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Jesús Bastero
Zenaida Uriz

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# COTYPE FOR $p$-BANACH SPACES

Jesús Bastero and Zenaida Uriz *

The theorem of Maurey and Pisier, concerning the type and cotype of Banach spaces, is one of more important tools in the modern local theory of Banach spaces. Several different proofs of this theorem have been obtained (see [6],[7],[8]). In the $p$-Banach case, Kalton has proved the corresponding version for the type in an unpublished paper [4] and a weaker version for the cotype appears in [1].

The purpose of this paper is to prove the theorem of Maurey and Pisier for the cotype in $p$-Banach spaces, sharpening the partial results of [1]. More exactly, we will show the following theorem.

THEOREM 1: *If $X$ is a p-Banach space and $q(X)=\inf\{q;\ X$ is of q-Rademacher cotype$\}$, then $l^{q(X)}$ is finitely representable in $X$, when $q(X)<\infty$ and, for each $\epsilon>0$, $c_0$ is $(2^{1/p-1}+\epsilon)$ finitely representable in $X$, when $q(X)=\infty$.*

In order to prove this theorem we use an important and fundamental theorem due to Krivine [5] for $p=1$; its generalization to $p$-Banach case is not trivial, but goes through with few modifications which are necessary to replace Banach lattices by a more general situation.

All the spaces here considered are real. Let $0<p\leqslant 1$. A *p-convex norm* on a vector space $X$ is a function, denoted by $\|\cdot\|$, defined on $X$, with values in $\mathbb{R}_+$, that verifies:

$$\|x\|>0, \quad x\neq 0$$

$$\|ax\|=|a|\,\|x\|, \quad a\in\mathbb{R}, \quad x\in X$$

$$\|x+y\|^p\leqslant\|x\|^p+\|y\|^p, \quad x,\ y\in X$$

The subsets $\{x;\ \|x\|\leqslant 1/n\}$, for $n\in\mathbb{N}$, constitute a fundamental system of neighbourhoods of zero for a metric linear topology on $X$. If $X$ is complete with respect to this topology, we say that $(X,\|\cdot\|)$ is a *p-Banach space*.

* The contribution of second author to this paper forms part of her doctoral thesis.

DEFINITION 2: A $p$-Banach space $X$ is of cotype $q$, $0 < q \leqslant \infty$, if there exists a constant $C > 0$ such that

$$\left(\sum_1^n \|x_i\|^q\right)^{1/q} \leqslant C\int_0^1 \|\sum_1^n r_i(t)x_i\| \mathrm{d}t$$

for all $x_1, \dots, x_n \in X$, where $\{r_i(t)\}_1^n$ are the Rademacher functions on [0,1].

By extending a result of Kahane (see [3]), all the $L^q(X)$-norms are equivalent ($0 < q < \infty$) on the linear space spanned by the elements $r_i(t)x_i$, $1 \leqslant i$, $x_i \in X$, and so, up changing the constant $C$, the exponent in the integral may be replaced by any other exponent in $(0, \infty)$. All $p$-Banach spaces are of cotype $\infty$. $L^r$ is of cotype $q$, for all $q \geqslant \max\{2, r\}$, If we denote $q(X) = \inf\{q;\ X \text{ is of cotype } q\}$ then $2 \leqslant q(X) \leqslant \infty$.

In [1], it is defined $q_X = \inf\{q \geqslant 0;$ the identity in $X$ is $(q, 1)$-summing$\}$ (the identity in $X$ is $(q, 1)$-summing if there exists a constant $C > 0$, so that,

$$\left(\sum_1^n \|x_i\|^q\right)^{1/q} \leqslant C\|\sum_1^n r_i(t)x_i\|_{L^\infty(X)}$$

when $x_1, \dots, x_n \in X$), and, there, it is proved that $l^{q_X}$ is finitely representable in $X$, for $q_X < \infty$; hence, $q_X = \sup\{q;$ the embedding $l^q \to c_0$ is finitely factorizable through $X\}$, when $q_X < \infty$. As a trivial corollary of Theorem 1, we have

COROLLARY 3: $q_X = q(X)$.

Now, we are going to prove the Theorem 1. Before, we shall construct an auxiliary sequence, $(\varphi_q(n))_n$, related with the cotype constant of a space. Let $X$ be a $p$-Banach space. For every $q$, $2 \leqslant q < \infty$, we define $\varphi_q(n)$ as being the smallest positive constant $\varphi$ such that

$$\left(\sum_1^n \|x_1\|^q\right)^{1/q} \leqslant \varphi\left(\int_0^1 \|\sum_1^n r_i(t)x_i\|^q \mathrm{d}t\right)^{1/q}$$

for each $n$ elements $x_1, \dots, x_n$ of $X$. It is easily obtained that $\varphi_q(1) = 1$ and the sequence $(\varphi_q(n))_n$ is submultiplicative. For $p$-Banach spaces, the arguments work as they do for Banach spaces and, then, the following lemma is inmediately deduced as it is done in [6] Lemmas 1.2 and 1.3).

LEMMA 4: *If* $0 < \alpha < 1 - q/q(X)$, $\lim_{n\to\infty} n^\alpha/\varphi_q^q(n) = 0$ (*we put* $1/q(X) = 0$, *if* $q(X) = \infty$).

Next proposition is essentially based on [7] and the main ideas appearing there have been adapted here by using similar arguments to those of [1]. Because of that, we only will sketch the proof.

PROPOSITION 5: *Let $X$ be a p-Banach space. For each $q$, $2 \leqslant q < q(X)$, $\delta$, $0 < \delta < 1$, and $n \in \mathbb{N}$ there exist $n$ vectors $x_1^{q,n,\delta}, \ldots, x_n^{q,n,\delta}$ belonging to $X$ such that we have*:

$$1 - \delta \leqslant \| x_i^{q,n,\delta} \| \leqslant 1, \quad 1 \leqslant i \leqslant n \tag{i}$$

$$\left( \int_0^1 \| \sum_{i \in H} r_i(t) x_i \|^q \mathrm{d}t \right)^{1/q} \leqslant 2^{1/p} |H|^{1/q} \tag{ii}$$

*if $H$ is a finite subset of $\{1, 2, \ldots, n\}$.*

SKETCH OF THE PROOF: Let $2 \leqslant q < q(X)$ and $0 < \delta < 1$. Select $0 < \alpha < < 1 - q/q(X)$. For adequate $\epsilon > 0$ we may choose a large enough $N \in \mathbb{N}$ such that

$$n < N^\alpha < \varphi_q^q(N) \epsilon (1 - \epsilon)^q \frac{1 - (1 - \delta)^q}{2^{q(1/p - 1)}}$$

Then, there exist $x_1, \ldots, x_N \in X$, verifying $\max_{1 \leqslant i \leqslant N} \| x_i \| = 1$ and

$$\left( \sum_{i=1}^N \| x_i \|^q \right)^{1/q} > \varphi_q(N)(1 - \epsilon) \left( \int_0^1 \| \sum_{i=1}^N r_i(t) x_i \|^q \mathrm{d}t \right)^{1/q}$$

Reasoning as in [7] and [1], it is possible to extract a subset of $\{x_1, \ldots, x_N\}$ which has cardinality bigger or equal than $n$ and satisfies the conclusions of the proposition after a normalization.

By passing to consecutive ultrapowers we obtain:

PROPOSITION 6: *Let $X$ be a p-Banach space. There exist an ultrapower $Y$ of $X$ and a normalized, invariant for spreading sequence $(x_n)_n$ in $Y$, such that, for each finite subset $H$ of $\mathbb{N}$*

$$\| \sum_{i \in H} r_i(t) x_i \|_{L^{q(X)}([0,1]; X)} \leqslant 2^{1/p} |H|^{1/q(X)}$$

PROOF: By using the same arguments of [1], proposition 10, the theorem of Egoroff and the theorem 3.2 of [4], we can easily get the result.

NOTE: The sequence obtained in the preceding proposition verifies $\| x_1 - x_2 \| > 0$. Indeed, if $x_1 = x_n$ for all $n \in \mathbb{N}$, then by Khintchine's inequality,

$$n^{1/2} \leqslant Cn^{1/q(X)}$$

for some absolute constant (or $n \leqslant 2^{1/p}$ when $q(X) = \infty$), which only holds if $q(X) = 2$. But, there is nothing to prove when $q(X) = 2$, because this is the Dvoretzky-Rogers theorem.

In this point the proof of the Theorem 1 splits in two parts: the first one for $q(X) = \infty$ and the other one for $q(X) < \infty$.

**Case q(X) = ∞**

We only need to construct an invariant for spreading and for signs sequence (see [4]) having closed span isomorphic to $c_0$, because we could apply the result of [9] in order to prove that $c_0$ is $(2^{1/p-1} + \epsilon)$-isomorphic to a subspace of the closed linear span of this sequence.

PROPOSITION 7: *Let X be a p-Banach space such that $q(X) = \infty$. There exist a normalized invariant for spreading and signs sequence, $(e_n)_n$, in an ultrapower of X and constants $c, C > 0$, satisfying*

$$c \max_{1 \leqslant i \leqslant n} |a_i| \leqslant \left\| \sum_1^n a_i e_i \right\| \leqslant C \max_{1 \leqslant i \leqslant n} |a_i|$$

*for each $n \in N$ and every $a_1, \ldots, a_n \in \mathbb{R}$.*

PROOF: Let $(x_n)_n$ be the sequence stated in Proposition 6. Put $z_n = x_{2n+1} - x_n$. It is clear that

$$\sup_n \left\| \sum_1^n z_i \right\| \leqslant 2^{1/p}.$$

Moreover

$$2^{-1/p} \|z_1\| \max_{1 \leqslant i \leqslant n} |a_i| \leqslant \left\| \sum_1^n a_i z_i \right\|$$

when $a_1, \ldots, a_n \in \mathbb{R}$. We can construct $(e_n)_n$ from $(z_n)_n$ as it is done in Theorem I.1 of [5]. Now, it easily follows that

$$2^{-1/p} \|z_1\| \max_{1 \leqslant i \leqslant n} |a_i| \leqslant \left\| \sum_1^n a_i e_i \right\| \leqslant 2^{2/p}(2^p - 1)^{-1/p} \max_{1 \leqslant i \leqslant n} |a_i|$$

when $a_1, \ldots, a_n \in \mathbb{R}$.

## Case q(X) < ∞

We have to obtain an invariant for spreading and for signs sequence in some ultrapower of $X$ and then apply the following theorem of Krivine (see [5]), which also holds in our context:

THEOREM 8: *Let $Y$ be a p-Banach space and let $(e_n)_n$ be a sign-invariant, spreading sequence in $Y$, such that, there exist $r$, $p \leqslant r < \infty$, and $c > 0$ verifying*

$$c \| \sum_1^n x_i \| \geqslant \left( \sum_1^n \| x_i \|^r \right)^{1/r} \tag{*}$$

*where the $x_i$'s $(1 \leqslant i \leqslant n)$ are pairwise disjoint finite linear combinations of $e_n$'s. If*

$$2^{-1/q} = \inf \left\{ \lambda > 0;\ \lim_{n \to \infty} \lambda^n \| \sum_1^{2^n} e_i \| = \infty \right\}$$

*then $l^q$ is finitely representable in $Y$.*

NOTE: If $q(X) < \infty$ and $(e_n)_n$ is a sign-invariant spreading sequence in $X$, then the preceding property (*) holds for some $r > q(X)$.

The following lemma has been remarked to us by Bernard Maurey.

LEMMA 9: *If $(e_n)_n$ is an invariant for spreading sequence in a p-Banach space $X$, then the sequence $(e_{2n-1} - e_{2n})_n$ is unconditional.*

PROOF: Let $u_n = e_{2n-1} - e_{2n}$ and $v_n = e_{4n-3} - e_{4n-2} + e_{4n-1} - e_{4n}$. The sequences $(u_n)_n$ and $(v_n)_n$ are invariant for spreading. Let $\alpha_1, \dots, \alpha_n$ be real numbers and $\epsilon_i = \pm 1$, $1 \leqslant i \leqslant n$,

$$\begin{aligned} \| \sum_1^n \epsilon_i \alpha_i u_i \|^p &= \| \sum_1^n \epsilon_i \alpha_i ( e_{5i-4} - e_{5i} ) \|^p \\ &\leqslant \| \sum_1^n \epsilon_i \alpha_i ( e_{5i-4} - e_{5i-3} + e_{5i-2} - e_{5i-1} ) \|^p \\ &\quad + \| \sum_1^n \epsilon_i \alpha_i ( e_{5i-3} - e_{5i-2} + e_{5i-1} - e_{5i} ) \|^p \\ &= 2 \| \sum_1^n \epsilon_i \alpha_i v_i \|^p \end{aligned}$$

Now, by defining $\bar{u}_i = e_{6i-4} - e_{6i-1}$ $(1 \leqslant i \leqslant n)$, $\bar{\bar{u}}_i = e_{6i-3} - e_{6i-2}$ if $\epsilon_i = 1$ and $\bar{\bar{u}}_i = e_{6i-5} - e_{6i}$ if $\epsilon_i = -1$ $(1 \leqslant i \leqslant n)$, we have

$$\begin{aligned}
\|\sum_i^n \epsilon_i \alpha_i v_i\|^p &= \|\sum_{\epsilon_i = 1} \alpha_i v_i - \sum_{\epsilon_i = -1} \alpha_i v_i\|^p \\
&= \|\sum_{\epsilon_i = 1} \alpha_i (e_{6i-4} - e_{6i-3} + e_{6i-2} - e_{6i-1}) \\
&\quad - \sum_{\epsilon_i = -1} \alpha_i (e_{6i-5} - e_{6i-4} + e_{6i-1} - e_{6i})\|^p \\
&= \|\sum_{i=1}^n \alpha_i (\bar{u}_i - \bar{\bar{u}}_i)\|^p \\
&\leqslant \|\sum_1^n \alpha_i \bar{u}_i\|^p + \|\sum_1^n \alpha_i \bar{\bar{u}}_i\|^p \\
&= 2\|\sum_1^n \alpha_i u_i\|^p
\end{aligned}$$

This implies that the sequence $(u_n)_n$ is $4^{1/p}$-unconditional.

NOTE: The constant of unconditionality obtained in the preceding proposition is not the best possible for Banach spaces. Other classical arguments give 2 as constant, but they do not work if $p < 1$ (see [2]).

We return, now, to the proof of the theorem in the case $q(X) < \infty$.

By Proposition 6 there exists a normalized spreading sequence $(x_n)_n$ in some ultrapower $Y$ of $X$ such that

$$\|\sum_1^n r_i(t) x_i\|_{L^{q(X)}([0,1];\, X)} \leqslant 2^{1/p} n^{1/q(X)}$$

By applying Lemma 9 we have

$$\sup_{\epsilon_i = \pm 1} \|\sum_1^n \epsilon_i (x_{2i-1} - x_{2i})\| \leqslant 4^{1/p} \|\sum_1^n \eta_i (x_{2i-1} - x_{2i})\|$$

for all choice of $\eta_i = \pm 1$. Then,

$$\sup_{\epsilon_i = \pm 1} \| \sum_1^n \epsilon_i (x_{2i-1} - x_{2i}) \|$$

$$\leqslant 4^{1/p} \| \sum_1^n r_i(t)(x_{2i-1} - x_{2i}) \|_{L^{q(X)}([0,1], X)}$$

$$\leqslant 8^{1/p} \| \sum_1^n r_i(t) x_i \|_{L^{q(X)}([0,1]; X)}$$

$$\leqslant 16^{1/p} n^{1/q(X)}.$$

As $\|x_{2i} - x_{2i}\| = \|x_1 - x_2\|$ we may proceed as proposition 11 of [1] and we obtain a normalized, sign-invariant and spreading sequence $(v_n)_n$ in some ultrapower of $X$ such that

$$\| \sum_1^n v_i \| \leqslant C n^{1/q(X)} \quad (n \in N, C = C(p))$$

As $q(X) < \infty$, $c_0$ is not finitely representable in $X$ and then, the sequence $(v_n)_n$ satisfies the conditions of the theorem of Krivine (Theorem 8). Also $n^{1/q} \leqslant \|\Sigma_1^n v_i\|$, for all $n \in N$ and for all $q > q(X)$, hence, $l^q(X)$ is finitely representable in $X$.

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Jesús Bastero and Zenaida Uriz
Sección de Mathemáticas
Facultad de Ciencias
Universidad de Zaragoza
Zaragoza
Spain