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BEST APPROXIMANTS FROM NON-ARCHIMEDEAN STONE-WEIERSTRASS SUBSPACES

Maria Zoraide M. Costa Soares

0. Introduction

Let X be a locally compact Hausdorff space, let $(F, |\cdot|)$ be a non-archimedean non-trivially valued division ring and $(E, \|\cdot\|)$ a normed space over $(F, |\cdot|)$.

We say that $g: X \rightarrow E$ vanishes at infinity if, for each $\epsilon > 0$, the set $\{x \in X; \|g(x)\| \geq \epsilon\}$ is compact.

We denote by $\mathcal{C}(X; E)$ the vector space of all continuous functions from X into E . $\mathcal{C}_0(X; E)$ will denote the vector space of all continuous functions which vanish at infinity, equipped with the norm $f \mapsto \|f\| = \sup\{\|f(x)\|; x \in X\}$.

The vector subspace of $\mathcal{C}(X; F)$ consisting of all continuous functions $f: X \rightarrow F$ such that $f(X)$ has compact closure in F , is denoted by $\mathcal{C}^*(X; F)$.

If Δ is the equivalence relation determined by $A \subset \mathcal{C}(X; F)$, $\Delta(x) = \{y \in X; a(y) = a(x) \text{ for all } a \in A\}$ is the Δ -equivalence class containing x .

If $Y \subset X$ is any non-empty set, we denote by $f|_Y$ the mapping $y \in Y \rightarrow f(y)$. If \mathcal{F} is any family of mappings $f: X \rightarrow S$, we denote by $\mathcal{F}|_Y$ the set $\{f|_Y; f \in \mathcal{F}\}$.

In this paper, we extend some results of Machado and Prolla [3] to the case of non-archimedean normed spaces, and other results of Prolla [4].

If $A \subset \mathcal{C}^*(X; F)$ is a subalgebra and $W \subset \mathcal{C}_0(X; E)$ is a vector subspace which is an A -module, we proved in [5] that for each $f \in \mathcal{C}_0(X; E)$,

$$\text{dist}(f; W) = \sup_{x \in X} \text{dist}(f|_{\Delta(x)}; W|_{\Delta(x)}).$$

We extend this “localization formula” for set-valued mappings under an upper semicontinuity hypothesis (see Theorem 1.7 below) generalizing a result of Prolla [4].

In Approximation Theory, given a normed space $(N, \|\cdot\|)$ and a non-empty subset $W \subset N$, there are two main problems. The first one is to characterize the closure of W in N , i.e., the set of all $f \in N$ such that $\text{dist}(f; W) = 0$. When N is a normed space of functions, this leads to

Stone-Weierstrass type theorems by choosing appropriate algebraic conditions on W . (For example, W is an A -module, etc.).

The second problem arises when $\text{dist}(f; W) > 0$. Does there exist $g \in W$ such that

$$\|f - g\| = \text{dist}(f; W)?$$

More generally, if instead of a single f one deals with a bounded set $B \subset N$, does there exist $g \in W$ such that

$$\sup_{f \in B} \|f - g\| = \inf_{w \in W} \sup_{f \in B} \|f - w\|?$$

Such a g , when it exists is called a Chebyshev center of B in W . We present some results (see Theorems 3.8 and 3.9) when N is $\mathcal{C}_0(X; E)$ and W is a so-called Stone-Weierstrass subspace. (see Olech [2]).

When W is a $\mathcal{C}^*(X; F)$ -module (or more generally an A -module, for some separating subalgebra $A \subset \mathcal{C}^*(X; F)$) it is natural to ask whether approximation properties of $W(x) = \{w(x); w \in W\}$ in E , for every $x \in X$, will ensure the same for W in $\mathcal{C}_0(X; E)$. Theorem 3.10 and 3.11 are along this line: in 3.10 one assumes that, for each $s \in X$, and $v \in E$ there is some element $w(x)$ such that $\|v - w(x)\| = \text{dist}(v; W(x))$. Theorem 3.11 deals with the analogous question for Chebyshev centers.

This work represents part of the author's dissertation at the Universidade de Campinas.

1. Stone-Weierstrass theorems

Let $X, (F, |\cdot|)$ and $(E, \|\cdot\|)$ be as in the introduction.

1.1. DEFINITION: A carrier φ from X to E is a mapping from X into the non-empty subsets of E .

1.2. DEFINITION: Let φ be a carrier from X into E . We define the distance of φ from a function $g \in \mathcal{C}_0(X; E)$ to be

$$\text{dist}(\varphi; g) = \sup_{x \in X} \left\{ \sup_{y \in \varphi(x)} \|y - g(x)\| \right\}$$

and the distance of φ from a subset $W \subset \mathcal{C}_0(X; E)$ to be

$$\text{dist}(\varphi; W) = \inf\{\text{dist}(\varphi; g); g \in W\}.$$

1.3. DEFINITION: Let φ a carrier of X into E . We say that φ is upper semicontinuous (u.s.c.) with respect to $W \subset \mathcal{C}_0(X; E)$, if given $w \in W$ and

$r > 0$, for each $x \in X$ such that $\varphi(x) \in B(w(x); r)$ and each $\epsilon > 0$, there is a neighborhood U of x such that $\varphi(y) \in B(w(y); r + \epsilon)$ for all $y \in U$. (If $v \in E$ and $s > 0$ we denote by $B(v; s)$ the set $\{u \in E; \|u - v\| < s\}$).

1.4. EXAMPLE: If $f \in \mathcal{C}_0(X; E)$, then $\varphi(x) = \{f(x)\}$, $x \in X$, is upper semicontinuous with respect to any $W \subset \mathcal{C}_0(X; E)$. Indeed, for each $w \in W$ and $r > 0$, the set

$$\{x \in X; \varphi(x) \subset B(w(x); r)\} = \{x \in X; \|f(x) - w(x)\| < r\}$$

is open.

1.5. EXAMPLE: Let $N \subset \mathcal{C}_0(X; E)$ be a equicontinuous subset. Define a carrier φ from X into E by setting

$$\varphi(x) = \{f(x); f \in N\},$$

for all $x \in X$. We claim that φ is u.s.c. with respect to any $W \subset \mathcal{C}_0(X; E)$. Indeed, let $w \in W$, $r > 0$ and $x \in X$ with $\varphi(x) \subset B(w(x); r)$ be given. Let $\epsilon > 0$. If N is equicontinuous then $N - \{w\}$ is equicontinuous too, and there is a neighborhood U of x such that $\|f(y) - w(y) - (f(x) - w(x))\| < \epsilon$ for all $y \in U$.

Hence, for all $y \in U$

$$\begin{aligned} \|f(y) - w(y)\| &= \|f(y) - w(y) - (f(x) - w(x)) \\ &\quad + (f(x) - w(x))\| \\ &\leq \|f(y) - w(y) - (f(x) - w(x))\| \\ &\quad + \|f(x) - w(x)\| \\ &< \epsilon + r. \end{aligned}$$

1.6. DEFINITION: Let φ be a carrier of X into E and let $W \subset \mathcal{C}_0(X; E)$. We say that φ *vanishes at infinity with respect to W* , if for each $w \in W$ and $\epsilon > 0$ the set

$$\{x \in X; \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset\}$$

is relatively compact, i.e. has compact closure.

1.7. THEOREM: Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$; let $A \subset \mathcal{C}^*(X; F)$ be a subalgebra and $W \subset \mathcal{C}_0(X; E)$ a vector subspace which is an A -module. For any carrier φ of X into E which is

upper semicontinuous and vanishes at infinity with respect to W , we have:

$$\text{dist}(\varphi; W) = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

PROOF: Let

$$\lambda = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

We always have $\lambda \leq \text{dist}(\varphi; W)$.

Let $\epsilon > 0$ and $x \in X$ be given; there exists $g_x \in W$ such that

$$\text{dist}(\varphi|_{\Delta(x)}; g_x|_{\Delta(x)}) < \lambda + \epsilon.$$

This implies that

$$\|t - g_x(y)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(y) \text{ and } y \in \Delta(x).$$

Since φ is upper semicontinuous with respect to W , there is an open neighborhood U_x of x such that

$$\|t - g_x(z)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(z) \text{ and } z \in U_x.$$

Clearly, $\Delta(x) \subset U_x$.

Since φ vanishes at infinity with respect to W , the closure K_x of

$$S_x = \{y \in X; \varphi(y) \cap (E \setminus B(g_x(y); \lambda + \epsilon)) \neq \emptyset\}$$

is compact. We claim that $\Delta(x) \cap K_x = \emptyset$. Indeed, assume $z \in \Delta(x) \cap K_x$. Since $\Delta(x) \subset U_x$ and K_x is the closure of S_x , there is some $y \in U_x \cap S_x$. But $\varphi(y) \subset B(g_x(y); \lambda + \epsilon)$ for all $y \in U_x$ and so y cannot be in S_x .

By Lemma 2.4, [5], there exists a finite set $\{x_1, x_2, \dots, x_n\} \subset X$ such that for each $0 < \delta < 1$, there are functions $a_1, a_2, \dots, a_n \in A_0$ satisfying:

- (1) $|a_i(x)| \leq 1$ for all $x \in X; i = 1, \dots, n$;
- (2) $|a_i(t)| < \delta$ for all $t \in K_{x_i}; i = 1, \dots, n$;
- (3) $\sum_{i=1}^n a_i(x) = 1$ for all $x \in X$;

where A_0 is the subalgebra generated by A and the constant functions.

We choose $\delta > 0$ such that

$$\delta \cdot \max_{1 \leq i \leq n} \|t - g_{x_i}(x)\| < \lambda + \epsilon \quad \text{for all } t \in \varphi(x)$$

and to this δ let $a_1, a_2, \dots, a_n \in A$ be given satisfying (1) to (3).

Define

$$g = \sum_{i=1}^n a_i g_{x_i}.$$

Then $g \in W$, and for each $x \in X$ and $t \in \varphi(x)$, we have:

$$\begin{aligned} \|t - g(x)\| &= \left\| \sum_{i=1}^n a_i(x)t - \sum_{i=1}^n a_i(x)g_{x_i}(x) \right\| \\ &= \left\| \sum_{i=1}^n a_i(x)(t - g_{x_i}(x)) \right\|. \end{aligned}$$

If $x \in K_{x_i}$, then

$$\begin{aligned} \|a_i(x)(t - g_{x_i}(x))\| &< \delta \cdot \|t - g_{x_i}(x)\| \\ &\leq \delta \cdot \max_{1 \leq i \leq n} \|t - g_{x_i}(x)\| < \lambda + \epsilon. \end{aligned}$$

If $x \notin K_{x_i}$, then

$$\|a_i(x)(t - g_{x_i}(x))\| \leq 1 \cdot \|t - g_{x_i}(x)\| < \lambda + \epsilon.$$

Hence, for all $x \in X$ and $t \in \varphi(x)$,

$$\begin{aligned} \|t - g(x)\| &= \left\| \sum_{i=1}^n a_i(x)(t - g_{x_i}(x)) \right\| \\ &\leq \max_{1 \leq i \leq n} \|a_i(x)(t - g_{x_i}(x))\| \\ &< \lambda + \epsilon. \end{aligned}$$

Then,

$$\text{dist}(\varphi; g) \leq \lambda + \epsilon.$$

A fortiori, $\text{dist}(\varphi; W) \leq \lambda + \epsilon$. Since $\epsilon > 0$ was arbitrary,

$$\text{dist}(\varphi; W) \leq \lambda = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

1.8. DEFINITION: A family of functions $N \subset \mathcal{C}_0(X; E)$ is said to *vanish collectively at infinity* if, for each $\epsilon > 0$, there is a compact subset $K \subset X$ such that $\|f(x)\| < \epsilon$ for all $x \notin K$ and $f \in N$.

1.9. **EXAMPLE:** Let $N \subset \mathcal{C}_0(X; E)$ be a totally bounded subset. Then N vanishes collectively at infinity. Indeed, let $\epsilon > 0$ be given. There exists a finite set $\{f_1, f_2, \dots, f_n\} \subset N$ such that, for each $f \in N$, there is $1 \leq i \leq n$ with $\|f - f_i\| < \epsilon/2$. For each $1 \leq i \leq n$, there is a compact subset $K_i \subset X$ such that $\|f_i(x)\| < \epsilon/2$ for all $x \notin K_i$. Let K be the union $K_1 \cup K_2 \cup \dots \cup K_n$. Then for all $x \notin K$ and $f \in N$, $\|f(x)\| < \epsilon$.

1.10. **PROPOSITION:** Let $N \subset \mathcal{C}_0(X; E)$ be a family which vanishes collectively at infinity and let $W \subset \mathcal{C}_0(X; E)$. The carrier

$$\varphi(x) = \{f(x); f \in N\}, \quad x \in X,$$

vanishes at infinity with respect to W .

PROOF: If $N \subset \mathcal{C}_0(X; E)$ vanishes collectively at infinity and $w \in \mathcal{C}_0(X; E)$, then $G = \{f - w; f \in N\}$ vanishes collectively at infinity too.

Let $\epsilon > 0$ and $K \subset X$ be a compact set such that

$$\|f(x) - w(x)\| < \epsilon$$

for all $x \notin K$ and $f \in N$.

Then $\varphi(x) \subset B(w(x); \epsilon)$ for all $x \notin K$ and

$$X \setminus \{x \in X; \varphi(x) \subset B(w(x); \epsilon)\} \subset K$$

and so the set

$$\{x \in X; \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset\}$$

is relatively compact.

1.11. **THEOREM:** Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, \|\cdot\|)$; let $A \subset \mathcal{C}^*(X; F)$ be a subalgebra; let $W \subset \mathcal{C}_0(X; E)$ be a vector subspace which is an A -module; and $N \subset \mathcal{C}_0(X; E)$ a totally bounded subset and define for all $x \in X$, $\varphi(x) = \{f(x); f \in N\}$. Then,

$$\text{dist}(\varphi; W) = \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}).$$

PROOF: By Example 1.5, φ is upper semicontinuous, and by Example 1.9, N vanishes collectively at infinity and by Proposition 1.10, φ vanishes at infinity with respect to any $W \subset \mathcal{C}_0(X; E)$. It remains to apply Theorem 1.7.

2. Chebyshev centers

2.1. **DEFINITION:** Let $(N, \|\cdot\|)$ be a normed space over $(F, |\cdot|)$, $W \subset N$ and B be a non-empty bounded subset of N . The *relative Chebyshev*

radius of B (with respect to W) is, by definition, the number

$$\text{rad}_W(B) = \inf \left\{ \sup_{f \in B} \|w - f\|; w \in W \right\}.$$

If $W = N$, then we write

$$\text{rad}_N(B) = \text{rad}(B)$$

and call it the *Chebyshev radius of B* .

The elements $w_0 \in W$ where the infimum is attained are called *relative Chebyshev centers of B (with respect to W)*, and we denote by $\text{cent}_W(B)$ the set of all such $w_0 \in W$.

If $W = N$, then we write $\text{cent}_N(B) = \text{cent } B$ and call it the set of *Chebyshev centers of B* .

We say that W has the *relative Chebyshev center property in N* if $\text{cent}_W(B) \neq \emptyset$ for all non-empty bounded sets $B \subset N$.

When $W = N$, and $\text{cent}(B) \neq \emptyset$ for every non-empty bounded subset $B \subset N$, i.e. if N has the relative Chebyshev center property in N , we say that N *admits Chebyshev centers*.

Let $M \subset N$ be a closed linear subspace and $f \in N$. A *best approximant* of f in M is any element $g \in M$ such that

$$\|f - g\| = \inf_{h \in M} \|f - h\| = \text{dist}(f; M).$$

We denote by $P_M(f)$ the set of all best approximants of f in M . If $P_M(f)$ contains at least one element for all $f \in N$, M is called *proximal*.

The main problems of *best (simultaneous) approximation theory* are the following (in decreasing order of generality):

PROBLEM I: Let $W \subset N$ be given. Determine if W has the relative Chebyshev center property in N . In particular, determine if N admits Chebyshev centers.

PROBLEM II: Let $W \subset N$ be given. Determine the class B of all non-empty bounded sets $B \subset N$ such that $\text{cent}_W(B) \neq \emptyset$.

PROBLEM III: Let $W \subset N$ be given. Determine if W is proximal in N , i.e., determine if the class B of Problem II contains all sets of the form $B = \{f\}$, $f \in N$.

Suppose that N is $\mathcal{C}_0(X; E)$ equipped with the sup-norm and let $W \subset \mathcal{C}_0(X; E)$. To each non-empty and bounded set $B \subset \mathcal{C}_0(X; E)$, we

define the carrier

$$\varphi_B(x) = \{f(x); f \in B\}$$

for all $x \in X$. It follows that

$$\text{dist}(\varphi_B; W) = \text{rad}_W(B).$$

Consequently, by Theorem 1.11, we have the following formula of localizability for the Chebyshev radius.

2.2. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$; let $A \subset \mathcal{C}^*(X; F)$ be a subalgebra and $W \subset \mathcal{C}_0(X; E)$ a vector subspace which is an A -module. For each non-empty and totally bounded subset $B \subset \mathcal{C}_0(X; E)$ we have*

$$\text{rad}_W(B) = \sup_{x \in X} \text{rad}_W|_{\Delta(x)}(B|_{\Delta(x)}).$$

2.3. DEFINITION: Let Δ be an equivalence relation in X . We say that a carrier φ from X into E is Δ -bounded if

$$\varphi(\Delta(x)) = \cup \{ \varphi(t); t \in \Delta(x) \}$$

is a bounded subset of E , for all $x \in X$. Let us define

$$\delta(\varphi) = \sup_{x \in X} \text{rad}(\varphi(\Delta(x)))$$

2.4. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $A \subset \mathcal{C}^*(X; F)$ a subalgebra. Let $W \subset \mathcal{C}_0(X; E)$ be an A -module such that for each $x \in X$ and $z \in E$, there is some $w \in W$ such that $w(t) = z$ for all $t \in \Delta(x)$. Then for any Δ -bounded carrier φ from X into E which is upper semicontinuous and vanishes at infinity with respect to W , we have:*

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

PROOF: By Theorem 1.7, we have:

$$\begin{aligned} \text{dist}(\varphi; W) &= \sup_{x \in X} \text{dist}(\varphi|_{\Delta(x)}; W|_{\Delta(x)}) \\ &= \sup_{x \in X} \inf_{w \in W} \text{dist}(\varphi|_{\Delta(x)}; w) \\ &= \sup_{x \in X} \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\|. \end{aligned}$$

Let $x \in X$. For each $z \in E$, choose $w_z \in W$ such that $w_z(t) = z$ for all $t \in \Delta(x)$. Then

$$\begin{aligned} & \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \\ & \leq \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w_z(t)\| \\ & = \sup_{y \in \varphi(\Delta(x))} \|y - z\|. \end{aligned}$$

Since $z \in E$ was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \leq \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \|y - z\|.$$

Hence,

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

3. Stone-Weierstrass subspaces

3.1. DEFINITION: A vector subspace $W \subset \mathcal{C}_0(X; E)$ is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space Y and a proper continuous surjection $\pi: X \rightarrow Y$ such that

$$W = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

We denote by W_π the Stone-Weierstrass subspace determined by π .

If $W_\pi \subset \mathcal{C}_0(X; E)$ is a Stone-Weierstrass subspace, then

$$A_\pi = \{\varphi \circ \pi; \varphi \in \mathcal{C}^*(X; F)\}$$

is a subalgebra of $\mathcal{C}^*(X; F)$ which contains the constants and

$$\{\pi^{-1}(y); y \in Y\}$$

is the set of equivalence classes modulo A_π . Therefore, W_π is an A_π -module.

Clearly W_π is closed in $\mathcal{C}_0(X; E)$.

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that $\Delta(W_\pi) \subset W_\pi$, where $\Delta(W_\pi)$ is the Stone-Weierstrass hull of W_π in $\mathcal{C}_0(X; E)$.

Let $f \in \Delta(W_\pi)$. We will prove that f is constant on the sets $\pi^{-1}(y)$ for all $y \in Y$.

Let t and t' be in X such that $\pi(t) = \pi(t')$. Then $g(t) = g(t')$ for all $g \in W_\pi$. Then, the pair $(t, t') \in \Delta_{W_\pi}$.

If $\delta(t, t') = 0$ then $\delta_{t|W_\pi} = \delta_{t'|W_\pi} = 0$ and by hypothesis $f \in \Delta(W_\pi)$, then we have $f(t) = 0 \cdot f(t') = 0$.

If $\delta(t, t') = 1$ then $0 \neq \delta_{t|W_\pi} = \delta_{t'|W_\pi}$ and since $f \in \Delta(W_\pi)$ we have $f(t) = 1 \cdot f(t') = f(t')$.

Therefore, $f \in W_\pi$.

Let $f \in \mathcal{C}_0(X; E)$ be given. Since π is proper, $\pi^{-1}(y)$ is compact and then $f(\pi^{-1}(y))$ is compact, hence bounded in E , for each $y \in Y$. Let us define

$$\delta(f) = \sup_{y \in Y} \text{rad}(f(\pi^{-1}(y))).$$

If $w \in W_\pi$ then

$$\|f - w\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t) - w(t)\| \geq \delta(f).$$

Hence

$$\delta(f) \leq \text{dist}(f; W_\pi).$$

3.2. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $W_\pi \subset \mathcal{C}_0(X; E)$ a Stone-Weierstrass subspace. Then, for all $f \in \mathcal{C}_0(X; E)$*

$$\text{dist}(f; W_\pi) = \delta(f).$$

PROOF: By Theorem 2.4, $\text{dist}(f; W_\pi) \leq \delta(f)$ and by remarks made before we have $\delta(f) \leq \text{dist}(f; W_\pi)$.

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset $B \subset \mathcal{C}_0(X; E)$ and the associated carrier φ_B from X into E defined by

$$\varphi_B(x) = \{f(x); f \in B\} \quad \text{for all } x \in X.$$

Since B is bounded, it follows that φ_B is Δ -bounded for any equivalence relation Δ on X .

For each $y \in Y$ define

$$B(\pi^{-1}(y)) = \cup \{f(\pi^{-1}(y)); f \in B\}$$

and

$$\delta(B) = \sup \{\text{rad}(B(\pi^{-1}(y))); y \in Y\}.$$

then $\delta(B) = \delta(\varphi_B)$, and by Theorem 2.4,

$$\text{rad}_{W_\pi}(B) \leq \delta(B)$$

because W_π is a Stone-Weierstrass subspace.

Conversely, each $w \in W_\pi$ is constant on $\pi^{-1}(y)$ for every $y \in Y$. Thus

$$\begin{aligned} \text{dist}(\varphi_B; w) &= \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi_B(t)} \|z - w(t)\| \\ &\geq \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi_B(t)} \|z - v\| \\ &= \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{f \in B} \|f(t) - v\| \\ &= \sup_{y \in Y} \text{rad}(B(\pi^{-1}(y))) = \delta(B). \end{aligned}$$

Hence

$$\delta(B) \leq \text{dist}(\varphi_B; W_\pi) = \text{rad}_{W_\pi}(B).$$

We have thus proved the following.

3.3. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $W_\pi \subset \mathcal{C}_0(X; E)$ a Stone-Weierstrass subspace. Then, for any bounded and equicontinuous subset $B \subset \mathcal{C}_0(X; E)$, we have*

$$\text{rad}_{W_\pi}(B) = \sup_{y \in Y} \text{rad}(B(\pi^{-1}(y))).$$

3.4. DEFINITION: Let X and Z be two topological spaces. A set valued mapping φ from X into Z is said to be *lower semicontinuous* if $\{x \in X; \varphi(x) \cap G \neq \emptyset\}$ is open in X for every open subset $G \subset Z$.

A continuous mapping $f: X \rightarrow Z$ is called a *continuous selection* for a carrier φ if $f(x) \in \varphi(x)$ for all $x \in X$.

The following result is a consequence of Michael [1], Theorem 2, page 233.

3.5. THEOREM: *Let X be a 0-dimensional compact T_1 -space and let $(E, \|\cdot\|)$ be a Banach space over a non-trivially valued division ring $(F, |\cdot|)$. Every lower semicontinuous carrier φ from X into the non-empty, closed subsets of E admits a continuous selection.*

3.6. REMARK: Let X be a 0-dimensional, Hausdorff and locally compact space. The Alexandroff compactification, X_ω , of X is 0-dimensional and Hausdorff space. There is a linear isometry of $\mathcal{C}_0(X; E)$ into $\mathcal{C}(X_\omega; E)$.

Let X be a locally compact T_1 -space, and π a proper continuous surjection of X onto another locally compact T_1 -space Y . Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$. Let $B \subset \mathcal{C}_0(X; E)$ be a bounded non-empty subset which is equicontinuous and vanishes collectively at infinity. For each $x \in E$ let be given a closed vector subspace $W(x) \subset E$. Let $\delta > 0$ be given.

Let us define two set valued mappings φ_ω and ψ_ω on Y_ω and X_ω respectively, by setting for any $y \in Y$

$$\varphi_\omega(y) = \left\{ s \in E; \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s\| \leq \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\};$$

and for any $x \in X$

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E; \sup_{f \in B} |f(x) - s| \leq \delta \right\}$$

$$\psi_\omega(\omega) = \{0\}.$$

3.7. LEMMA: *Under the preceding hypothesis, the set valued mappings φ_ω and ψ_ω are lower semicontinuous on Y_ω and X_ω respectively.*

PROOF: a) Let $g \subset E$ be open such that $\varphi_\omega(y_0) \cap G \neq \emptyset$. If $y_0 \in Y$, we choose $s_0 \in \varphi_\omega(y_0) \cap G$, then

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y_0)} |f(x) - s_0| \leq \delta.$$

Since $\pi^{-1}(y_0)$ is a compact subset of X , there exists a finite open covering V_1, V_2, \dots, V_n of $\pi^{-1}(y_0)$, with

$$V_i \cap \pi^{-1}(y_0) \neq \emptyset, \quad 1 \leq i \leq n,$$

such that

$$x, x' \in V_i \Rightarrow \|f(x) - f(x')\| < \delta$$

for all $f \in B$. This is possible because the set $B \subset \mathcal{C}_0(X; E)$ is equicontinuous.

Let $x \in X$. For each $z \in E$, choose $w_z \in W$ such that $w_z(t) = z$ for all $t \in \Delta(x)$. Then

$$\begin{aligned} & \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \\ & \leq \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w_z(t)\| \\ & = \sup_{y \in \varphi(\Delta(x))} \|y - z\|. \end{aligned}$$

Since $z \in E$ was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - w(t)\| \leq \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \|y - z\|.$$

Hence,

$$\text{dist}(\varphi; W) \leq \delta(\varphi).$$

3. Stone-Weierstrass subspaces

3.1. DEFINITION: A vector subspace $W \subset \mathcal{C}_0(X; E)$ is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space Y and a proper continuous surjection $\pi: X \rightarrow Y$ such that

$$W = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

We denote by W_π the Stone-Weierstrass subspace determined by π .

If $W_\pi \subset \mathcal{C}_0(X; E)$ is a Stone-Weierstrass subspace, then

$$A_\pi = \{\varphi \circ \pi; \varphi \in \mathcal{C}^*(X; F)\}$$

is a subalgebra of $\mathcal{C}^*(X; F)$ which contains the constants and

$$\{\pi^{-1}(y); y \in Y\}$$

is the set of equivalence classes modulo A_π . Therefore, W_π is an A_π -module.

Clearly W_π is closed in $\mathcal{C}_0(X; E)$.

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that $\Delta(W_\pi) \subset W_\pi$, where $\Delta(W_\pi)$ is the Stone-Weierstrass hull of W_π in $\mathcal{C}_0(X; E)$.

Let $f \in \Delta(W_\pi)$. We will prove that f is constant on the sets $\pi^{-1}(y)$ for all $y \in Y$.

Let t and t' be in X such that $\pi(t) = \pi(t')$. Then $g(t) = g(t')$ for all $g \in W_\pi$. Then, the pair $(t, t') \in \Delta_{W_\pi}$.

If $\delta(t, t') = 0$ then $\delta_{t|W_\pi} = \delta_{t'|W_\pi} = 0$ and by hypothesis $f \in \Delta(W_\pi)$, then we have $f(t) = 0 \cdot f(t') = 0$.

If $\delta(t, t') = 1$ then $0 \neq \delta_{t|W_\pi} = \delta_{t'|W_\pi}$ and since $f \in \Delta(W_\pi)$ we have $f(t) = 1 \cdot f(t') = f(t')$.

Therefore, $f \in W_\pi$.

Let $f \in \mathcal{C}_0(X; E)$ be given. Since π is proper, $\pi^{-1}(y)$ is compact and then $f(\pi^{-1}(y))$ is compact, hence bounded in E , for each $y \in Y$. Let us define

$$\delta(f) = \sup_{y \in Y} \text{rad}(f(\pi^{-1}(y))).$$

If $w \in W_\pi$ then

$$\|f - w\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t) - w(t)\| \geq \delta(f).$$

Hence

$$\delta(f) \leq \text{dist}(f; W_\pi).$$

3.2. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $W_\pi \subset \mathcal{C}_0(X; E)$ a Stone-Weierstrass subspace. Then, for all $f \in \mathcal{C}_0(X; E)$*

$$\text{dist}(f; W_\pi) = \delta(f).$$

PROOF: By Theorem 2.4, $\text{dist}(f; W_\pi) \leq \delta(f)$ and by remarks made before we have $\delta(f) \leq \text{dist}(f; W_\pi)$.

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset $B \subset \mathcal{C}_0(X; E)$ and the associated carrier φ_B from X into E defined by

$$\varphi_B(x) = \{f(x); f \in B\} \quad \text{for all } x \in X.$$

Since B is bounded, it follows that φ_B is Δ -bounded for any equivalence relation Δ on X .

For each $y \in Y$ define

$$B(\pi^{-1}(y)) = \cup \{f(\pi^{-1}(y)); f \in B\}$$

and

$$\delta(B) = \sup \{ \text{rad}(B(\pi^{-1}(y))); y \in Y \}.$$

We claim that $\text{rad}(K) \leq \delta$. Indeed, let $g \in W_\pi$ be given. Then

$$\begin{aligned} \text{rad}(K) &= \inf_{z \in E} \sup_{x \in \pi^{-1}(y)} \|f(x) - z\| \\ &\leq \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \leq \|f - g\|. \end{aligned}$$

Since g was arbitrary,

$$\text{rad}(K) \leq \inf_{g \in W_\pi} \|f - g\|.$$

It follows that $s_0 \in \varphi_\omega(y)$ and hence $\varphi_\omega(y) \neq \emptyset$ for all $y \in Y$.

By Lemma 3.7 applied to $B = \{f\}$, φ_ω is lower semicontinuous.

By Theorem 3.5, there is $g_\omega \in \mathcal{C}(Y_\omega; E)$ with $g_\omega(y) \in \varphi_\omega(y)$ for all $y \in Y_\omega$, furthermore $g_\omega(\omega) = 0$. Let $g \in \mathcal{C}_0(X; E)$ be the restriction of g_ω to Y . Then $g(y) \in \varphi(y)$ for all $y \in Y$. Let $w = g \circ \pi$. Then $w \in W_\pi$ and, for any $x \in X$ let $y = \pi(x)$. Then

$$\|f(x) - w(x)\| = \|f(x) - g(y)\| \leq \delta.$$

Hence

$$\|f - w\| \leq \text{dist}(f; W_\pi).$$

This ends the proof that W_π is proximal in $\mathcal{C}_0(X; E)$.

3.9. THEOREM: *Let X be a 0-dimensional, locally compact T_1 -space. Let $(E, \|\cdot\|)$ be a non-archimedean Banach space over $(F, |\cdot|)$. If E admits Chebyshev centers, and $W_\pi \subset \mathcal{C}_0(X; E)$ is a Stone-Weierstrass subspace, then $\text{cent}_{W_\pi}(B) \neq \emptyset$ for every non-empty bounded subset $B \subset \mathcal{C}_0(X; E)$ which is equicontinuous and vanishes collectively at infinity.*

PROOF: Let $\pi: X \rightarrow Y$ be the continuous and proper mapping of X onto a locally compact Hausdorff space Y such that

$$W_\pi = \{g \circ \pi; g \in \mathcal{C}_0(Y; E)\}.$$

Let $B \subset \mathcal{C}_0(X; E)$ be a non-empty bounded subset which is equicontinuous.

Let $\delta = \text{rad}_{W_\pi}(B)$:

CASE I: $\delta > 0$. Consider $Y_\omega = Y \cup \{\omega\}$ the compactification of Alexandroff of Y .

For each $y \in Y$, let

$$\varphi_\omega(y) = \left\{ s \in E; \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s\| \leq \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\}.$$

Let us prove that φ_ω is a carrier from Y_ω into the non-empty closed subsets of E . Let $y \in Y_\omega$ be given. If $y = \omega$ then $\varphi_\omega(y) = \{0\}$ and hence $\varphi_\omega(y)$ is non-empty and closed. If $y \in Y$ then $\varphi_\omega(y)$ is closed in E . Since $B \subset \mathcal{C}_0(X; E)$ is bounded,

$$B(y) = \{f(x); x \in \pi^{-1}(y), f \in B\}$$

is bounded in E , and by hypothesis $\text{cent}(B(y)) \neq \emptyset$, i.e., there exists $s_0 \in E$ such that

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - s_0\| = \text{rad}(B(y)).$$

To each $g \in W_\pi$, we have

$$\text{rad}(B(y)) \leq \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\|$$

because g is constant on $\pi^{-1}(y)$. Hence

$$\begin{aligned} \text{rad}(B(y)) &\leq \sup_{y \in Y} \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \\ &= \sup_{f \in B} \sup_{y \in Y} \sup_{x \in \pi^{-1}(y)} \|f(x) - g(x)\| \\ &= \sup_{f \in B} \|f - g\|. \end{aligned}$$

Since g was arbitrary,

$$\text{rad}(B(y)) \leq \inf_{g \in W_\pi} \sup_{f \in B} \|f - g\| = \text{rad}_{W_\pi}(B) = \delta.$$

Therefore, $s_0 \in \varphi_\omega(y)$ and $\varphi_\omega(y)$ is non-empty.

By Lemma 3.7, φ_ω is lower semicontinuous.

By Theorem 3.5, there is $g_\omega \in \mathcal{C}(Y_\omega; E)$ with $g_\omega(y) \in \varphi_\omega(y)$ for all $y \in Y_\omega$. Notice that $g_\omega(\omega) = 0$. Let $g \in \mathcal{C}_0(X; E)$ be the restriction of g_ω

to Y . Then $g(y) \in \varphi_\omega(y)$ for all $y \in Y$. Let $w = g \circ \pi$. Then $w \in W_\pi$ and for any $x \in X$, let $y = \pi(x)$. Then for any $f \in B$ we have

$$\|f(x) - w(x)\| = \|f(x) - g(y)\| \leq \sup_{t \in \pi^{-1}(y)} \|f(t) - g(y)\| \leq \delta.$$

Hence

$$\sup_{f \in B} \|f - w\| \leq \delta, \quad \text{and so } w \in \text{cent}_{W_\pi}(B).$$

CASE II: $\delta = 0$.

Now $\text{rad}_{W_\pi}(B) = 0$ implies $B = \{f\}$ and $\text{dist}(f; W_\pi) = \text{rad}_{W_\pi}(B) = 0$. therefore $f \in W_\pi$ and there is nothing to prove.

3.10. THEOREM: *Let X be a 0-dimensional, locally compact T_1 -space. Let $(E, \|\cdot\|)$ be a non-archimedean Banach space over $(F, |\cdot|)$. Let $A \subset \mathcal{C}^*(X; F)$ be a separating subalgebra and let $W \subset \mathcal{C}_0(X; E)$ be a closed vector subspace which is an A -module such that $W(x)$ is proximal in E for every $x \in X$. Then, W is proximal in $\mathcal{C}_0(X; E)$.*

PROOF: Let $f \in \mathcal{C}_0(X; E)$ be given with $f \notin W$. Then

$$\delta = \text{dist}(f; W) > 0,$$

because W is closed. Consider $X_\omega = X \cup \{\omega\}$ the compactification of Alexandroff of X . For each $x \in X$, let

$$\psi_\omega(x) = W(x) \cap \{s \in E; \|f(x) - s\| < \delta\}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

Let us prove that ψ_ω is a carrier from X_ω into the non-empty closed subset of E . Indeed, let $x \in X_\omega$. If $x = \omega$ then $\psi_\omega(x) = \{0\}$ and then $\psi_\omega(x)$ is non-empty and closed. If $x \in X$, there exists $w \in W$ such that

$$\|w(x) - f(x)\| \leq \text{dist}(f(x); W(x)) \leq \delta$$

and hence $\psi_\omega(x) \neq \emptyset$ and closed since $W(x)$ is proximal.

By Lemma 3.7 applied with $B = \{f\}$, ψ_ω is lower semicontinuous.

By Theorem 3.5, there exists $g_\omega \in \mathcal{C}(X_\omega; E)$ such that $g_\omega(x) \in \psi_\omega(x)$ for all $x \in X_\omega$, furthermore $g_\omega(\omega) = 0$.

Let $g \in \mathcal{C}_0(X; E)$ be the restriction of g_ω to X . Hence $g(x) \in W(x)$. By Theorem 2.5 [5], $g \in \overline{W}$. Since W is closed, $g \in W$. On the other hand

$$\|f(x) - g(x)\| \leq \delta = \text{dist}(f; W)$$

for all $x \in X$, and therefore

$$\|f - g\| \leq \text{dist}(f; W),$$

i.e., W is proximal in $\mathcal{C}_0(X; E)$.

3.11. THEOREM: *Let X and E as Theorem 3.10. Let $A \subset \mathcal{C}^*(X; F)$ be a separating subalgebra and let $W \subset \mathcal{C}_0(X; E)$ be a closed vector subspace which is an A -module and such that $W(x)$ has the relative Chebyshev center property in E , for every $x \in X$. Then*

$$\text{cent}_W(B) \neq \emptyset,$$

for every non-empty equicontinuous and bounded $B \subset \mathcal{C}_0(X; E)$ which vanishes collectively at infinity.

PROOF: Let $B \subset \mathcal{C}_0(X; E)$ be a non-empty bounded subset which is equicontinuous at every point of X and vanishes at infinity. Let $\delta = \text{rad}_W(B)$. If $\delta = 0$, then B is a singleton $\{f\}$ with $f \in W$ and there is nothing to prove. We may assume that $\delta > 0$.

Let X_ω be the compactification of Alexandroff of X . To each $x \in X$,

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E; \sup_{f \in B} \|f(x) - s\| \leq \delta \right\}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

We will prove that ψ_ω is a carrier from X_ω into the nonempty closed subsets of E . Indeed. Let $x \in X_\omega$. If $x = \omega$ then $\psi_\omega(x) = \{0\} \neq \emptyset$ and $\psi_\omega(x)$ is closed in E . If $x \neq \omega$, we define $B(x) = \{f(x); f \in B\}$, then $B(x)$ is bounded in E and by hypothesis there is some $w \in W$ such that

$$\sup_{f \in B} \|f(x) - w(x)\| \leq \text{rad}_{W(x)}(B(x)).$$

Now

$$\begin{aligned} \text{rad}_{W(x)}(B(x)) &= \inf_{w \in W} \sup_{f \in B} \|f(x) - w(x)\| \\ &\leq \inf_{w \in W} \sup_{f \in B} \|f - w\| = \delta. \end{aligned}$$

Hence $\psi_\omega(x) \neq \emptyset$. Clearly, $\psi_\omega(x)$ is closed. By Lemma 3.7, ψ_ω is lower semicontinuous.

