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A GENERAL NOTION OF EXTREME SUBSET

Marek Lassak

The purpose of this paper is a general view on various definitions of extreme subsets and extreme elements as used in several branches of mathematics. Using terms of set operations, we present a universal scheme common for many such definitions and we discuss it under some weak assumptions.

Let X be an arbitrary set and let $\mathcal{P}(X)$ denote the family of all subsets of X . Any function Φ mapping a subfamily \mathcal{D}_Φ of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ is called a *set operation* in X or simply an *operation*. \mathcal{D}_Φ is the domain of Φ .

In [16] the following definition has been proposed

DEFINITION: If $A \subset B \subset X$ and if for any $K \in \mathcal{D}_\Phi \cap \mathcal{P}(B)$ and any $x \in A \cap \Phi(K)$ there exists $M \in \mathcal{D}_\Phi \cap \mathcal{P}(A \cap K)$ such that $x \in \Phi(M)$, then A is said to be a Φ -*extreme subset* of B .

In the case $A = \{a\}$ of the above definition, a will be called a Φ -*extreme element* of B .

Examples of Φ -extreme elements are extreme points of a set of a real linear space (Φ maps any set $\{x, y\}$ onto the segment joining x and y), isolated points of a set of any topological space (Φ is the closure operation), terminal points of a set of a metric space (Φ maps any set $\{x, y\}$ onto the set of all points lying metrically between x and y). For examples of Φ -extreme subsets and other examples of Φ -extreme elements see the last paragraph of this paper.

1. Set operations

This paragraph is of preparatory nature.

An operation Φ in X is called:

- *full*, if $\mathcal{D}_\Phi = \mathcal{P}(X)$,
- *isotonic*, if for any $G, H \in \mathcal{D}_\Phi$ from $G \subset H$ it follows $\Phi(G) \subset \Phi(H)$,
- *enlarging*, if $G \subset \Phi(G)$ for any $G \in \mathcal{D}_\Phi$,
- *idempotent*, if $\Phi(\Phi(A)) = \Phi(A)$ for any $A \in \mathcal{D}_\Phi$ such that $\Phi(A) \in \mathcal{D}_\Phi$,
- a *closure operation*, if it is full, isotonic, enlarging, and idempotent,

- *additive*, if it is full and $\Phi(G \cup H) = \Phi(G) \cup \Phi(H)$ for any $G, H \in \mathcal{P}(X)$,
- *cover finite*, if $\Phi(G) \subset \cup \{ \Phi(F); F \in \mathcal{D}_\Phi \cap \mathcal{P}(G), |F| < \infty \}$ for any $G \in \mathcal{D}_\Phi$,
- *domain finite*, if $\Phi(G) = \cup \{ \Phi(F); F \in \mathcal{D}_\Phi \cap \mathcal{P}(G), |F| < \infty \}$ for any $G \in \mathcal{D}_\Phi$.

Full isotonic, full enlarging, full idempotent, full domain finite, and closure operations are considered by many authors (see [1],[6],[10],[11],[20],[21] for instance).

The smallest number $k \geq 0$ such that

$$\Phi(G) \subset \cup \{ \Phi(F); F \in \mathcal{D}_\Phi \cap \mathcal{P}(G), |F| \leq k \} \quad \text{for any } G \in \mathcal{D}_\Phi$$

is called *Caratheodory's number* of Φ and it is denoted by $c(\Phi)$. If such k does not exist, then we put $c(\Phi) = \infty$.

Obviously, if Φ is isotonic, then the inclusion in the above definition can be replaced by the equality. Note that any operation with finite Caratheodory's number is cover finite.

If $\Phi(G) = \Phi(H)$, then G and H are called *Φ -equivalent*. If additionally $G \subset H$, then we call G a *Φ -equivalent subset* of H .

We call $K \in \mathcal{D}_\Phi$ a *Φ -stable set* if $\Phi(K) = K$.

Remember that a closed under arbitrary intersections family $\mathcal{C} \subset \mathcal{P}(X)$ is called a *closure system* over X . If moreover $\phi \in \mathcal{C}$, then (X, \mathcal{C}) is usually called a *convexity structure*. Let the symbol $h_{\mathcal{C}}(A)$ denote the intersection of all sets of \mathcal{C} which contain a given set $A \in \mathcal{P}(X)$.

It is well known (see e.g. [6], p. 43) that there exists one-to-one correspondence between closure systems over X and closure operations in X : (\rightarrow) if \mathcal{C} is a closure system over X , then $h_{\mathcal{C}}$ is a closure operation, (\leftarrow) if Φ is a closure operation, then the family \mathcal{C}_Φ of Φ -stable sets is a closure system over X and $h_{\mathcal{C}_\Phi} = \Phi$.

If Φ is a closure operation, then $\Phi(K)$ is called the *closure* of K .

For a given operation Φ in X we define an auxiliary operation

$$\Phi^\cup(G) = \cup \{ \Phi(D); D \in \mathcal{D}_\Phi \cap \mathcal{P}(G) \}.$$

If Φ is full, then let (comp. [11], p. 311)

$$\Phi^\omega(G) = \bigcup_{n=0}^{\infty} \Phi^n(G),$$

where $\Phi^0(G) = G$, $\Phi^{i+1}(G) = \Phi(\Phi^i(G))$, $i = 0, 1, \dots$.

Obviously, both of the operations Φ^\cup , Φ^ω are full. Moreover, Φ^\cup is isotonic. If Φ is isotonic, then $\Phi^\cup(G) = \Phi(G)$ for any $G \in \mathcal{D}_\Phi$. If Φ is full and isotonic, then $\Phi^\cup = \Phi$.

2. General properties of Φ -extreme subsets

Observe that the definition of Φ -extreme subset A of B can be shortly expressed by the formula

$$A \cap \Phi(K) \subset \Phi \cup (A \cap K) \quad \text{for any } K \in \mathcal{D}_\Phi \cap \mathcal{P}(B). \quad (1)$$

The following five statements of Theorem 1 generalize well known properties of the classic extreme subsets in real linear spaces.

THEOREM 1: *For each operation Φ in X we have:*

(a) *Any finite intersection of Φ -extreme subsets of a set B is a Φ -extreme subset of B . This is also true for arbitrary intersections provided Φ is cover finite.*

(b) *Any union of Φ -extreme subsets of a set B is also a Φ -extreme subset of B .*

(c) *If B is a Φ -extreme subset of C and A is a Φ -extreme subset of B , then A is a Φ -extreme subset of C .*

(d) *If $A \subset B \subset C$ and if A is a Φ -extreme subset of C , then A is a Φ -extreme subset of B .*

(e) *Sets B , ϕ and (if $\phi \in D_\Phi$) all subsets of $B \cap \Phi(\phi)$ are Φ -extreme subsets of B .*

PROOF: We prove only the first part. The other ones are left to the reader.

Let A_1, A_2 be Φ -extreme subsets of B . Suppose, $x \in (A_1 \cap A_2) \cap \Phi(K)$, where $K \in \mathcal{D}_\Phi \cap \mathcal{P}(B)$. Since A_1 is a Φ -extreme subset of B and $x \in A_1 \cap \Phi(K)$, there exists $M_1 \in \mathcal{D}_\Phi \cap \mathcal{P}(A_1 \cap K)$ such that $x \in \Phi(M_1)$. Similarly, from $x \in A_2 \cap \Phi(M_1)$ and $M_1 \in \mathcal{D}_\Phi \cap \mathcal{P}(B)$ we get that $x \in \Phi(M)$ for some $M \in \mathcal{D}_\Phi \cap \mathcal{P}(A_2 \cap M_1)$. Obviously, $M \in \mathcal{D}_\Phi \cap \mathcal{P}(A_1 \cap A_2 \cap K)$. Thus $A_1 \cap A_2$ is a Φ -extreme subset of B . Consequently, any finite intersection of Φ -extreme subsets of B is also Φ -extreme.

Now, let Φ be cover finite and $A_\lambda, \lambda \in \Lambda$, be Φ -extreme subsets of B . Put $A = \bigcap \{A_\lambda, \lambda \in \Lambda\}$. Suppose, $K \in \mathcal{D}_\Phi \cap \mathcal{P}(B)$ and $x \in A \cap \Phi(K)$. Since Φ is cover finite, $x \in \Phi(M)$ for a finite $M \in \mathcal{D}_\Phi \cap \mathcal{P}(K)$. Without loss of generality, we can assume that M is minimum (in respect to inclusion) set of $\mathcal{D}_\Phi \cap \mathcal{P}(K)$ for which $x \in \Phi(M)$. Moreover, $x \in A_\lambda$ and A_λ is a Φ -extreme subset of B for any $\lambda \in \Lambda$. Thus for any $\lambda \in \Lambda$ there exists $M_\lambda \in \mathcal{D}_\Phi \cap \mathcal{P}(A_\lambda \cap M) \subset \mathcal{D}_\Phi \cap \mathcal{P}(K)$ such that $x \in \Phi(M_\lambda)$. Since M is minimal and $M_\lambda \subset M$, we have $M_\lambda = M$ for all $\lambda \in \Lambda$. From $M_\lambda \subset A_\lambda, \lambda \in \Lambda$, we get $M \subset A$. Hence $M \in \mathcal{D}_\Phi \cap \mathcal{P}(A \cap K)$. Thus A is a Φ -extreme subset of B .

Note that in general case the intersection of infinitely many Φ -extreme subsets may fail to be Φ -extreme (comp. e.g. Example 4).

PROPOSITION 1: *Let Φ be a full isotonic operation [respectively: a cover*

finite operation, an operation with finite Caratheodory's number, an additive operation]. A subset A of B is Φ -extreme if and only if the below condition (2) [respectively: (3), (4), (5)] holds:

$$A \cap \Phi(K) \subset \Phi(A \cap K) \quad \text{for any } K \in \mathcal{P}(B), \quad (2)$$

$$A \cap \Phi(K) \subset \Phi^\cup(A \cap K) \quad \text{for any finite } K \in \mathcal{D}_\Phi \cap \mathcal{P}(B), \quad (3)$$

$$A \cap \Phi(K) \subset \Phi^\cup(A \cap K) \quad \text{for any } K \in \mathcal{D}_\Phi \cap \mathcal{P}(B)$$

$$\text{with } |K| \leq c(\Phi), \quad (4)$$

$$A \cap \Phi(B \setminus A) \subset \Phi(\phi). \quad (5)$$

PROOF: The first three statements result easily from (1), because $\Phi^\cup = \Phi$ for any full isotonic operation, and from the definitions of cover finite operation and Caratheodory's number.

We prove the last part. As additive, the operation Φ is isotonic and full. So it is sufficient to show that (2) and (5) are equivalent.

Immediately, (5) results from (2) putting $K = B \setminus A$.

Assume, (5) holds. Therefore $A \cap \Phi(B \setminus A) \subset A \cap \Phi(\phi)$. Let K be a subset of B . Since Φ is isotonic, $\Phi(K \setminus A) \subset \Phi(B \setminus A)$ and $\Phi(\phi) \subset \Phi(K \cap A)$. Moreover, Φ is additive. Thus we get that

$$\begin{aligned} A \cap \Phi(K) &= A \cap \Phi[(K \cap A) \cup (K \setminus A)] \\ &= A \cap [\Phi(K \cap A) \cup \Phi(K \setminus A)] \\ &= [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(K \setminus A)] \\ &\subset [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(B \setminus A)] \\ &\subset [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(\phi)] \\ &= A \cap \Phi(K \cap A) \subset \Phi(K \cap A), \end{aligned}$$

which ends the proof.

Condition (2) is very useful and it can be applied to full isotonic operations, particularly to the operation Φ^\cup for any Φ . This is why we formulate the following proposition whose proof is left to the reader.

PROPOSITION 2: A is a Φ -extreme subset of B if and only if A is a Φ^\cup -extreme subset of B .

3. Φ -extreme and Φ^ω -extreme subsets

THEOREM 2: *Any Φ -extreme subset of B is also Φ^ω -extreme provided Φ is full domain finite and B is Φ^ω -stable (particularly: Φ -stable).*

PROOF: Let A be a Φ -extreme subset of B . Since Φ is full and (as domain finite) isotonic, Φ^ω is also full and isotonic. By the first part of Proposition 1, to verify that A is a Φ^ω -extreme subset of B it is sufficient to show the inclusion $A \cap \Phi^\omega(H) \subset \Phi^\omega(A \cap H)$ for any $H \in \mathcal{P}(B)$.

Let $x \in A \cap \Phi^\omega(H)$. Let m be the smallest number such that $x \in \Phi^m(H)$. Recurrently, define finite sets H_m, \dots, H_0 as follows. Put $H_m = \{x\}$. Obviously, $H_m \subset \Phi^m(H)$. Suppose, a finite subset H_n of $\Phi^n(H)$ is defined, where $m \geq n > 0$. Since Φ is full domain finite, there exists a finite set $G_{n-1} \subset \Phi^{n-1}(H)$ such that $H_n \subset \Phi(G_{n-1})$. Consequently, there exists a finite minimal (in respect to inclusion) subset H_{n-1} of $\Phi^{n-1}(H)$ such that $H_n \subset \Phi(H_{n-1})$. So H_m, \dots, H_0 are defined. Of course, $H_k \subset \Phi(H_{k-1})$ for $k = 1, \dots, m$ and $H_k \subset \Phi^k(H)$ for $k = 0, \dots, m$.

Since Φ is isotonic, any Φ^k is also isotonic. Therefore

$$H_k \subset \Phi^k(H) \subset \Phi^k(B) \subset \Phi^\omega(B) = B \quad \text{for } k = 0, \dots, m.$$

Obviously, $H_m \subset A$. Assume $H_n \subset A$, where $m \geq n > 0$. Putting $K = H_{n-1}$ in (2) we obtain $A \cap \Phi(H_{n-1}) \subset \Phi(A \cap H_{n-1})$. Since $H_n \subset A$ and $H_n \subset \Phi(H_{n-1})$, we have $H_n \subset \Phi(A \cap H_{n-1})$. Moreover, H_{n-1} is a minimal subset of $\Phi^{n-1}(H)$ such that $H_n \subset \Phi(H_{n-1})$. Therefore $H_{n-1} \subset A \cap H_{n-1}$ and consequently, $H_{n-1} \subset A$. Thus H_m, \dots, H_0 are subsets of A . Particularly, $H_0 \subset A$.

From $H_k \subset \Phi(H_{k-1})$, $k = 1, \dots, m$, from $H_0 \subset A$ and $H_0 \subset \Phi^0(H) = H$, thanks the isotonicity of Φ' , we get

$$\begin{aligned} \{x\} &= H_m \subset \Phi(H_{m-1}) \subset \Phi^2(H_{m-2}) \subset \dots \subset \Phi^m(H_0) \\ &= \Phi^m(H_0 \cap A) \subset \Phi^m(H \cap A) \subset \Phi^\omega(H \cap A). \end{aligned}$$

PROPOSITION 3: *Any Φ^ω -extreme subset of B is also Φ -extreme provided: Φ is full, $c(\Phi) < \infty$, and $\Phi(F)$ is Φ -stable for $|F| \leq c(\Phi)$.*

PROOF: Let A be a Φ^ω -extreme subset of B . By (1) we obtain that

$$A \cap \Phi^\omega(K) \subset (\Phi^\omega)^\cup(A \cap K) \quad \text{for any } K \in \mathcal{P}(B).$$

If $|K| \leq c(\Phi)$, then $\Phi^\omega(K) = \Phi(K)$. Moreover, $\Phi^\omega(M) = \Phi(M)$ for $M \in \mathcal{P}(A \cap K)$. Consequently, $(\Phi^\omega)^\cup(A \cap K) = \Phi^\cup(A \cap K)$. Hence (4) is satisfied which, in virtue of Proposition 1, ends the proof.

4. Φ -extreme elements

If $\phi \in \mathcal{D}_\Phi$, then any element of $\Phi(\phi)$ is a Φ -extreme element of each set to which it belongs (comp. (e) in Theorem 1). Such elements are called *trivially Φ -extreme*. For an arbitrary operation Φ in X , we denote by $E_\Phi(B)$ the set of all *non-trivially Φ -extreme* elements, i.e. of all Φ -extreme elements of B which do not belong to $\Phi(\phi)$.

PROPOSITION 4: *If $a \in E_\Phi(B)$, then*

$$a \in \Phi(K) \Rightarrow a \in K \quad \text{for any } K \in \mathcal{D}_\Phi \cap \mathcal{P}(B). \quad (6)$$

For a full enlarging operation Φ , the conditions $a \in E_\Phi(B)$ and (6) are equivalent.

Let us observe (for use in Example 2) that if Φ is cover finite, than (6) is equivalent to the condition

$$a \in \Phi(K) \Rightarrow a \in K \quad \text{for any finite } K \in \mathcal{D}_\Phi \cap \mathcal{P}(B). \quad (7)$$

PROPOSITION 5: *If Φ is a full operation and $a \in E_\Phi(B)$, then*

$$a \in B \setminus \Phi(B \setminus \{a\}). \quad (8)$$

For full enlarging isotonic operations, the conditions $a \in E_\Phi(B)$ and (8) are equivalent.

PROPOSITION 6: *If Φ is full and enlarging, then $a \in E_\Phi(B)$ implies*

$$\Phi(B \setminus \{a\}) \neq \Phi(B). \quad (9)$$

If Φ is a closure operation, $a \in E_\Phi(B)$ is equivalent to (9).

PROPOSITION 7: *Let Φ be a full, enlarging, isotonic operation and let a belong to a Φ -stable set B . Then $a \in E_\Phi(B)$ if and only if*

$$B \setminus \{a\} \text{ is } \Phi\text{-stable}. \quad (10)$$

We omit the proofs of Propositions 4–7 as rather tedious. Simply examples show that all assumptions about Φ are necessary there. In the case of closure operations, an additional characterization of non-trivially Φ -extreme elements is given in the first part of Proposition 9.

With the help of Proposition 4 one can easily obtain the following

PROPOSITION 8: *Let Φ be an enlarging operation and $B \in \mathcal{D}_\Phi$. Then $E_\Phi(B)$ is contained in each Φ -equivalent subset of B . Moreover, if $E_\Phi(B)$*

is a Φ -equivalent subset of B , then it is the smallest Φ -equivalent subset of B .

PROPOSITION 9: For any closure operation Φ and any $B \in \mathcal{P}(X)$ we have:

- (a) $E_\Phi(B)$ coincides with the intersection of all Φ -equivalent subsets of B .
- (b) If there exists the smallest Φ -equivalent subset of B , then it is equal to $E_\Phi(B)$.
- (c) $\Phi(E_\Phi(B)) = \Phi(B)$ if and only if the family of all Φ -equivalent subsets of B is a closure system over B .
- (d) $E_\Phi(\Phi(B)) \subset E_\Phi(B)$.

PROOF: (a) Let $a \notin E_\Phi(B)$. In virtue of Proposition 8 it is sufficient to show that a does not belong to some Φ -equivalent subset of B . The searched subset is $B \setminus \{a\}$ because (9) does not hold.

(b) If there exists the smallest Φ -equivalent subset of B , then it coincides with the intersection of all Φ -equivalent subsets of B . Consequently, it is equal to $E_\Phi(B)$ as we have shown in (a).

(c) This statement easy results from (b) and from Proposition 8.

(d) Let $a \in E_\Phi(\Phi(B))$. By (a) we obtain that a belongs to any Φ -equivalent subset $\Phi(B)$. Particularly, $a \in B$. Since a is a Φ -extreme element of $\Phi(B)$ and $B \subset \Phi(B)$, we infer from part (d) of Theorem 1 that a is also a Φ -extreme element of B . From $a \in E_\Phi(\Phi(B))$ it follows $a \notin \Phi(B)$. Consequently, $a \in E_\Phi(B)$.

For B being Φ -stable, the reader can observe a connection of statement (a) of Proposition 9 with the operation j_h considered in [24] and consequently, a connection of Examples 1 and 4 presented at the end of this paper with Corollaries 2.3 and 2.4 of [24].

PROPOSITION 10: For any closure operation Φ and any set $B \in \mathcal{P}(X)$ the following conditions are equivalent:

$$E(\Phi(B)) = E_\Phi(B), \quad (11)$$

$$\text{for any } a \in B \setminus \Phi(B \setminus \{a\}) \text{ the set } \Phi(B) \setminus \{a\} \text{ is } \Phi\text{-stable}, \quad (12)$$

$$\begin{aligned} &\text{for any } a \in B \setminus \Phi(B \setminus \{a\}) \text{ there exists a } \Phi\text{-stable set } S \\ &\text{such that } S \cup \{a\} \text{ is } \Phi\text{-stable, } a \notin S \text{ and } \Phi(B \setminus \{a\}) \subset S. \end{aligned} \quad (13)$$

PROOF: (11) \Rightarrow (12). Let $a \in B \setminus \Phi(B \setminus \{a\})$ in (12). From Proposition 5 we obtain that $a \in E_\Phi(B) = E_\Phi(\Phi(B))$. Since $\Phi(B)$ is Φ -stable, by Proposition 7 we get that the set $\Phi(B) \setminus \{a\}$ is Φ -stable.

(12) \Rightarrow (11). Let $a \in E_\Phi(B)$. By Proposition 5 and by our assumption (12), the set $\Phi(B) \setminus \{a\}$ is Φ -stable. From Proposition 7 we infer that $a \in E_\Phi(\Phi(B))$. Thus $E_\Phi(B) \subset E_\Phi(\Phi(B))$. The inverse inclusion has been shown in Proposition 9.

(12) \Rightarrow (13). Put $S = \Phi(B) \setminus \{a\}$ in (12). Since Φ is enlarging, $a \in \Phi(B)$. Consequently, $S \cup \{a\} = \Phi(B)$. By the idempotence of Φ , the set $S \cup \{a\}$ is Φ -stable. From $a \notin \Phi(B \setminus \{a\})$ and from the isotonicity of Φ we get that $\Phi(B \setminus \{a\}) \subset \Phi(B) \setminus \{a\} = S$.

(13) \Rightarrow (12). Thanks to the correspondence between closure operations and closure systems, it is sufficient to show that $\Phi(B) \setminus \{a\}$ is an intersection of Φ -stable sets, i.e. that for any $c \notin \Phi(B) \setminus \{a\}$ there exists a Φ -stable set K_c such that $c \notin K_c$ and $\Phi(B) \setminus \{a\} \subset K_c$. Obviously, if our c is different from a , then one can put $K_c = \Phi(B)$. In the case $c = a$ put $K_a = S$. Since $S \cup \{a\}$ is Φ -stable and since Φ is enlarging and isotonic, from $\Phi(B \setminus \{a\}) \subset S$ we infer that

$$\Phi(B) \subset \Phi[\Phi(B \setminus \{a\}) \cup \{a\}] \subset S \cup \{a\}.$$

Consequently, $\Phi(B) \setminus \{a\} \subset S = K_a$. Of course, $a \notin K_a$.

The proof is complete.

Note that any Φ -stable set C such that $a \notin C$ can be presented in the form $\Phi(B \setminus \{a\})$, where $B = C \cup \{a\}$. Consequently, using standard techniques, from Proposition 10 we obtain

THEOREM 3: *If Φ is a closure operation, then the following conditions are equivalent:*

any set and its closure have identical non-trivially Φ -extreme elements, (14)

for any Φ -stable set C and $a \notin C$ the set $\Phi(C \cup \{a\}) \setminus \{a\}$ is Φ -stable, (15)

for any Φ -stable set C and $a \notin C$ there exists a Φ -stable set S such that $S \cup \{a\}$ is Φ -stable, $a \notin S$ and $C \subset S$, (16)

Φ -equivalent sets have identical non-trivially Φ -extreme elements. (17)

From Theorem 1.4 of [6], p. 46, and from the correspondence between closure operations and closure systems it results that if Φ is a domain finite closure operation, then any Φ -stable set not containing an element $x \in X$ can be enlarged to a maximal Φ -stable set not containing x . Consequently, the equivalence of (14) and (16) implies

THEOREM 4: *Let Φ be a domain finite closure operation. Any set and its closure have identical non-trivially Φ -extreme elements if and only if for any*

$x \in X$ and any maximal Φ -stable set C not containing x , the set $C \cup \{x\}$ is Φ -stable.

5. Some examples

Usually, Φ -extreme subsets and elements are considered for Φ defined on pairs of elements of X . This way one defines the notions of extreme subset and extreme point of a set of a real linear space, extreme support hyperplanes (comp. [3], p. 15), terminal points ([2], p. 53) and subsets [16] of a set of a metric space, extreme points of a subset of a partially ordered set ([7], comp. also Example 3 below), extreme points and subsets in various axiomatic convexity spaces (see e.g. [5]). Also extreme rays of a cone can be defined on this way. The equivalence of some formulas used in such definitions is presented in Proposition 11, where instead $\Phi(\{x, y\})$ we simply write $\Phi\{x, y\}$.

PROPOSITION 11: *Let Φ and $\bar{\Phi}$ be two operations defined on all pairs of (not necessarily different) elements of X such that $x, y \notin \Phi\{x, y\}$, $\Phi\{x, x\} = \phi$ and $\bar{\Phi}\{x, y\} = \{x, y\} \cup \Phi\{x, y\}$ for any $x, y \in X$. For any subset A of B the following conditions are equivalent:*

- (a) A is a Φ -extreme subset of B ,
- (b) A is a $\bar{\Phi}$ -extreme subset of B ,
- (c) for every $x, y \in B$ from $A \cap \Phi\{x, y\} \neq \phi$ it results $x, y \in A$,
- (d) for every $x, y \in B$ and $a \in A \cap \bar{\Phi}\{x, y\}$ it is $x = a$, or $y = a$, or $x, y \in A$,
- (e) $A \cap \Phi\{x, y\} = \phi$ for each $x \in B \setminus A$ and each $y \in B$,
- (f) $A \cap \bar{\Phi}\{x, y\} \subset \{y\}$ for each $x \in B \setminus A$ and each $y \in B$.

EXAMPLE 1: Let L be a real linear space and let $\text{segm}\{x, y\} = \{(1 - \alpha)x + \alpha y; 0 < \alpha < 1\}$. Obviously, the classical notion of extreme subset of a set is just the notion of segm -extreme subset. Consider other kinds of extremeness in L . Let conv denote the operation of convex hull. From the equality $(\overline{\text{segm}}^\cup)^\omega = \text{conv}$ (comp. [4]) and by Propositions 2, 3, 5 and Theorems 2, 4 (as it results from Theorem 3.1 of [11], for any maximal convex set C not containing a given point a , the set $C \cup \{a\}$ is also convex) we conclude that conv -extreme subsets of any convex set coincide with extreme subsets and that conv -extreme points of any set B are identical with extreme points of $\text{conv } B$. A generalization of conv -extremeness is presented in Example 2. A subset A of a convex set B is called a *semi-extreme subset* of B if $B \setminus A$ is convex (comp. [12], p. 32). As in [22], pp. 186–187, this notion can be extended to arbitrary (i.e. not necessary convex) set B : if $A \subset B$ and $A \cap \text{conv}(B \setminus A) = \phi$, then we call A a *semi-extreme subset* of B . Simple examples show that the intersection of two semi-extreme subsets may not be a semi-extreme subset. From (a) in Theorem 1 we infer that the notion of semi-extreme subset is not a case of our scheme of Φ -extremeness. One can extend the definition of

semi-extreme subsets for operations as follows. If $A \subset B \subset X$ and if for any $K \in \mathcal{D}_\Phi \cap \mathcal{P}(B \setminus A)$ and any $x \in A \cap \Phi(K)$ there exists $M \in \mathcal{D}_\Phi \cap \mathcal{P}(A \cap K)$ such that $x \in \Phi(M)$, then A is called a Φ -semi-extreme subset of B . The reader can easily check up that for Φ -semi-extreme subsets there hold analogical properties as (b), (d) and (e) of Theorem 1 and that conv-semi-extreme subsets are just semi-extreme ones. Also relative extreme points [15] and relative extreme subsets [17] are studied. Remember that a subset A of B is said to be an *extreme subset* of B relative to C if for any $x \in B$, $y \in C$, from $A \cap \text{segm}\{x, y\} \neq \emptyset$ it follows $x \in A$. This is a special case of our notion of Φ -extreme subset, where \mathcal{D}_Φ consists of all one-point sets and $\Phi(\{x\}) = \bigcup_{y \in C} \text{segm}\{x, y\}$.

EXAMPLE 2: Consider two kinds of extremeness in a convexity structure (X, \mathcal{C}) . As in many papers concerning convexity structures, the hull operation (i.e. the closure operation) $h_{\mathcal{C}}$ generated by \mathcal{C} will be simply denoted by the same symbol \mathcal{C} . In [22], pp. 186–187, there was introduced a notion of extreme subset for convexity structures for which, as for a special case of the definition of Φ -semi-extreme subset presented in Example 1, we use the term \mathcal{C} -semi-extreme subset or semi-extreme subset for short. Let $A \subset B \subset X$. We call A a \mathcal{C} -semi-extreme subset of B if

$$A \cap \mathcal{C}(B \setminus A) = \emptyset.$$

Simultaneously, it is natural to consider also \mathcal{C} -extreme (shortly: extreme) subsets. By Proposition 1 it is clear that (2) may be used as a definition: if

$$A \cap \mathcal{C}(K) \subset \mathcal{C}(A \cap K) \quad \text{for any } K \subset B,$$

then A is called a \mathcal{C} -extreme subset of B . In the case $A = \{a\}$, a is called a \mathcal{C} -extreme point of B . In two special cases, the notion of \mathcal{C} -extreme point $a \in B$ has been introduced earlier: using (7) for \mathcal{C} being domain finite ([8], p. 151) and with the help of (10) when $B \in \mathcal{C}$ (see [13], p. 127 and [14], p. 119). From Proposition 5 we obtain a simply characterization of \mathcal{C} -extremeness of a point a of B , namely

$$a \notin \mathcal{C}(B \setminus \{a\}).$$

It enables us to observe a surprising connection of the notions of \mathcal{C} -extreme point and \mathcal{C} -independent set ([21], p. 38, [18], p. 174, [9], p. 27, [13], p. 120) which can be simply expressed by defining a \mathcal{C} -independent set as one with all points \mathcal{C} -extreme. Other properties of \mathcal{C} -extreme subsets and points are given in Theorems 1, 3, 4 and Propositions 4, 6, 8, 9, 10. Moreover, from (3), (4) and from the equality $\mathcal{C}^\cup = \mathcal{C}$ they result

characterizations of \mathcal{C} -extreme subsets for \mathcal{C} being domain finite and for \mathcal{C} with finite Caratheodory's number. Obviously, any \mathcal{C} -extreme subset of B is a \mathcal{C} -semi-extreme one of B . Moreover, \mathcal{C} -semi-extreme and \mathcal{C} -extreme points of arbitrary set coincide.

EXAMPLE 3: Let P be a set partially ordered by a relation \leq . For any $x, z \in P$ put $\bar{\Sigma}\{x, z\} = \{y; x \leq y \leq z \text{ or } z \leq y \leq x\}$ and $\Sigma\{x, z\} = \bar{\Sigma}\{x, z\} \setminus \{x, z\}$. A set $C \subset P$ is called *order convex* [7] if $\bar{\Sigma}\{x, z\} \subset C$ for any $x, z \in C$. Since the family of order convex subsets is a closure system over P , there is defined the corresponding closure (hull) operation Ω . By *order extreme subsets* we understand $\bar{\Sigma}$ -extreme subsets, i.e., Σ -extreme subsets (see Proposition 11). Order extreme elements have been introduced and discussed in [7]. From the equality $\bar{\Sigma}^\cup = \Omega$ (see [7]) and from Proposition 2 and Theorem 4 we obtain that order extreme and Ω -extreme subsets of arbitrary set coincide and that any set and its order convex hull have identical order extreme elements. As an example of order extreme elements, one can take \leq_a -maximal elements (see [23], pp. 18 and 30) in a semi-regular topological convexity structure and, more general, in any convexity structure such that for any a, b, c from $c \in \mathcal{C}\{a, b\}$ and $b \in \mathcal{C}\{a, c\}$ it results $b = c$. Similarly as in Examples 1 and 2 we can consider *relative order extreme subsets* (i.e. relative Σ -extreme subsets) and *Ω -semi-extreme subsets*. It is easy to test that order extreme subsets relative a set and relative its order convex hull are identical and that A is an Ω -semi-extreme subset of B if and only if A is an order extreme subset of B relative to $B \setminus A$.

EXAMPLE 4: Denote by cl the closure operation in a topological space T . From the last part of Proposition 1 we obtain that a subset A of B is cl -extreme if and only if $A \cap cl(B \setminus A) = \emptyset$. Particularly, cl -extreme points of B coincide with isolated points of B . Note that cl -extreme subsets of T are just open sets.

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