. Now $\varphi^{-1}(\varphi(P(X)) \cap S_0) = l + \varphi^{-1}(C_1)$ where $\varphi^{-1}(C_1)$ is no longer regarded as a global section of the fibering p: $P(X) \to \Omega_X$ as in [6], but rather as a section of ρ over a curve in Ω_X, where we use the aforementioned fact that l is the only line in S. Then among the points of intersection in $C_1 \cdot l$ is a *possible* point of intersection of l with $\varphi^{-1}(C_1)$, and the remaining points are the intersections of l with *at most* the other lines in X meeting l. Therefore $(C_1 \cdot l)_{S_0} \leqslant N_0 + 1$, afortiori $d + 3 \leqslant N_0 + 1$, which proves (4.5).

Let H_1 be the hypersurface of degree d which cuts out $\varphi(P(X)) \subset X$, and let $l \subset X$ be any line. Since $l \subset X$, we have $(H_1 \circ l)_{\mathbb{P}^5} = ((H_1 \cdot X) \cdot l)_X$. Furthermore $d = (H_1 \cdot l)_{\mathbb{P}^5}$, moreover $H_1 \cap X = \varphi(P(X))$. In summary:

(4.6) PROPOSITION: $d = (\varphi(P(X)) \circ l)_X$.

This concludes §4.

§5. The fundamental computational lemma (F.C.L.)

In this section we will arrive at a version of the F.C.L. in [6] for Φ_*: $H_2(\Omega_X, \mathbb{Q}) \to H_4(X, \mathbb{Q})$ where Φ_* is studied on the homology level via Poincaré duality. As in §4, X will be a generic quintic. Now recalling the diagram in (4.1) together with (4.2), there is a diagram:

$$\begin{array}{ccccc} & & \Omega_X \times \Omega_X & & \\ & & \cup & & \\ & & D & & \\ & \swarrow^{p} & & \searrow^{\psi} & \\ \Omega_X & & & & \Gamma \underset{\approx}{\overset{k}{\rightarrow}} \operatorname{sing}(\varphi(P(X))) \end{array} \tag{5.1}$$

Define $\Gamma_0 = \{ y \in \Gamma \mid k$ is regular at y & $k(y) \notin \operatorname{sing}(\operatorname{sing} \varphi(P(X)))\}$. Clearly Γ_0 is smooth and Zariski open in Γ. Next define $D_0 = \psi^{-1}(\Gamma_0)$, $\Sigma_0 = D - D_0$, and note that $j(D_0) = D_0$ and Σ_0 is closed in D. Note that $\Sigma = p(\Sigma_0) \subset \Omega_X$ is closed and of codimension $\geqslant 1$. Define $\Omega_{X,0} = \Omega_X - \Sigma$. We can desingularize the diagram in (5.1) to:

$$\begin{array}{ccc} & \tilde{D} & \\ \swarrow^{p} & & \searrow^{\sigma_1} \\ \Omega_X & & \tilde{\Gamma}, \end{array} \tag{5.2}$$

where all maps are morphisms, and $\tilde{D}$, $\tilde{\Gamma}$ are smooth. Diagrams (5.1) & (5.2) are analogous to the diagrams on *p*. 46&47 in [6], indeed we have even tried to retain similar notation. Let i_0: $\Omega_{X,0} \hookrightarrow \Omega_X$ be the inclusion, and set $H_2(\Omega_X, \mathbb{Q})_\Sigma = i_{0,*}(H_2(\Omega_{X,0}, \mathbb{Q})) \subset H_2(\Omega_X, \mathbb{Q})$. We can now state:

(5.3) Theorem (F.C.L.): *Let* $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})_\Sigma$. *Then* $(\Phi_*(\gamma_1) \cdot \Phi_*(\gamma_2))_X = (d - N_0)(\gamma_1 \cdot \gamma_2)_{\Omega_X} + (i\gamma_1 \cdot \gamma_2)_{\Omega_X}$.

Proof: Except for dimensions of cycles in question, the proof of (5.3) is formally identical to the proof of the F.C.L. in [6; p. 45], which begins on p. 46 of [6], and involves the integral invariants N_0, and d of (4.6).

(5.4) For the remainder of this section, we will occupy ourselves with the problem of reformulating (5.3) so as to not involve the particular algebraic cycle $\Sigma \subset \Omega_X$.

We will now fulfill a promise made earlier:

(5.5) Proposition: Φ_*: $H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$ *is surjective.*

Proof: We will use the notation following (0.2) where $\Delta \subset U_0$ is a polydisk centered at $0 \in \Delta$, $X = X_0 \in \coprod_{v \in \Delta} X_v$. Let i_v: $X_v \hookrightarrow \coprod_{v \in \Delta} X_v$ be the inclusion morphism. Let X be transcendentally generic. Now because Δ is uncountable, any $\gamma \in H^{2,2}(X, \mathbb{Q})$ will have a horizontal displacement in $\coprod_{v\in\Delta} H^4(X_v, \mathbb{Q})$ which is also of Hodge type (2, 2), i.e. $i_v^* \circ (i_0^*)^{-1}(\gamma) \in H^{2,2}(X_v, \mathbb{Q})$ for all $v \in \Delta$. However it is a general fact (using Lefschetz pencils) that such $\gamma \in \mathbb{Q}\omega \wedge \omega$, hence X transcendentally generic $\Rightarrow H^{2,2}(X, \mathbb{Q}) = \mathbb{Q}\omega \wedge \omega$. This means that the only algebraic cocycle in $H^4(X, \mathbb{Q})$ is a $\mathbb{Q}$ multiple of $\omega \wedge \omega$. Since Φ_* preserves algebraicity, clearly Φ_* is surjective for transcendental X. Now it can be easily seen that the cylinder homomorphisms $\Phi_{v,*}$: $H^2(\Omega_{X_v}, \mathbb{C}) \to H^4(X_v, \mathbb{C})$ piece together to form a morphism $\overline{\Phi}$: $\coprod_{v\in\Delta} H^2(\Omega_{X_v}, \mathbb{C}) \to \coprod_{v\in\Delta} H^4(X_v, \mathbb{C})$ of (trivial) analytic vector bundles over Δ. From the above discussion $\overline{\Phi}$ is fiberwise surjective on a uncountable dense subset of Δ, hence by analytic considerations, must be surjective over Δ. Q.E.D.

Let $k_0: \Sigma \hookrightarrow \Omega_X$ be the inclusion. Our next result is:

(5.6) PROPOSITION:

$$H_2(\Omega_X, \mathbb{Q})_\Sigma = \left\{ \begin{matrix} \gamma \in H_2(\Omega_X, \mathbb{Q}) \mid (\gamma \cdot k_{0,*}(\alpha))_{\Omega_X} = 0 \\ \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}) \end{matrix} \right\}.$$

PROOF: It follows from [3; ch. 27] that there is a commutative diagram: (for our purposes $H^2(\Omega_X, \mathbb{C})$ will be viewed as deRham cohomology)

$$\begin{array}{ccccc} & & H^2(\Omega_X, \mathbb{Q}) & \overset{k_0^*}{\to} & H^2(\Sigma, \mathbb{Q}) \\ & & D_P \uparrow \wr & & D_A \uparrow \wr \\ H_2(\Omega_{X,0}, \mathbb{Q}) & \overset{i_{0,*}}{\to} & H_2(\Omega_X, \mathbb{Q}) & \overset{f_*}{\to} & H_2(\Omega_X, \Omega_{X,0}) \end{array} \tag{5.7}$$

where D_P and D_A are respectively Poincaré and Alexander duality. Now for

$$\begin{aligned} &\gamma \in H_2(\Omega_X, \mathbb{Q}),\ f_*(\gamma) \\ &\quad = 0 \Leftrightarrow k_0^* \circ D_P(\gamma) = 0 \\ &\qquad \Leftrightarrow \int_{k_{0,*}(\alpha)} D_P(\gamma) = 0 \quad \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}) \\ &\qquad \Leftrightarrow (\gamma \cdot k_{0,*}(\alpha))_{\Omega_X} = 0 \quad \text{for all } \alpha \in H_2(\Sigma, \mathbb{Q}). \end{aligned}$$

Now recall the Lefschetz (1, 1) theorem which states that $H^{1,1}(\Omega_X, \mathbb{Z})$ is generated by the fundamental classes of algebraic curves in Ω_X. We introduce the following notation:

(5.8) Definition:

(i) The transcendental cohomology, $H_T^2(\Omega_X, \mathbb{Q})$, is given by: $H_T^2(\Omega_X, \mathbb{Q}) = \{\gamma \in H^2(\Omega_X, \mathbb{Q}) \mid \gamma \wedge H^{1,1}(\Omega_X, \mathbb{Q}) = 0\}$.

(ii) $H_\Sigma^2(\Omega_X, \mathbb{Q}) = D_P(H_2(\Omega_X, \mathbb{Q})_\Sigma)$.

(5.9) Corollary: $H_T^2(\Omega_X, \mathbb{Q}) \subset H_\Sigma^2(\Omega_X, \mathbb{Q})$.

Proof: Compare (5.6) to (5.8)(i).

According to (5.9), it is clear that one can formulate a version of (5.3) for cocycles in $H_T^2(\Omega_X, \mathbb{Q})$, however there is another subspace in $H_\Sigma^2(\Omega_X, \mathbb{Q})$ which contains $H_T^2(\Omega_X, \mathbb{Q})$ and best suits our purposes. Recall the definition of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ in (0.3). There is an equivalent definition of $H_A^{1,1}(\Omega_X, \mathbb{Q})$ using the notation in the proof of (5.5) and the Lefschetz (1, 1) Theorem.

(5.10) Definition:

(i) $$H_A^{1,1}(\Omega_X, \mathbb{Q}) = \left\{ \begin{array}{l|l} \text{algebraic cocycles} & \text{a horizontal} \\ \gamma \in H^2(\Omega_X, \mathbb{Q}) & \text{deformation of } \gamma \text{ in} \\ & \coprod_{v \in \Delta} H^2(\Omega_{X_v}, \mathbb{Q}) \text{ is} \\ & \text{algebraic} \end{array} \right\}.$$

(ii) $$H_P^2(\Omega_X, \mathbb{Q}) = \{\gamma \in H^2(\Omega_X, \mathbb{Q}) \mid \gamma \wedge H_A^{1,1}(\Omega_X, \mathbb{Q}) = 0\}.$$

(5.11) Remarks: From the general theory of Hilbert schemes, $H_A^{1,1}(\Omega_X, \mathbb{Q})$ is independent of the choice of polydisk $\Delta \subset U_0$, dim $H_A^{1,1}(\Omega_{X_v}, \mathbb{Q})$ is constant over $v \in U_0$, and $H_A^{1,1}(\Omega_X, \mathbb{Q}) = H^{1,1}(\Omega_X, \mathbb{Q})$ for transcendentally generic X.

(5.12) Proposition: $H_T^2(\Omega_X, \mathbb{Q}) \subset H_P^2(\Omega_X, \mathbb{Q}) \subset H_\Sigma^2(\Omega_X, \mathbb{Q})$.

Proof: The inclusion $H_T^2(\Omega_X, \mathbb{Q}) \subset H_P^2(\Omega_X, \mathbb{Q})$ is obvious from the definitions, moreover is an equality for transcendentally generic X ((5.11)). Next as X varies, i.e. $v \in U_0$ varies, Σ also varies algebraically, hence $[\Sigma] \in H_A^{1,1}(\Omega_X, \mathbb{Q})$, therefore the second inclusion follows from (5.6), (5.8)(ii)&(5.10)(ii).

(5.13) Remarks:

(i) The well known properties of the pairing $H^2(\Omega_X, \mathbb{C}) \times H^2(\Omega_X, \mathbb{C}) \to \mathbb{C}$ imply $H^2(\Omega_X, \mathbb{Q}) = H_P^2(\Omega_X, \mathbb{Q}) \oplus H_A^{1,1}(\Omega_X, \mathbb{Q})$ is an orthogonal decomposition under $\wedge$.

(ii) As X varies, i.e. $v \in U_0$ varies, the incidence correspondence $D \subset \Omega_X \times \Omega_X$ also varies algebraically. Therefore $i(H_A^{1,1}(\Omega_X, \mathbb{Q})) \subset H_A^{1,1}(\Omega_X, \mathbb{Q})$.

We need the following:

(5.14) LEMMA: $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \subset \mathrm{Prim}^4(X, \mathbb{Q})$.

PROOF: Let H_1, H_2 be generic hyperplanes in $\mathbb{P}^5$, $X_s = H_1 \cap H_2 \cap X$, $Y_s = X_s \cap \varphi(P(X))$. Note that $[X_s] = \omega \wedge \omega \in H^{2,2}(X, \mathbb{Z})$, and Y_s is a curve in $S = H_1 \cap \varphi(P(X))$. By (3.9), Y_s induces a corresponding curve C_1 in Ω_X, given by the formula $C_1 = \rho_* \circ \varphi^*(Y_s)$. Since Y_s varies algebraically as X varies, clearly $[C_1] \in H_A^{1,1}(\Omega_X, \mathbb{Q})$. Now let $\gamma \in H_2(\Omega_X, \mathbb{Q})$ be given so that $D_P(\gamma) \in H_P^2(\Omega_X, \mathbb{Q})$. From the techniques of the proof of (5.6), it is clear that γ can be chosen to be supported on Ω_X-supp(C_1). Therefore, on the cycle level, $\Phi_*(\gamma) \cap Y_s = 0$, hence $(\Phi_*(\gamma) \cdot X_s)_X = 0$. By translating this in terms of cohomology, $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \wedge \omega \wedge \omega = 0$. But $\wedge\,\omega$: $H^6(X, \mathbb{Q}) \to H^8(X, \mathbb{Q})$ is an isomorphism, hence $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \wedge \omega = 0$, i.e. $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) \subset \mathrm{Prim}^4(X, \mathbb{Q})$. Q.E.D.

There is another needed result:

(5.15) LEMMA: *Let* $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})$. *Then* $(i\gamma_1 \cdot \gamma_2)_{\Omega_X} = (\gamma_1 \cdot i\gamma_2)_{\Omega_X}$.

PROOF: Using the notation of (5.2), together with the definition of j, there is a commutative diagram:

$$\begin{array}{ccc} \tilde{D} & \xrightarrow{\tilde{j}} & \tilde{D} \\ g\downarrow & & \downarrow g \\ D & \xrightarrow{j} & D \end{array} \qquad (5.16)$$

where $\tilde{j}$ is biregular, and g is a birational morphism. Define $\tilde{p} = p \circ g$: $\tilde{D} \to \Omega_X$. It is easy to verify that the correspondence defined by $\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*$ is the same as $p_* \circ j \circ p^* = D$, hence $\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^* = i$. Now by applying the projection formula 3 times we have: (Note $\tilde{j}^* = \tilde{j}_*$)

$$\begin{aligned}(i\gamma_1 \cdot \gamma_2)_{\Omega_X} &= (\tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*(\gamma_1) \cdot \gamma_2)_{\Omega_X} \\ &= (\gamma_1 \cdot \tilde{p}_* \circ \tilde{j}_* \circ \tilde{p}^*(\gamma_2))_{\Omega_X} \\ &= (\gamma_1 \cdot i\gamma_2)_{\Omega_X}.\end{aligned}$$

(5.17) COROLLARY: $i(H_P^2(\Omega_X, \mathbb{Q})) \subset H_P^2(\Omega_X, \mathbb{Q})$.

PROOF: Otherwise there exists $\gamma_1 \in H_P^2(\Omega_X, \mathbb{Q})$, $\gamma_2 \in H_A^{1,1}(\Omega_X, \mathbb{Q})$ such that $i(\gamma_1) \wedge \gamma_2 \neq 0$. But $i(\gamma_1) \wedge \gamma_2 = \gamma_1 \wedge i(\gamma_2)$ by (5.15) $= 0$ by (5.13), a contradiction.

We can formulate (5.3) for $H_P^2(\Omega_X, \mathbb{Q})$:

(5.18) PROPOSITION: *Given* $\gamma_1, \gamma_2 \in H_2(\Omega_X, \mathbb{Q})$ *with* $D_P(\gamma_1)$, $D_P(\gamma_2)$ *in* $H_P^2(\Omega_X, \mathbb{Q})$. *Then* $(\Phi_*(\gamma_1)\cdot\Phi_*(\gamma_2))_X = (d-N_0)(\gamma_1\cdot\gamma_2)_{\Omega_X} + (i\gamma_1\cdot\gamma_2)_{\Omega_X}$.

PROOF: Use (5.3) & (5.12).

Combining everything together so far we arrive at the final result of this section:

(5.19) THEOREM. *The following subspaces are the same*:

(*i*) $S_1 = \{\gamma \in H_P^2(\Omega_X, \mathbb{Q}) \mid \Phi_*(\gamma) = 0\}$

(*ii*) $S_2 = \{\gamma \in H_P^2(\Omega_X, \mathbb{Q}) \mid (d-N_0)\gamma + i(\gamma) = 0\}$

(*iii*) $S_3 = (i + 119\cdot I)H_P^2(\Omega_X, \mathbb{Q})$.

PROOF: $S_1 = S_2$ follows immediately from (5.18) and (5.5). Next (3.11), (5.14), (5.17) imply $S_3 \subset S_1$. We first justify the claim: $\{\ker(i+119\cdot I)\} \cap S_1 = 0$. If $\gamma \in \ker(i+119\cdot I)\} \cap S_1$, then $i(\gamma) + 119\gamma = (d-N_0)\gamma + i(\gamma) = 0$, hence $(119-(d-N_0))\gamma = 0$, $\Rightarrow \gamma = 0$ by (4.5), which proves the claim. Using the claim, it is clear that the homomorphism $(i+119\cdot I)$: $S_1 \to S_3$ is injective, hence an isomorphism as $S_3 \subset S_1$. (5.19) now follows.

§6. A quadratic relation and the proof of the main theorem

We now attend to the proof of the main theorem ((0.6)). Let $r = d - N_0$, and set $Q(i) = (rI+i)(i+119\cdot I) = i^2 + (119+r)i + r\cdot 119\cdot I$. We prove:

(6.1) PROPOSITION:

(*i*) $Q(i)$: $H_P^2(\Omega_X, \mathbb{Q}) \to H_P^2(\Omega_X, \mathbb{Q})$ *is the zero morphism.*

(*ii*) i: $H_P^2(\Omega_X, \mathbb{Q}) \to H_P^2(\Omega_X, \mathbb{Q})$ *is an isomorphism.*

PROOF: Part (i) is an immediate consequence of (5.19). For part (ii), note that $i(\gamma) = Q(i)(\gamma) = 0 \Rightarrow r\cdot 199\gamma = 0$, afortiori $\gamma = 0$. Q.E.D.

Note that for any $\gamma \in H_A^{1,1}(\Omega_X, \mathbb{Q})$, $\Phi_*(\gamma)$ *has the property that under a horizontal displacement in* $\coprod_{v\in\Delta} H^4(X_v, \mathbb{Q})$, $\Phi_*(\gamma)$ is still algebraic. One concludes from the proof of (5.5) that $\Phi_*(\gamma) \in \mathbb{Q}\,\omega\wedge\omega$. Therefore $\Phi_*(H_A^{1,1}(\Omega_X, \mathbb{Q})) = \mathbb{Q}\,\omega\wedge\omega$, hence:

(6.2) COROLLARY: $\Phi_*(H_P^2(\Omega_X, \mathbb{Q})) = \mathrm{Prim}^4(X, \mathbb{Q})$.

PROOF: Use the above remark, (5.5)&(5.14).

Combining (6.2) with (5.19)&(6.1), we arrive at our main result.

(6.3) THEOREM:

(*i*) i *respects the decomposition* $H^2(\Omega_X, \mathbb{Q}) = H_P^2(\Omega_X, \mathbb{Q}) \oplus H_A^{1,1}(\Omega_X, \mathbb{Q})$, *moreover* i: $H_P^2(\Omega_X, \mathbb{Q}) \to H_P^2(\Omega_X, \mathbb{Q})$ *is an isomorphism.*