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## BEST APPROXIMANTS FROM NON-ARCHIMEDEAN STONE-WEIERSTRASS SUBSPACES

Maria Zoraide M. Costa Soares

### 0. Introduction

Let  $X$  be a locally compact Hausdorff space, let  $(F, |\cdot|)$  be a non-archimedean non-trivially valued division ring and  $(E, \|\cdot\|)$  a normed space over  $(F, |\cdot|)$ .

We say that  $g: X \rightarrow E$  vanishes at infinity if, for each  $\epsilon > 0$ , the set  $\{x \in X; \|g(x)\| \geq \epsilon\}$  is compact.

We denote by  $\mathcal{C}(X; E)$  the vector space of all continuous functions from  $X$  into  $E$ .  $\mathcal{C}_0(X; E)$  will denote the vector space of all continuous functions which vanish at infinity, equipped with the norm  $f \mapsto \|f\| = \sup\{\|f(x)\|; x \in X\}$ .

The vector subspace of  $\mathcal{C}(X; F)$  consisting of all continuous functions  $f: X \rightarrow F$  such that  $f(X)$  has compact closure in  $F$ , is denoted by  $\mathcal{C}^*(X; F)$ .

If  $\Delta$  is the equivalence relation determined by  $A \subset \mathcal{C}(X; F)$ ,  $\Delta(x) = \{y \in X; a(y) = a(x) \text{ for all } a \in A\}$  is the  $\Delta$ -equivalence class containing  $x$ .

If  $Y \subset X$  is any non-empty set, we denote by  $f|_Y$  the mapping  $y \in Y \rightarrow f(y)$ . If  $\mathcal{F}$  is any family of mappings  $f: X \rightarrow S$ , we denote by  $\mathcal{F}|_Y$  the set  $\{f|_Y; f \in \mathcal{F}\}$ .

In this paper, we extend some results of Machado and Prolla [3] to the case of non-archimedean normed spaces, and other results of Prolla [4].

If  $A \subset \mathcal{C}^*(X; F)$  is a subalgebra and  $W \subset \mathcal{C}_0(X; E)$  is a vector subspace which is an  $A$ -module, we proved in [5] that for each  $f \in \mathcal{C}_0(X; E)$ ,

$$\text{dist}(f; W) = \sup_{x \in X} \text{dist}(f|_{\Delta(x)}; W|_{\Delta(x)}).$$

We extend this “localization formula” for set-valued mappings under an upper semicontinuity hypothesis (see Theorem 1.7 below) generalizing a result of Prolla [4].

In Approximation Theory, given a normed space  $(N, \|\cdot\|)$  and a non-empty subset  $W \subset N$ , there are two main problems. The first one is to characterize the closure of  $W$  in  $N$ , i.e., the set of all  $f \in N$  such that  $\text{dist}(f; W) = 0$ . When  $N$  is a normed space of functions, this leads to

Stone-Weierstrass type theorems by choosing appropriate algebraic conditions on  $W$ . (For example,  $W$  is an  $A$ -module, etc.).

The second problem arises when  $\text{dist}(f; W) > 0$ . Does there exist  $g \in W$  such that

$$\|f - g\| = \text{dist}(f; W)?$$

More generally, if instead of a single  $f$  one deals with a bounded set  $B \subset N$ , does there exist  $g \in W$  such that

$$\sup_{f \in B} \|f - g\| = \inf_{w \in W} \sup_{f \in B} \|f - w\|?$$

Such a  $g$ , when it exists is called a Chebyshev center of  $B$  in  $W$ . We present some results (see Theorems 3.8 and 3.9) when  $N$  is  $\mathcal{C}_0(X; E)$  and  $W$  is a so-called Stone-Weierstrass subspace. (see Olech [2]).

When  $W$  is a  $\mathcal{C}^*(X; F)$ -module (or more generally an  $A$ -module, for some separating subalgebra  $A \subset \mathcal{C}^*(X; F)$ ) it is natural to ask whether approximation properties of  $W(x) = \{w(x); w \in W\}$  in  $E$ , for every  $x \in X$ , will ensure the same for  $W$  in  $\mathcal{C}_0(X; E)$ . Theorem 3.10 and 3.11 are along this line: in 3.10 one assumes that, for each  $s \in X$ , and  $v \in E$  there is some element  $w(x)$  such that  $\|v - w(x)\| = \text{dist}(v; W(x))$ . Theorem 3.11 deals with the analogous question for Chebyshev centers.

This work represents part of the author's dissertation at the Universidade de Campinas.

## 1. Stone-Weierstrass theorems

Let  $X, (F, |\cdot|)$  and  $(E, \|\cdot\|)$  be as in the introduction.

1.1. DEFINITION: A carrier  $\varphi$  from  $X$  to  $E$  is a mapping from  $X$  into the non-empty subsets of  $E$ .

1.2. DEFINITION: Let  $\varphi$  be a carrier from  $X$  into  $E$ . We define the distance of  $\varphi$  from a function  $g \in \mathcal{C}_0(X; E)$  to be

$$\text{dist}(\varphi; g) = \sup_{x \in X} \left\{ \sup_{y \in \varphi(x)} \|y - g(x)\| \right\}$$

and the distance of  $\varphi$  from a subset  $W \subset \mathcal{C}_0(X; E)$  to be

$$\text{dist}(\varphi; W) = \inf\{\text{dist}(\varphi; g); g \in W\}.$$

1.3. DEFINITION: Let  $\varphi$  a carrier of  $X$  into  $E$ . We say that  $\varphi$  is upper semicontinuous (u.s.c.) with respect to  $W \subset \mathcal{C}_0(X; E)$ , if given  $w \in W$  and

$r > 0$ , for each  $x \in X$  such that  $\varphi(x) \in B(w(x); r)$  and each  $\epsilon > 0$ , there is a neighborhood  $U$  of  $x$  such that  $\varphi(y) \in B(w(y); r + \epsilon)$  for all  $y \in U$ . (If  $v \in E$  and  $s > 0$  we denote by  $B(v; s)$  the set  $\{u \in E; \|u - v\| < s\}$ ).

1.4. EXAMPLE: If  $f \in \mathcal{C}_0(X; E)$ , then  $\varphi(x) = \{f(x)\}$ ,  $x \in X$ , is upper semicontinuous with respect to any  $W \subset \mathcal{C}_0(X; E)$ . Indeed, for each  $w \in W$  and  $r > 0$ , the set

$$\{x \in X; \varphi(x) \subset B(w(x); r)\} = \{x \in X; \|f(x) - w(x)\| < r\}$$

is open.

1.5. EXAMPLE: Let  $N \subset \mathcal{C}_0(X; E)$  be a equicontinuous subset. Define a carrier  $\varphi$  from  $X$  into  $E$  by setting

$$\varphi(x) = \{f(x); f \in N\},$$

for all  $x \in X$ . We claim that  $\varphi$  is u.s.c. with respect to any  $W \subset \mathcal{C}_0(X; E)$ . Indeed, let  $w \in W$ ,  $r > 0$  and  $x \in X$  with  $\varphi(x) \subset B(w(x); r)$  be given. Let  $\epsilon > 0$ . If  $N$  is equicontinuous then  $N - \{w\}$  is equicontinuous too, and there is a neighborhood  $U$  of  $x$  such that  $\|f(y) - w(y) - (f(x) - w(x))\| < \epsilon$  for all  $y \in U$ .

Hence, for all  $y \in U$

$$\begin{aligned} \|f(y) - w(y)\| &= \|f(y) - w(y) - (f(x) - w(x)) \\ &\quad + (f(x) - w(x))\| \\ &\leq \|f(y) - w(y) - (f(x) - w(x))\| \\ &\quad + \|f(x) - w(x)\| \\ &< \epsilon + r. \end{aligned}$$

1.6. DEFINITION: Let  $\varphi$  be a carrier of  $X$  into  $E$  and let  $W \subset \mathcal{C}_0(X; E)$ . We say that  $\varphi$  *vanishes at infinity with respect to  $W$* , if for each  $w \in W$  and  $\epsilon > 0$  the set

$$\{x \in X; \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset\}$$

is relatively compact, i.e. has compact closure.

1.7. THEOREM: Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$ ; let  $A \subset \mathcal{C}^*(X; F)$  be a subalgebra and  $W \subset \mathcal{C}_0(X; E)$  a vector subspace which is an  $A$ -module. For any carrier  $\varphi$  of  $X$  into  $E$  which is

upper semicontinuous and vanishes at infinity with respect to  $W$ , we have:

$$\text{dist}(\varphi; W) = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

PROOF: Let

$$\lambda = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

We always have  $\lambda \leq \text{dist}(\varphi; W)$ .

Let  $\epsilon > 0$  and  $x \in X$  be given; there exists  $g_x \in W$  such that

$$\text{dist}(\varphi|_{\Delta(x)}; g_x|_{\Delta(x)}) < \lambda + \epsilon.$$

This implies that

$$\|t - g_x(y)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(y) \text{ and } y \in \Delta(x).$$

Since  $\varphi$  is upper semicontinuous with respect to  $W$ , there is an open neighborhood  $U_x$  of  $x$  such that

$$\|t - g_x(z)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(z) \text{ and } z \in U_x.$$

Clearly,  $\Delta(x) \subset U_x$ .

Since  $\varphi$  vanishes at infinity with respect to  $W$ , the closure  $K_x$  of

$$S_x = \{y \in X; \varphi(y) \cap (E \setminus B(g_x(y); \lambda + \epsilon)) \neq \emptyset\}$$

is compact. We claim that  $\Delta(x) \cap K_x = \emptyset$ . Indeed, assume  $z \in \Delta(x) \cap K_x$ . Since  $\Delta(x) \subset U_x$  and  $K_x$  is the closure of  $S_x$ , there is some  $y \in U_x \cap S_x$ . But  $\varphi(y) \subset B(g_x(y); \lambda + \epsilon)$  for all  $y \in U_x$  and so  $y$  cannot be in  $S_x$ .

By Lemma 2.4, [5], there exists a finite set  $\{x_1, x_2, \dots, x_n\} \subset X$  such that for each  $0 < \delta < 1$ , there are functions  $a_1, a_2, \dots, a_n \in A_0$  satisfying:

- (1)  $|a_i(x)| \leq 1$  for all  $x \in X; i = 1, \dots, n$ ;
- (2)  $|a_i(t)| < \delta$  for all  $t \in K_{x_i}; i = 1, \dots, n$ ;
- (3)  $\sum_{i=1}^n a_i(x) = 1$  for all  $x \in X$ ;

where  $A_0$  is the subalgebra generated by  $A$  and the constant functions.

We choose  $\delta > 0$  such that

$$\delta \cdot \max_{1 \leq i \leq n} \|t - g_{x_i}(x)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(x)$$

and to this  $\delta$  let  $a_1, a_2, \dots, a_n \in A$  be given satisfying (1) to (3).

Define

$$g = \sum_{i=1}^n a_i g_{x_i}.$$

Then  $g \in W$ , and for each  $x \in X$  and  $t \in \varphi(x)$ , we have:

$$\begin{aligned} \|t - g(x)\| &= \left\| \sum_{i=1}^n a_i(x)t - \sum_{i=1}^n a_i(x)g_{x_i}(x) \right\| \\ &= \left\| \sum_{i=1}^n a_i(x)(t - g_{x_i}(x)) \right\|. \end{aligned}$$

If  $x \in K_{x_i}$ , then

$$\begin{aligned} \|a_i(x)(t - g_{x_i}(x))\| &< \delta \cdot \|t - g_{x_i}(x)\| \\ &\leq \delta \cdot \max_{1 \leq i \leq n} \|t - g_{x_i}(x)\| < \lambda + \epsilon. \end{aligned}$$

If  $x \notin K_{x_i}$ , then

$$\|a_i(x)(t - g_{x_i}(x))\| \leq 1 \cdot \|t - g_{x_i}(x)\| < \lambda + \epsilon.$$

Hence, for all  $x \in X$  and  $t \in \varphi(x)$ ,

$$\begin{aligned} \|t - g(x)\| &= \left\| \sum_{i=1}^n a_i(x)(t - g_{x_i}(x)) \right\| \\ &\leq \max_{1 \leq i \leq n} \|a_i(x)(t - g_{x_i}(x))\| \\ &< \lambda + \epsilon. \end{aligned}$$

Then,

$$\text{dist}(\varphi; g) \leq \lambda + \epsilon.$$

A fortiori,  $\text{dist}(\varphi; W) \leq \lambda + \epsilon$ . Since  $\epsilon > 0$  was arbitrary,

$$\text{dist}(\varphi; W) \leq \lambda = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

**1.8. DEFINITION:** A family of functions  $N \subset \mathcal{C}_0(X; E)$  is said to *vanish collectively at infinity* if, for each  $\epsilon > 0$ , there is a compact subset  $K \subset X$  such that  $\|f(x)\| < \epsilon$  for all  $x \notin K$  and  $f \in N$ .

1.9. **EXAMPLE:** Let  $N \subset \mathcal{C}_0(X; E)$  be a totally bounded subset. Then  $N$  vanishes collectively at infinity. Indeed, let  $\epsilon > 0$  be given. There exists a finite set  $\{f_1, f_2, \dots, f_n\} \subset N$  such that, for each  $f \in N$ , there is  $1 \leq i \leq n$  with  $\|f - f_i\| < \epsilon/2$ . For each  $1 \leq i \leq n$ , there is a compact subset  $K_i \subset X$  such that  $\|f_i(x)\| < \epsilon/2$  for all  $x \notin K_i$ . Let  $K$  be the union  $K_1 \cup K_2 \cup \dots \cup K_n$ . Then for all  $x \notin K$  and  $f \in N$ ,  $\|f(x)\| < \epsilon$ .

1.10. **PROPOSITION:** Let  $N \subset \mathcal{C}_0(X; E)$  be a family which vanishes collectively at infinity and let  $W \subset \mathcal{C}_0(X; E)$ . The carrier

$$\varphi(x) = \{f(x); f \in N\}, \quad x \in X,$$

vanishes at infinity with respect to  $W$ .

**PROOF:** If  $N \subset \mathcal{C}_0(X; E)$  vanishes collectively at infinity and  $w \in \mathcal{C}_0(X; E)$ , then  $G = \{f - w; f \in N\}$  vanishes collectively at infinity too.

Let  $\epsilon > 0$  and  $K \subset X$  be a compact set such that

$$\|f(x) - w(x)\| < \epsilon$$

for all  $x \notin K$  and  $f \in N$ .

Then  $\varphi(x) \subset B(w(x); \epsilon)$  for all  $x \notin K$  and

$$X \setminus \{x \in X; \varphi(x) \subset B(w(x); \epsilon)\} \subset K$$

and so the set

$$\{x \in X; \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset\}$$

is relatively compact.

1.11. **THEOREM:** Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, \|\cdot\|)$ ; let  $A \subset \mathcal{C}^*(X; F)$  be a subalgebra; let  $W \subset \mathcal{C}_0(X; E)$  be a vector subspace which is an  $A$ -module; and  $N \subset \mathcal{C}_0(X; E)$  a totally bounded subset and define for all  $x \in X$ ,  $\varphi(x) = \{f(x); f \in N\}$ . Then,

$$\text{dist}(\varphi; W) = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

**PROOF:** By Example 1.5,  $\varphi$  is upper semicontinuous, and by Example 1.9,  $N$  vanishes collectively at infinity and by Proposition 1.10,  $\varphi$  vanishes at infinity with respect to any  $W \subset \mathcal{C}_0(X; E)$ . It remains to apply Theorem 1.7.

## 2. Chebyshev centers

2.1. **DEFINITION:** Let  $(N, \|\cdot\|)$  be a normed space over  $(F, |\cdot|)$ ,  $W \subset N$  and  $B$  be a non-empty bounded subset of  $N$ . The *relative Chebyshev*

radius of  $B$  (with respect to  $W$ ) is, by definition, the number

$$\text{rad}_W(B) = \inf \left\{ \sup_{f \in B} \|w - f\|; w \in W \right\}.$$

If  $W = N$ , then we write

$$\text{rad}_N(B) = \text{rad}(B)$$

and call it the *Chebyshev radius of  $B$* .

The elements  $w_0 \in W$  where the infimum is attained are called *relative Chebyshev centers of  $B$  (with respect to  $W$ )*, and we denote by  $\text{cent}_W(B)$  the set of all such  $w_0 \in W$ .

If  $W = N$ , then we write  $\text{cent}_N(B) = \text{cent } B$  and call it the set of *Chebyshev centers of  $B$* .

We say that  $W$  has the *relative Chebyshev center property in  $N$*  if  $\text{cent}_W(B) \neq \emptyset$  for all non-empty bounded sets  $B \subset N$ .

When  $W = N$ , and  $\text{cent}(B) \neq \emptyset$  for every non-empty bounded subset  $B \subset N$ , i.e. if  $N$  has the relative Chebyshev center property in  $N$ , we say that  $N$  *admits Chebyshev centers*.

Let  $M \subset N$  be a closed linear subspace and  $f \in N$ . A *best approximant* of  $f$  in  $M$  is any element  $g \in M$  such that

$$\|f - g\| = \inf_{h \in M} \|f - h\| = \text{dist}(f; M).$$

We denote by  $P_M(f)$  the set of all best approximants of  $f$  in  $M$ . If  $P_M(f)$  contains at least one element for all  $f \in N$ ,  $M$  is called *proximal*.

The main problems of *best (simultaneous) approximation theory* are the following (in decreasing order of generality):

**PROBLEM I:** Let  $W \subset N$  be given. Determine if  $W$  has the relative Chebyshev center property in  $N$ . In particular, determine if  $N$  admits Chebyshev centers.

**PROBLEM II:** Let  $W \subset N$  be given. Determine the class  $B$  of all non-empty bounded sets  $B \subset N$  such that  $\text{cent}_W(B) \neq \emptyset$ .

**PROBLEM III:** Let  $W \subset N$  be given. Determine if  $W$  is proximal in  $N$ , i.e., determine if the class  $B$  of Problem II contains all sets of the form  $B = \{f\}$ ,  $f \in N$ .

Suppose that  $N$  is  $\mathcal{C}_0(X; E)$  equipped with the sup-norm and let  $W \subset \mathcal{C}_0(X; E)$ . To each non-empty and bounded set  $B \subset \mathcal{C}_0(X; E)$ , we



define the carrier

$$\varphi_B(x) = \{f(x); f \in B\}$$

for all  $x \in X$ . It follows that

$$\text{dist}(\varphi_B; W) = \text{rad}_W(B).$$

Consequently, by Theorem 1.11, we have the following formula of localizability for the Chebyshev radius.

**2.2. THEOREM:** *Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$ ; let  $A \subset \mathcal{C}^*(X; F)$  be a subalgebra and  $W \subset \mathcal{C}_0(X; E)$  a vector subspace which is an  $A$ -module. For each non-empty and totally bounded subset  $B \subset \mathcal{C}_0(X; E)$  we have*

$$\text{rad}_W(B) = \sup_{x \in X} \text{rad}_W|_{\Delta(x)}(B|_{\Delta(x)}).$$

**2.3. DEFINITION:** Let  $\Delta$  be an equivalence relation in  $X$ . We say that a carrier  $\varphi$  from  $X$  into  $E$  is  $\Delta$ -bounded if

$$\varphi(\Delta(x)) = \cup \{ \varphi(t); t \in \Delta(x) \}$$

is a bounded subset of  $E$ , for all  $x \in X$ . Let us define

$$\delta(\varphi) = \sup_{x \in X} \text{rad}(\varphi(\Delta(x)))$$

**2.4. THEOREM:** *Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$  and  $A \subset \mathcal{C}^*(X; F)$  a subalgebra. Let  $W \subset \mathcal{C}_0(X; E)$  be an  $A$ -module such that for each  $x \in X$  and  $z \in E$ , there is some  $w \in W$  such that  $w(t) = z$  for all  $t \in \Delta(x)$ . Then for any  $\Delta$ -bounded carrier  $\varphi$  from  $X$  into  $E$  which is upper semicontinuous and vanishes at infinity with respect to  $W$ , we have:*

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

**PROOF:** By Theorem 1.7, we have:

$$\begin{aligned} \text{dist}(\varphi; W) &= \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}) \\ &= \sup_{x \in X} \inf_{w \in W} \text{dist}(\varphi|_{\Delta(x)}; w) \\ &= \sup_{x \in X} \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\|. \end{aligned}$$

Let  $x \in X$ . For each  $z \in E$ , choose  $w_z \in W$  such that  $w_z(t) = z$  for all  $t \in \Delta(x)$ . Then

$$\begin{aligned} & \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \\ & \leq \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w_z(t)\| \\ & = \sup_{y \in \varphi(\Delta(x))} \|y - z\|. \end{aligned}$$

Since  $z \in E$  was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \leq \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \|y - z\|.$$

Hence,

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

### 3. Stone-Weierstrass subspaces

3.1. DEFINITION: A vector subspace  $W \subset \mathcal{C}_0(X; E)$  is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space  $Y$  and a proper continuous surjection  $\pi: X \rightarrow Y$  such that

$$W = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

We denote by  $W_\pi$  the Stone-Weierstrass subspace determined by  $\pi$ .

If  $W_\pi \subset \mathcal{C}_0(X; E)$  is a Stone-Weierstrass subspace, then

$$A_\pi = \{\varphi \circ \pi; \varphi \in \mathcal{C}^*(X; F)\}$$

is a subalgebra of  $\mathcal{C}^*(X; F)$  which contains the constants and

$$\{\pi^{-1}(y); y \in Y\}$$

is the set of equivalence classes modulo  $A_\pi$ . Therefore,  $W_\pi$  is an  $A_\pi$ -module.

Clearly  $W_\pi$  is closed in  $\mathcal{C}_0(X; E)$ .

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that  $\Delta(W_\pi) \subset W_\pi$ , where  $\Delta(W_\pi)$  is the Stone-Weierstrass hull of  $W_\pi$  in  $\mathcal{C}_0(X; E)$ .

Let  $f \in \Delta(W_\pi)$ . We will prove that  $f$  is constant on the sets  $\pi^{-1}(y)$  for all  $y \in Y$ .

Let  $t$  and  $t'$  be in  $X$  such that  $\pi(t) = \pi(t')$ . Then  $g(t) = g(t')$  for all  $g \in W_\pi$ . Then, the pair  $(t, t') \in \Delta_{W_\pi}$ .

If  $\delta(t, t') = 0$  then  $\delta_{t|W_\pi} = \delta_{t'|W_\pi} = 0$  and by hypothesis  $f \in \Delta(W_\pi)$ , then we have  $f(t) = 0 \cdot f(t') = 0$ .

If  $\delta(t, t') = 1$  then  $0 \neq \delta_{t|W_\pi} = \delta_{t'|W_\pi}$  and since  $f \in \Delta(W_\pi)$  we have  $f(t) = 1 \cdot f(t') = f(t')$ .

Therefore,  $f \in W_\pi$ .

Let  $f \in \mathcal{C}_0(X; E)$  be given. Since  $\pi$  is proper,  $\pi^{-1}(y)$  is compact and then  $f(\pi^{-1}(y))$  is compact, hence bounded in  $E$ , for each  $y \in Y$ . Let us define

$$\delta(f) = \sup_{y \in Y} \text{rad}(f(\pi^{-1}(y))).$$

If  $w \in W_\pi$  then

$$\|f - w\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t) - w(t)\| \geq \delta(f).$$

Hence

$$\delta(f) \leq \text{dist}(f; W_\pi).$$

**3.2. THEOREM:** *Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$  and  $W_\pi \subset \mathcal{C}_0(X; E)$  a Stone-Weierstrass subspace. Then, for all  $f \in \mathcal{C}_0(X; E)$*

$$\text{dist}(f; W_\pi) = \delta(f).$$

**PROOF:** By Theorem 2.4,  $\text{dist}(f; W_\pi) \leq \delta(f)$  and by remarks made before we have  $\delta(f) \leq \text{dist}(f; W_\pi)$ .

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset  $B \subset \mathcal{C}_0(X; E)$  and the associated carrier  $\varphi_B$  from  $X$  into  $E$  defined by

$$\varphi_B(x) = \{f(x); f \in B\} \quad \text{for all } x \in X.$$

Since  $B$  is bounded, it follows that  $\varphi_B$  is  $\Delta$ -bounded for any equivalence relation  $\Delta$  on  $X$ .

For each  $y \in Y$  define

$$B(\pi^{-1}(y)) = \cup \{f(\pi^{-1}(y)); f \in B\}$$

and

$$\delta(B) = \sup \{ \text{rad}(B(\pi^{-1}(y))); y \in Y \}.$$

then  $\delta(B) = \delta(\varphi_B)$ , and by Theorem 2.4,

$$\text{rad}_{W_\pi}(B) \leq \delta(B)$$

because  $W_\pi$  is a Stone-Weierstrass subspace.

Conversely, each  $w \in W_\pi$  is constant on  $\pi^{-1}(y)$  for every  $y \in Y$ . Thus

$$\begin{aligned} \text{dist}(\varphi_B; w) &= \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi_B(t)} \|z - w(t)\| \\ &\geq \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi_B(t)} \|z - v\| \\ &= \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{f \in B} \|f(t) - v\| \\ &= \sup_{y \in Y} \text{rad}(B(\pi^{-1}(y))) = \delta(B). \end{aligned}$$

Hence

$$\delta(B) \leq \text{dist}(\varphi_B; W_\pi) = \text{rad}_{W_\pi}(B).$$

We have thus proved the following.

**3.3. THEOREM:** *Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$  and  $W_\pi \subset \mathcal{C}_0(X; E)$  a Stone-Weierstrass subspace. Then, for any bounded and equicontinuous subset  $B \subset \mathcal{C}_0(X; E)$ , we have*

$$\text{rad}_{W_\pi}(B) = \sup_{y \in Y} \text{rad}(B(\pi^{-1}(y))).$$

**3.4. DEFINITION:** Let  $X$  and  $Z$  be two topological spaces. A set valued mapping  $\varphi$  from  $X$  into  $Z$  is said to be *lower semicontinuous* if  $\{x \in X; \varphi(x) \cap G \neq \emptyset\}$  is open in  $X$  for every open subset  $G \subset Z$ .

A continuous mapping  $f: X \rightarrow Z$  is called a *continuous selection* for a carrier  $\varphi$  if  $f(x) \in \varphi(x)$  for all  $x \in X$ .

The following result is a consequence of Michael [1], Theorem 2, page 233.

**3.5. THEOREM:** *Let  $X$  be a 0-dimensional compact  $T_1$ -space and let  $(E, \|\cdot\|)$  be a Banach space over a non-trivially valued division ring  $(F, |\cdot|)$ . Every lower semicontinuous carrier  $\varphi$  from  $X$  into the non-empty, closed subsets of  $E$  admits a continuous selection.*

3.6. REMARK: Let  $X$  be a 0-dimensional, Hausdorff and locally compact space. The Alexandroff compactification,  $X_\omega$ , of  $X$  is 0-dimensional and Hausdorff space. There is a linear isometry of  $\mathcal{C}_0(X; E)$  into  $\mathcal{C}(X_\omega; E)$ .

Let  $X$  be a locally compact  $T_1$ -space, and  $\pi$  a proper continuous surjection of  $X$  onto another locally compact  $T_1$ -space  $Y$ . Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$ . Let  $B \subset \mathcal{C}_0(X; E)$  be a bounded non-empty subset which is equicontinuous and vanishes collectively at infinity. For each  $x \in E$  let be given a closed vector subspace  $W(x) \subset E$ . Let  $\delta > 0$  be given.

Let us define two set valued mappings  $\varphi_\omega$  and  $\psi_\omega$  on  $Y_\omega$  and  $X_\omega$  respectively, by setting for any  $y \in Y$

$$\varphi_\omega(y) = \left\{ s \in E; \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s\| \leq \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\};$$

and for any  $x \in X$

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E; \sup_{f \in B} |f(x) - s| \leq \delta \right\}$$

$$\psi_\omega(\omega) = \{0\}.$$

3.7. LEMMA: Under the preceding hypothesis, the set valued mappings  $\varphi_\omega$  and  $\psi_\omega$  are lower semicontinuous on  $Y_\omega$  and  $X_\omega$  respectively.

PROOF: a) Let  $g \subset E$  be open such that  $\varphi_\omega(y_0) \cap G \neq \emptyset$ . If  $y_0 \in Y$ , we choose  $s_0 \in \varphi_\omega(y_0) \cap G$ , then

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y_0)} |f(x) - s_0| \leq \delta.$$

Since  $\pi^{-1}(y_0)$  is a compact subset of  $X$ , there exists a finite open covering  $V_1, V_2, \dots, V_n$  of  $\pi^{-1}(y_0)$ , with

$$V_i \cap \pi^{-1}(y_0) \neq \emptyset, \quad 1 \leq i \leq n,$$

such that

$$x, x' \in V_i \Rightarrow \|f(x) - f(x')\| < \delta$$

for all  $f \in B$ . This is possible because the set  $B \subset \mathcal{C}_0(X; E)$  is equicontinuous.

Let  $x \in X$ . For each  $z \in E$ , choose  $w_z \in W$  such that  $w_z(t) = z$  for all  $t \in \Delta(x)$ . Then

$$\begin{aligned} & \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \\ & \leq \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w_z(t)\| \\ & = \sup_{y \in \varphi(\Delta(x))} \|y - z\|. \end{aligned}$$

Since  $z \in E$  was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \leq \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \|y - z\|.$$

Hence,

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

### 3. Stone-Weierstrass subspaces

3.1. DEFINITION: A vector subspace  $W \subset \mathcal{C}_0(X; E)$  is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space  $Y$  and a proper continuous surjection  $\pi: X \rightarrow Y$  such that

$$W = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

We denote by  $W_\pi$  the Stone-Weierstrass subspace determined by  $\pi$ .

If  $W_\pi \subset \mathcal{C}_0(X; E)$  is a Stone-Weierstrass subspace, then

$$A_\pi = \{\varphi \circ \pi; \varphi \in \mathcal{C}^*(X; F)\}$$

is a subalgebra of  $\mathcal{C}^*(X; F)$  which contains the constants and

$$\{\pi^{-1}(y); y \in Y\}$$

is the set of equivalence classes modulo  $A_\pi$ . Therefore,  $W_\pi$  is an  $A_\pi$ -module.

Clearly  $W_\pi$  is closed in  $\mathcal{C}_0(X; E)$ .

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that  $\Delta(W_\pi) \subset W_\pi$ , where  $\Delta(W_\pi)$  is the Stone-Weierstrass hull of  $W_\pi$  in  $\mathcal{C}_0(X; E)$ .

Let  $f \in \Delta(W_\pi)$ . We will prove that  $f$  is constant on the sets  $\pi^{-1}(y)$  for all  $y \in Y$ .

Let  $t$  and  $t'$  be in  $X$  such that  $\pi(t) = \pi(t')$ . Then  $g(t) = g(t')$  for all  $g \in W_\pi$ . Then, the pair  $(t, t') \in \Delta_{W_\pi}$ .

If  $\delta(t, t') = 0$  then  $\delta_{t|W_\pi} = \delta_{t'|W_\pi} = 0$  and by hypothesis  $f \in \Delta(W_\pi)$ , then we have  $f(t) = 0 \cdot f(t') = 0$ .

If  $\delta(t, t') = 1$  then  $0 \neq \delta_{t|W_\pi} = \delta_{t'|W_\pi}$  and since  $f \in \Delta(W_\pi)$  we have  $f(t) = 1 \cdot f(t') = f(t')$ .

Therefore,  $f \in W_\pi$ .

Let  $f \in \mathcal{C}_0(X; E)$  be given. Since  $\pi$  is proper,  $\pi^{-1}(y)$  is compact and then  $f(\pi^{-1}(y))$  is compact, hence bounded in  $E$ , for each  $y \in Y$ . Let us define

$$\delta(f) = \sup_{y \in Y} \text{rad}(f(\pi^{-1}(y))).$$

If  $w \in W_\pi$  then

$$\|f - w\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t) - w(t)\| \geq \delta(f).$$

Hence

$$\delta(f) \leq \text{dist}(f; W_\pi).$$

**3.2. THEOREM:** *Let  $(E, \|\cdot\|)$  be a non-archimedean normed space over  $(F, |\cdot|)$  and  $W_\pi \subset \mathcal{C}_0(X; E)$  a Stone-Weierstrass subspace. Then, for all  $f \in \mathcal{C}_0(X; E)$*

$$\text{dist}(f; W_\pi) = \delta(f).$$

**PROOF:** By Theorem 2.4,  $\text{dist}(f; W_\pi) \leq \delta(f)$  and by remarks made before we have  $\delta(f) \leq \text{dist}(f; W_\pi)$ .

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset  $B \subset \mathcal{C}_0(X; E)$  and the associated carrier  $\varphi_B$  from  $X$  into  $E$  defined by

$$\varphi_B(x) = \{f(x); f \in B\} \quad \text{for all } x \in X.$$

Since  $B$  is bounded, it follows that  $\varphi_B$  is  $\Delta$ -bounded for any equivalence relation  $\Delta$  on  $X$ .

For each  $y \in Y$  define

$$B(\pi^{-1}(y)) = \cup \{f(\pi^{-1}(y)); f \in B\}$$

and

$$\delta(B) = \sup \{ \text{rad}(B(\pi^{-1}(y))); y \in Y \}.$$

We claim that  $\text{rad}(K) \leq \delta$ . Indeed, let  $g \in W_\pi$  be given. Then

$$\begin{aligned} \text{rad}(K) &= \inf_{z \in E} \sup_{x \in \pi^{-1}(y)} \|f(x) - z\| \\ &\leq \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \leq \|f - g\|. \end{aligned}$$

Since  $g$  was arbitrary,

$$\text{rad}(K) \leq \inf_{g \in W_\pi} \|f - g\|.$$

It follows that  $s_0 \in \varphi_\omega(y)$  and hence  $\varphi_\omega(y) \neq \emptyset$  for all  $y \in Y$ .

By Lemma 3.7 applied to  $B = \{f\}$ ,  $\varphi_\omega$  is lower semicontinuous.

By Theorem 3.5, there is  $g_\omega \in \mathcal{C}(Y_\omega; E)$  with  $g_\omega(y) \in \varphi_\omega(y)$  for all  $y \in Y_\omega$ , furthermore  $g_\omega(\omega) = 0$ . Let  $g \in \mathcal{C}_0(X; E)$  be the restriction of  $g_\omega$  to  $Y$ . Then  $g(y) \in \varphi(y)$  for all  $y \in Y$ . Let  $w = g \circ \pi$ . Then  $w \in W_\pi$  and, for any  $x \in X$  let  $y = \pi(x)$ . Then

$$\|f(x) - w(x)\| = \|f(x) - g(y)\| \leq \delta.$$

Hence

$$\|f - w\| \leq \text{dist}(f; W_\pi).$$

This ends the proof that  $W_\pi$  is proximal in  $\mathcal{C}_0(X; E)$ .

**3.9. THEOREM:** *Let  $X$  be a 0-dimensional, locally compact  $T_1$ -space. Let  $(E, \|\cdot\|)$  be a non-archimedean Banach space over  $(F, |\cdot|)$ . If  $E$  admits Chebyshev centers, and  $W_\pi \subset \mathcal{C}_0(X; E)$  is a Stone-Weierstrass subspace, then  $\text{cent}_{W_\pi}(B) \neq \emptyset$  for every non-empty bounded subset  $B \subset \mathcal{C}_0(X; E)$  which is equicontinuous and vanishes collectively at infinity.*

**PROOF:** Let  $\pi: X \rightarrow Y$  be the continuous and proper mapping of  $X$  onto a locally compact Hausdorff space  $Y$  such that

$$W_\pi = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

Let  $B \subset \mathcal{C}_0(X; E)$  be a non-empty bounded subset which is equicontinuous.

Let  $\delta = \text{rad}_{W_\pi}(B)$ :

**CASE I:**  $\delta > 0$ . Consider  $Y_\omega = Y \cup \{\omega\}$  the compactification of Alexandroff of  $Y$ .



For each  $y \in Y$ , let

$$\varphi_\omega(y) = \left\{ s \in E; \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s\| \leq \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\}.$$

Let us prove that  $\varphi_\omega$  is a carrier from  $Y_\omega$  into the non-empty closed subsets of  $E$ . Let  $y \in Y_\omega$  be given. If  $y = \omega$  then  $\varphi_\omega(y) = \{0\}$  and hence  $\varphi_\omega(y)$  is non-empty and closed. If  $y \in Y$  then  $\varphi_\omega(y)$  is closed in  $E$ . Since  $B \subset \mathcal{C}_0(X; E)$  is bounded,

$$B(y) = \{f(x); x \in \pi^{-1}(y), f \in B\}$$

is bounded in  $E$ , and by hypothesis  $\text{cent}(B(y)) \neq \emptyset$ , i.e., there exists  $s_0 \in E$  such that

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s_0\| = \text{rad}(B(y)).$$

To each  $g \in W_\pi$ , we have

$$\text{rad}(B(y)) \leq \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\|$$

because  $g$  is constant on  $\pi^{-1}(y)$ . Hence

$$\begin{aligned} \text{rad}(B(y)) &\leq \sup_{y \in Y} \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \\ &= \sup_{f \in B} \sup_{y \in Y} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \\ &= \sup_{f \in B} \|f - g\|. \end{aligned}$$

Since  $g$  was arbitrary,

$$\text{rad}(B(y)) \leq \inf_{g \in W_\pi} \sup_{f \in B} \|f - g\| = \text{rad}_{W_\pi}(B) = \delta.$$

Therefore,  $s_0 \in \varphi_\omega(y)$  and  $\varphi_\omega(y)$  is non-empty.

By Lemma 3.7,  $\varphi_\omega$  is lower semicontinuous.

By Theorem 3.5, there is  $g_\omega \in \mathcal{C}(Y_\omega; E)$  with  $g_\omega(y) \in \varphi_\omega(y)$  for all  $y \in Y_\omega$ . Notice that  $g_\omega(\omega) = 0$ . Let  $g \in \mathcal{C}_0(X; E)$  be the restriction of  $g_\omega$

to  $Y$ . Then  $g(y) \in \varphi_\omega(y)$  for all  $y \in Y$ . Let  $w = g \circ \pi$ . Then  $w \in W_\pi$  and for any  $x \in X$ , let  $y = \pi(x)$ . Then for any  $f \in B$  we have

$$\|f(x) - w(x)\| = \|f(x) - g(y)\| \leq \sup_{t \in \pi^{-1}(y)} \|f(t) - g(y)\| \leq \delta.$$

Hence

$$\sup_{f \in B} \|f - w\| \leq \delta, \quad \text{and so } w \in \text{cent}_{W_\pi}(B).$$

CASE II:  $\delta = 0$ .

Now  $\text{rad}_{W_\pi}(B) = 0$  implies  $B = \{f\}$  and  $\text{dist}(f; W_\pi) = \text{rad}_{W_\pi}(B) = 0$ . therefore  $f \in W_\pi$  and there is nothing to prove.

**3.10. THEOREM:** *Let  $X$  be a 0-dimensional, locally compact  $T_1$ -space. Let  $(E, \|\cdot\|)$  be a non-archimedean Banach space over  $(F, |\cdot|)$ . Let  $A \subset \mathcal{C}^*(X; F)$  be a separating subalgebra and let  $W \subset \mathcal{C}_0(X; E)$  be a closed vector subspace which is an  $A$ -module such that  $W(x)$  is proximal in  $E$  for every  $x \in X$ . Then,  $W$  is proximal in  $\mathcal{C}_0(X; E)$ .*

**PROOF:** Let  $f \in \mathcal{C}_0(X; E)$  be given with  $f \notin W$ . Then

$$\delta = \text{dist}(f; W) > 0,$$

because  $W$  is closed. Consider  $X_\omega = X \cup \{\omega\}$  the compactification of Alexandroff of  $X$ . For each  $x \in X$ , let

$$\psi_\omega(x) = W(x) \cap \{s \in E; \|f(x) - s\| < \delta\}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

Let us prove that  $\psi_\omega$  is a carrier from  $X_\omega$  into the non-empty closed subset of  $E$ . Indeed, let  $x \in X_\omega$ . If  $x = \omega$  then  $\psi_\omega(x) = \{0\}$  and then  $\psi_\omega(x)$  is non-empty and closed. If  $x \in X$ , there exists  $w \in W$  such that

$$\|w(x) - f(x)\| \leq \text{dist}(f(x); W(x)) \leq \delta$$

and hence  $\psi_\omega(x) \neq \emptyset$  and closed since  $W(x)$  is proximal.

By Lemma 3.7 applied with  $B = \{f\}$ ,  $\psi_\omega$  is lower semicontinuous.

By Theorem 3.5, there exists  $g_\omega \in \mathcal{C}(X_\omega; E)$  such that  $g_\omega(x) \in \psi_\omega(x)$  for all  $x \in X_\omega$ , furthermore  $g_\omega(\omega) = 0$ .

Let  $g \in \mathcal{C}_0(X; E)$  be the restriction of  $g_\omega$  to  $X$ . Hence  $g(x) \in W(x)$ . By Theorem 2.5 [5],  $g \in \overline{W}$ . Since  $W$  is closed,  $g \in W$ . On the other hand

$$\|f(x) - g(x)\| \leq \delta = \text{dist}(f; W)$$

for all  $x \in X$ , and therefore

$$\|f - g\| \leq \text{dist}(f; W),$$

i.e.,  $W$  is proximal in  $\mathcal{C}_0(X; E)$ .

**3.11. THEOREM:** *Let  $X$  and  $E$  as Theorem 3.10. Let  $A \subset \mathcal{C}^*(X; F)$  be a separating subalgebra and let  $W \subset \mathcal{C}_0(X; E)$  be a closed vector subspace which is an  $A$ -module and such that  $W(x)$  has the relative Chebyshev center property in  $E$ , for every  $x \in X$ . Then*

$$\text{cent}_W(B) \neq \emptyset,$$

for every non-empty equicontinuous and bounded  $B \subset \mathcal{C}_0(X; E)$  which vanishes collectively at infinity.

**PROOF:** Let  $B \subset \mathcal{C}_0(X; E)$  be a non-empty bounded subset which is equicontinuous at every point of  $X$  and vanishes at infinity. Let  $\delta = \text{rad}_W(B)$ . If  $\delta = 0$ , then  $B$  is a singleton  $\{f\}$  with  $f \in W$  and there is nothing to prove. We may assume that  $\delta > 0$ .

Let  $X_\omega$  be the compactification of Alexandroff of  $X$ . To each  $x \in X$ ,

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E; \sup_{f \in B} \|f(x) - s\| \leq \delta \right\}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

We will prove that  $\psi_\omega$  is a carrier from  $X_\omega$  into the nonempty closed subsets of  $E$ . Indeed. Let  $x \in X_\omega$ . If  $x = \omega$  then  $\psi_\omega(x) = \{0\} \neq \emptyset$  and  $\psi_\omega(x)$  is closed in  $E$ . If  $x \neq \omega$ , we define  $B(x) = \{f(x); f \in B\}$ , then  $B(x)$  is bounded in  $E$  and by hypothesis there is some  $w \in W$  such that

$$\sup_{f \in B} \|f(x) - w(x)\| \leq \text{rad}_{W(x)}(B(x)).$$

Now

$$\begin{aligned} \text{rad}_{W(x)}(B(x)) &= \inf_{w \in W} \sup_{f \in B} \|f(x) - w(x)\| \\ &\leq \inf_{w \in W} \sup_{f \in B} \|f - w\| = \delta. \end{aligned}$$

Hence  $\psi_\omega(x) \neq \emptyset$ . Clearly,  $\psi_\omega(x)$  is closed. By Lemma 3.7,  $\psi_\omega$  is lower semicontinuous.

