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THE FIRST EIGENVALUE OF THE LAPLACIAN FOR A POSITIVELY CURVED HOMOGENEOUS RIEMANNIAN MANIFOLD

Hajime Urakawa

§0. Introduction

The purpose of this paper is to compute the first eigenvalue of the Laplacian for a certain positively curved homogeneous Riemannian manifold.

By a theorem of A. Lichnérowicz and M. Obata, if the Ricci curvature Ric_M of an n -dimensional compact Riemannian manifold M satisfies $\text{Ric}_M \geq n - 1$, then the first eigenvalue $\lambda_1(M)$ of the Laplacian of M is bigger than or equal to n , and the equality holds if and only if M is the standard sphere S^n of constant curvature one. Moreover, due to [L.Z.], [L.T.], [C], [B.B.G.], the following eigenvalue pinching theorem is known:

THEOREM: *Let M be a compact, n -dimensional Riemannian manifold whose sectional curvature $K_M \geq 1$. Then, there exists a constant $C(n) > 1$ depending only on n such that $C(n)n \geq \lambda_1(M) \geq n$ only if M is homeomorphic to S^n .*

On the other hand, due to [B], [W], [A.W.], [B.B 1,2] the classification of compact homogeneous Riemannian manifolds with positive sectional curvature is known. Therefore, it would be interesting to know the first eigenvalues $\lambda_1(M)$ of these positively curved homogeneous manifolds. In this paper, we give a comparatively sharp estimate of $\lambda_1(M)$ for such manifolds and as an application we determine $\lambda_1(M)$ of 7-dimensional positively curved homogeneous Riemannian manifolds $SU(3)/T_{k,l}$, and the manifold $F_4/\text{Spin}(8)$ of flags in the Cayley plane (cf. Theorem 2.1). Moreover, in the appendix, we give a complete list of $\lambda_1(M)$ of all compact simply connected irreducible Riemannian symmetric spaces, which was already given in [N1] for the classical cases. As a further application, we obtain a complete list of compact simply connected irreducible Riemannian symmetric spaces (cf. Theorem A.1) so that the identity map is stable as a harmonic map (cf. [Sm]).

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§1. Homogeneous Riemannian manifolds with positive curvature

In this section, following [A.W], [W], [B.B 1,2], we prepare the results of classifying simply connected homogeneous Riemannian manifolds with positive curvature.

Let G be a compact connected Lie group, and H a closed subgroup. Let \mathfrak{g} be a Lie algebra of G , and \mathfrak{h} the subalgebra of \mathfrak{g} corresponding to H .

DEFINITION 1.1 (cf. [A.W]): The pair (G, H) satisfies *Condition (II)* if there exists an $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)_0$ on \mathfrak{g} such that the orthogonal complement \mathfrak{v} of \mathfrak{h} in \mathfrak{g} has an orthogonal decomposition $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ with the following properties:

- (i) $[\mathfrak{v}_1, \mathfrak{v}_2] \subset \mathfrak{v}_2$, $[\mathfrak{v}_1, \mathfrak{v}_1] \subset \mathfrak{h} \oplus \mathfrak{v}_1$, $[\mathfrak{v}_2, \mathfrak{v}_2] \subset \mathfrak{h} \oplus \mathfrak{v}_1$, and
- (ii) for $X = X_1 + X_2$, $Y = Y_1 + Y_2$ with $X_i, Y_i \in \mathfrak{v}_i$, $i = 1, 2$, $[X, Y] = 0$ and $X \wedge Y \neq 0$ imply $[X_1, Y_1] \neq 0$.

Putting $\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{v}_1$, \mathfrak{k} is a subalgebra of \mathfrak{g} and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair of rank one (cf. [B.B 1, p. 58]). Furthermore, the connected Lie subgroup K of G corresponding to \mathfrak{k} is closed (cf. [B.B 1, p. 44]). In fact, the center \mathfrak{z} of \mathfrak{g} is included in \mathfrak{k} (cf. [A.W, p. 97]). Then, we have the decompositions $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, and $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}$, where \mathfrak{g}' is the semi-simple part of \mathfrak{g} , and \mathfrak{k}' is a subalgebra of \mathfrak{k} . Since $(\mathfrak{g}', \mathfrak{k}')$ is a symmetric pair with the semi-simple Lie algebra \mathfrak{g}' , the connected subgroup K' in G corresponding to \mathfrak{k}' is closed in the connected Lie group G' corresponding to \mathfrak{g}' (cf. [He, p. 179, the last part of the proof of Proposition 3.6]). Since G' is compact, K' is closed in G . Since the center Z of G is closed, the Lie group $Z \cdot K'$ is closed in G . Therefore its identity component K is closed.

DEFINITION 1.2 (cf. [A.W]): For $-1 < t < \infty$, we define an $\text{Ad}(H)$ -invariant inner product $(\cdot, \cdot)_t$ on $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ by setting

$$(X_1 + X_2, Y_1 + Y_2)_t := (1 + t)(X_1, Y_1)_0 + (X_2, Y_2)_0,$$

for $X_i, Y_i \in \mathfrak{v}_i$, $i = 1, 2$, and let g_t be a G -invariant Riemannian metric on G/H induced from $(\cdot, \cdot)_t$.

Moreover let h be a G -invariant Riemannian metric on G/K induced by the inner product $(\cdot, \cdot)_0$ on \mathfrak{v}_2 . Then, the natural projection $\pi: G/H \rightarrow G/K$ induces a Riemannian submersion $\pi: (G/H, g_t) \rightarrow (G/K, h)$ with totally geodesic fibers for all $-1 < t < \infty$ (cf. [B.B.B]).

Note that in case $\mathfrak{v}_1 = \{0\}$, $(G/H, g_0) = (G/K, h)$ is a Riemannian

symmetric space of rank one, and in case $v_2 = \{0\}$, Condition (II) implies the one such that the normally homogeneous Riemannian manifold $(G/H, g_0)$ has positive curvature.

THEOREM 1.3 (cf. [A.W, Theorem 2.4], [H, Corollary 2.2]): *Let (G, H) be a pair satisfying Condition (II), and $v_1 \neq \{0\}$, and $v_2 \neq \{0\}$. Let g_t , $-1 < t < \infty$ the G -invariant metric on G/H given in Definition 1.2. Then the Riemannian manifold $(G/H, g_t)$, $-1 < t < 0$, has positive curvature.*

THEOREM 1.4 (cf. [W], [B.B 1,2], [B]). *All compact simply connected homogeneous spaces G/H which are not homeomorphic to S^n and have positively curved G -invariant Riemannian metrics are displayed in the following table:*

(I) *In the case of normally homogeneous spaces,*

G/H	
(1)	$SU(n+1)/S(U(n) \times U(1)) = P^n(\mathbb{C}), n \geq 2$
(2)	$Sp(n+1)/Sp(n) \times Sp(1) = P^n(H), n \geq 2$
(3)	$F_4/Spin(9) = P^2(\text{Cay})$
(4)	$Sp(2)/SU(2)$

(II) *In the case of Condition (II) with $v_1 \neq \{0\}$ and $v_2 \neq \{0\}$,*

G/H	G/K
(5) $Sp(n)/Sp(n-1) \times T^1 \approx P^{2n-1}(\mathbb{C}), n \geq 2$	$Sp(n)/Sp(n-1) \times Sp(1)$
(6) $SU(5)/Sp(2) \times T^1$	$SU(5)/S(U(4) \times U(1))$
(7) $SU(3)/T^2$	$SU(3)/S(U(2) \times U(1))$
(8) $SU(3)/T^1$	$SU(3)/S(U(2) \times U(1))$
(9) $U(3)/T^2 \approx SU(3)/T^1$	$SU(3)/S(U(2) \times U(1))$
(10) $SU(3) \times SU(2)/T^1 \times \overline{SU(2)}$	$SU(3)/S(U(2) \times U(1))$
(11) $Sp(3)/SU(2) \times SU(2) \times SU(2)$	$Sp(3)/Sp(2) \times Sp(1)$
(12) $F_4/Spin(8)$	$F_4/Spin(9)$

REMARK 1: Here we denote by T^k , $k = 1, 2$, k -dimensional tori. In case of (7), T^2 is a maximal torus in $SU(3)$. In cases of (8), (9), the embeddings of T^1 and T^2 are given in [A.W], [B.B2] or §2. In case of (10), $T^1 \times \overline{SU(2)}$ is defined by $\{(t \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, x); t \in T^1, x \in SU(2)\}$ which is a closed subgroup in $SU(3) \times SU(2)$. Here T^1 is a torus in $SU(3)$

whose diagonal entries are $(e^{-2i\eta}, e^{i\eta}, e^{i\eta})$, $\eta \in \mathbb{R}$, and $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ is an element in $SU(3)$, x being in $SU(2)$.

REMARK 2: The examples (4), (5) and (6) are due to [B], and the ones (7) ~ (12) to [W], [A.W]. The inclusion $SU(2) \hookrightarrow Sp(2)$ in (4) is not canonical (cf. [B]). In the cases (5) and (6) the normally homogeneous Riemannian metric g_0 has positive curvature.

REMARK 3: The pairs (G, H) satisfying Condition (II) are classified in [B.B1, p. 59]. All simply connected homogeneous spaces G/H satisfying Condition (II) which are not homeomorphic to S^n appear in the following table Theorem 1.4.

§2. The first eigenvalue of the Laplacian

In this section, we prove the following theorem:

THEOREM 2.1: *Let G/H be a homogeneous space as in Theorem 1.4. Let $(\cdot, \cdot)_0$ be the $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} given by*

$$(X, Y)_0 = -B(X, Y), \quad X, Y \in \mathfrak{g},$$

for (1) ~ (12), except (9), where B is the Killing form of \mathfrak{g} . For (9), we define $(\cdot, \cdot)_0$ by

$$(X, Y)_0 = -6\text{Trace}(XY), \quad X, Y \in \mathfrak{u}(3).$$

We define the inner product $(\cdot, \cdot)_t$, $-1 < t \leq 0$, on the orthogonal complement \mathfrak{v} of \mathfrak{h} in \mathfrak{g} with respect to $(\cdot, \cdot)_0$ as in Definition 1.2 for (5) ~ (12), and we consider only $(\cdot, \cdot)_0$ for the cases (1) ~ (4). Then, we can estimate the first eigenvalue $\lambda_1(g_t)$ of the G -invariant Riemannian metric g_t on G/H corresponding to $(\cdot, \cdot)_t$ as follows:

G/H	$\lambda_1(g_t), -1 < t \leq 0$
(1) $SU(n+1)/S(U(n) \times U(1))$	$\lambda_1(g_0) = 1$
(2) $Sp(n+1)/Sp(n) \times Sp(1)$	$\lambda_1(g_0) = \frac{n+1}{n+2}$
(3) $F_4/\text{Spin}(9)$	$\lambda_1(g_0) = \frac{2}{3}$
(4) $Sp(2)/SU(2)$	$\frac{5}{12} \leq \lambda_1(g_0)$
(5) $Sp(n)/Sp(n-1) \times T^1$	$\frac{2n+1}{4(n+1)} \leq \lambda_1(g_t) \leq \frac{n}{n+1}$
(6) $SU(5)/Sp(2) \times T^1$	$\frac{12}{25} \leq \lambda_1(g_t) \leq 1$
(7) $SU(3)/T^2$	$\frac{4}{9} \leq \lambda_1(g_t) \leq 1$
(8) $SU(3)/T^1$	$\lambda_1(g_t) = 1$
(9) $U(3)/T^2$	$\frac{1}{2} \leq \lambda_1(g_t) \leq 1$
(10) $SU(3) \times SU(2)/T^1 \times \overline{SU(2)}$	$\frac{3}{8} \leq \lambda_1(g_t) \leq 1$
(11) $Sp(3)/SU(2) \times SU(2) \times SU(2)$	$\frac{7}{16} \leq \lambda_1(g_t) \leq \frac{3}{4}$
(12) $F_4/\text{Spin}(8)$	$\lambda_1(g_t) = \frac{2}{3}$

REMARK: The cases (1) ~ (3) are known, see [C.W].

We prepare for the proof of Theorem 2.1 with Lemma 2.2.

LEMMA 2.2: *Under the assumptions of Theorem 2.1, the first eigenvalue $\lambda_1(g_t)$ of $(G/H, g_t)$, $-1 < t \leq 0$, can be estimated as*

$$\lambda_1(g_0) \leq \lambda_1(g_t) \leq \lambda_1(G/K, h), \quad -1 < t \leq 0,$$

where $\lambda_1(G/K, h)$ is the first eigenvalue of $(G/K, h)$.

PROOF: Since $\pi; (G/H, g_t) \rightarrow (G/K, h)$ is a Riemannian submersion with totally geodesic fibers, the (positive) Laplacian Δ_{g_t}, Δ_h of $(G/H, g_t), (G/K, h)$ satisfy

$$\Delta_{g_t}(f \cdot \pi) = (\Delta_h f) \cdot \pi, \quad f \in C^\infty(G/K)$$

(cf. [B.B.B, p. 188]). Thus the spectrum $\text{Spec}(\Delta_{g_t})$ includes $\text{Spec}(\Delta_h)$, in particular, $\lambda_1(g_t) \leq \lambda_1(G/K, h)$ for all t .

For the remaining inequality, we put $p = \dim(v_1)$ and $q = \dim(v_2)$. Let $\{X_i\}_{i=1}^p, \{Y_i\}_{i=1}^q$ be orthonormal bases of v_1, v_2 , respectively. Then, since $\{X_i/\sqrt{t+1}\}_{i=1}^p, \{Y_i\}_{i=1}^q$ are orthonormal with respect to $(\cdot, \cdot)_t$, the Laplacian Δ_{g_t} of $(G/H, g_t)$ can be expressed (cf. [M.U, p. 476]) as

$$\Delta_{g_t} = -\frac{1}{t+1} \hat{\lambda} \left(\sum_{i=1}^p X_i^2 \right) + \hat{\lambda} \left(\sum_{i=1}^q Y_i^2 \right),$$

in particular,

$$\Delta_{g_0} = -\hat{\lambda} \left(\sum_{i=1}^p X_i^2 \right) - \hat{\lambda} \left(\sum_{i=1}^q Y_i^2 \right),$$

where $\hat{\lambda}$ is the canonical isomorphism of the algebra of $\text{Ad}(H)$ -invariant polynomials of $v = v_1 \oplus v_2$ into the space of G -invariant differential operators on G/H . Therefore we obtain

$$\Delta_{g_t} = \Delta_{g_0} + \left(1 - \frac{1}{t+1}\right) \hat{\lambda} \left(\sum_{i=1}^p X_i^2 \right). \quad (2.1)$$

Here because of $-1 < t \leq 0$, the operator $P := (1$

$-\frac{1}{t+1}) \hat{\lambda} \left(\sum_{i=1}^p X_i^2 \right) = \frac{t}{t+1} \hat{\lambda} \left(\sum_{i=1}^p X_i^2 \right)$ is non-negative, i.e., $\int_{G/H} (Pf) f$

$dv_{g_t} \geq 0$ for $f \in C^\infty(G/H)$, where dv_{g_t} is the volume element of $(G/H, g_t)$. Note that

$$dv_{g_t} = (t+1)^{p/2} dv_{g_0}. \quad (2.2)$$

Therefore, using (2.1), (2.2) and the Mini-Max Principle (cf. [B.U, Proposition 2.1]), we obtain $\lambda_1(g_t) \geq \lambda_1(g_0)$, $-1 < t \leq 0$. Q.E.D.

PROOF OF THEOREM 2.1: The case (8) will be shown in Lemma 2.3. The upper estimate of $\lambda_1(g_t)$ can be obtained by the inequality $\lambda_1(g_t) \leq \lambda_1(G/K, h)$ in Lemma 2.2 and Theorem 2.1, (1)~(3). For the lower estimate we use the inequalities

$$\lambda_1(G, g) \leq \lambda_1(g_0) \leq \lambda_1(g_t), \quad -1 < t \leq 0.$$

Here, $\lambda_1(G, g)$ is the first eigenvalue of (G, g) whose metric g is the bi-invariant one induced from the inner product $(\cdot, \cdot)_0$ on \mathfrak{g} . The computations of $\lambda_1(G, g)$ are accomplished in the appendix and note that $\lambda_1(U(n+1), g) = \frac{1}{2}$ (cf. [T, p. 307, Remark 2] or a direct computation, see Appendix). Here, the metric g on $U(n+1)$ is the bi-invariant Riemannian metric on $U(n+1)$ which is induced from the inner product $(X, Y)_0 := -2(n+1) \text{Trace}(XY)$, $X, Y \in \mathfrak{u}(n+1)$.

Case (8). A 1-dimensional torus $H = T^1$ in $G = SU(3)$ is conjugate in $SU(3)$ to

$$T_{k,l} = \left\{ \begin{pmatrix} e^{2\pi i k \theta} & & \\ & e^{2\pi i l \theta} & \\ & & e^{-2\pi i (k+l)\theta} \end{pmatrix}; \theta \in \mathbb{R} \right\},$$

where k, l are integers. We know by Lemma 3.1 and Theorem 3.2 in [A.W], that the pair $(SU(3), T^1)$ satisfies Condition (II) if and only if T^1 is conjugate in $SU(3)$ to $T_{k,l}$ with $kl(k+l) \neq 0$. Moreover, since $T_{k,l} = T_{mk, ml}$, $m \in \mathbb{Z} - (0)$, we can assume without loss of generality that $H = T^1 = T_{k,l}$ where $kl \neq 0$ and k, l are relatively prime.

Let $K = S(U(2) \times U(1)) = \left\{ \begin{pmatrix} x & \\ & \det x^{-1} \end{pmatrix}; x \in U(2) \right\}$, and $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ the corresponding Lie algebras of $G = SU(3), K, H$, respectively. Let $(\cdot, \cdot)_0$ be the inner product on \mathfrak{g} defined by

$$(X, Y)_0 = -B(X, Y) = -6\text{Trace}(XY), \quad X, Y \in \mathfrak{g} = \mathfrak{su}(3),$$

and we put $v_1 := \mathfrak{h}^\perp \cap \mathfrak{k}$, $v_2 := \mathfrak{k}^\perp$, and $v = \mathfrak{h}^\perp = v_1 \oplus v_2$, where $\mathfrak{k}^\perp, \mathfrak{h}^\perp$ are the orthogonal complements of $\mathfrak{k}, \mathfrak{h}$ in \mathfrak{g} with respect to $(\cdot, \cdot)_0$, respectively. We define the $\text{Ad}(H)$ -invariant inner product $(\cdot, \cdot)_t$, $-1 < t < \infty$, on $v = v_1 \oplus v_2$ as in Definition 1.2 and let g_t be the G -invariant

Riemannian metric on $G/H = SU(3)/T_{k,l}$ induced by $(\cdot, \cdot)_t$. Then, we have:

LEMMA 2.3: *Assume that $kl(k+l) \neq 0$. Then the first eigenvalue $\lambda_1(g_t)$ of $(SU(3)/T_{k,l}, g_t)$, $-1 < t \leq 0$, is given by $\lambda_1(g_t) = 1$ for every $-1 < t \leq 0$.*

PROOF: We already know that

$$\lambda_1(g_0) \leq \lambda_1(g_t) \leq \lambda_1(G/H, h) = 1.$$

So we only have to show $\lambda_1(g_0) = 1$. For this, we use Theorem 1 in [U] which tells us that the eigenvalues of the Laplacian of $(SU(3)/T_{k,l}, g_0)$ are given by

$$f(n_1, n_2) := \frac{1}{9}(m_1^2 + m_2^2 - m_1 m_2 + 3m_1),$$

where $m_1 := n_1 + n_2$, $m_2 := n_2$, and n_1 and n_2 run over the set of nonnegative integral satisfying $S_{n_1, n_2}^{k,l} \neq 0$. Here $S_{n_1, n_2}^{k,l}$ is the number of all the integer solutions (p', q, r) of the equations:

$$\begin{cases} kn_1 - ln_2 - (2k+l)p' + (-k+l)q + (k+2l)r = 0, & \text{and} \\ 0 \leq p' \leq n_1, 0 \leq q \leq n_2, & \text{and } 0 \leq r \leq p' + (n_2 - q). \end{cases}$$

Notice that, since our metric g_0 in this paper is 6 times the Riemannian metric in [U], the eigenvalues of $(SU(3)/T_{k,l}, g_0)$ are $\frac{1}{6}$ of the ones of the paper.

Then, we can easily check that

$$f(n_1, n_2) \geq 1 = f(1, 1),$$

except the cases $(n_1, n_2) = (1, 0)$ or $(0, 1)$. However, $S_{1,0}^{k,l} = S_{0,1}^{k,l} = 0$ due to the assumption $kl(k+l) \neq 0$. Therefore, we have the desired result. Q.E.D.

Appendix: The first eigenvalues of symmetric spaces

The table of the first eigenvalues of the Laplacian of compact simply connected irreducible Riemannian symmetric spaces has been already given by [N1] for the classical cases. In this appendix, we give a complete list including the exceptional cases.

At first let G be a compact simply connected simple Lie group, \mathfrak{g} its Lie algebra, and g the bi-invariant Riemannian metric on G induced from the inner product (\cdot, \cdot) on \mathfrak{g} given by

$$(X, Y) = -B(X, Y), \quad X, Y \in \mathfrak{g}, \quad (\text{A.1})$$

where B is the Killing form of \mathfrak{g} . We denote by the same notation the inner product on \mathfrak{g}^* canonically induced from (\cdot, \cdot) on \mathfrak{g} . Then it is known (cf. [Su]) that the spectrum of (G, g) can be given Freudenthal's formula as follows:

$$\begin{cases} \text{the eigenvalues:} & (\lambda + 2\rho, \lambda), \\ \text{their multiplicities;} & d_\lambda^2, \end{cases}$$

where 2ρ is the sum of all positive roots of the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} relative to a maximal abelian subalgebra \mathfrak{t} of \mathfrak{g} , and d_λ is the dimension of the irreducible unitary representation of G with highest weight λ , and λ varies over the set $D(G)$ of all dominant weights in the dual \mathfrak{t}^* of \mathfrak{t} . Therefore, we get

$$\lambda_1(G, g) = \min\{(\tilde{\omega}_i + 2\rho, \tilde{\omega}_i); \quad 1 \leq i \leq l\},$$

where $\{\tilde{\omega}_i\}_{i=1}^l$ are the fundamental weights of \mathfrak{g} corresponding to the fundamental root system $\{\alpha_1, \dots, \alpha_l\}$ of \mathfrak{g} .

By the last table in [Bo], we know $D(G)$, 2ρ , and the inner product (\cdot, \cdot) in \mathfrak{t}^* , so we get the following table of the first eigenvalue of the Laplacian of (G, g) :

In this Table A.1, the symbol X means that the identity map of (G, g) is unstable.

Next, the spectrum of the Laplacian of an irreducible Riemannian symmetric space G/K of compact type is given as follows. Let G be a compact simply connected simple Lie group, K the corresponding closed subgroup of G . Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K , respectively, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition. We give the inner product (\cdot, \cdot) on \mathfrak{p} by the restriction of (A.1), and let h be the G -invariant Riemannian metric on G/K induced from (\cdot, \cdot) . Then, it is known (cf. [Su]) that the spectrum of the Laplacian of $(G/K, h)$ is given by

$$\begin{cases} \text{the eigenvalues;} & (\lambda + 2\delta, \lambda), \\ \text{their multiplicities;} & d_\lambda. \end{cases}$$

Here, λ varies over the set $D(G, K)$ of highest weights of all spherical representations of (G, K) , which is determined by [Su] as follows.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace in \mathfrak{p} , and \mathfrak{h} , a maximal abelian subalgebra of \mathfrak{g} containing \mathfrak{a} . Let π be a δ -fundamental root system, say $\pi = \{\beta_1, \dots, \beta_l\}$, $l = \dim(\mathfrak{h})$, $\pi_0 = \{\beta \in \pi; \beta|_{\mathfrak{a}} \equiv 0\}$, and 2δ is the sum of all positive roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$ relative to π . We denote by $\{\mu_1, \dots, \mu_l\}$ the fundamental weights of \mathfrak{g} corresponding to π , and put

TABLE A.1. The first eigenvalue of the Laplacian of a compact simply connected simple Lie group.

type of G	$\lambda_1(G, g)$	
$A_l, l \geq 1$	$\frac{l(l+2)}{2(l+1)^2}$	X
$B_l, l \geq 2$	$\min\left\{\frac{l}{2l-1}, \frac{l(2l+1)}{8(2l-1)}\right\}$ $\frac{5}{12}, l=2$ $= \frac{21}{40}, l=3$ $\frac{l}{2l-1}, l \geq 4$	X
$C_l, l \geq 2$	$\frac{2l+1}{4l+4}$	X
$D_l, l \geq 3$	$\min\left\{\frac{2l-1}{4l-4}, \frac{l(2l-1)}{16(l-1)}\right\}$ $\frac{15}{32}, l=3$ $= \frac{2l-1}{4l-4}, l \geq 4$	X
E_6	$\frac{13}{18}$	
E_7	$\frac{57}{72}$	
E_8	1	
F_4	$\frac{2}{3}$	
G_2	$\frac{1}{2}$	

$q = \dim(\mathfrak{a})$. Define

$$M_i, 1 \leq i \leq q, = \begin{cases} 2\mu_i, & \text{if } p\beta_i = \beta_i \text{ and } (\beta_i, \pi_0) = \{0\}, \\ \mu_i, & \text{if } p\beta_i = \beta_i \text{ and } (\beta_i, \pi_0) \neq \{0\}, \\ \mu_i + \mu_{i'}, & \text{if } p\beta_i = \beta_{i'} \text{ and } \beta_i \neq \beta_{i'}, \end{cases}$$

where p is Satake's involution. Then we have

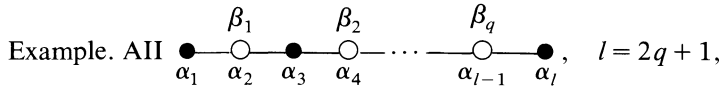
$$D(G, K) = \left\{ \sum_{i=1}^q m_i M_i; m_i \geq 0, m_i \in \mathbb{Z}, i = 1, \dots, q \right\}.$$

Then, since $(M_i + 2\delta, M_i) \geq 0$, we have

$$\lambda_1(G/K, h) = \min\{(M_i + 2\delta, M_i); i = 1, \dots, q\}.$$

Let $\{\alpha_1, \dots, \alpha_l\}, \{\tilde{\omega}_1, \dots, \tilde{\omega}_l\}, 2\rho$ be the fundamental root system, the corresponding fundamental weights, the sum of all positive roots, in the

last table in [Bo], respectively. Here, we should notice that the order in [Bo] is not in general the σ -order. Now $\{\beta_1, \dots, \beta_q\}$, $q = \dim \mathfrak{a}$, in the δ -fundamental system $\pi = \{\beta_1, \dots, \beta_l\}$ is given in the table in [Wr, p. 30–32] which is written as $\{\alpha_1, \dots, \alpha_{l^+}\}$. And $\{\beta_{q+1}, \dots, \beta_l\}$ are some black circles in the Satake diagram π . Ignoring the distinction between black circles and white circles in the Satake diagram, we get the Dynkin diagram $\{\alpha_1, \dots, \alpha_l\}$.



Then, define a one-to-one mapping ϕ from $\{\alpha_1, \dots, \alpha_l\}$ onto $\{\beta_1, \dots, \beta_l\}$ sending $\alpha_i (1 \leq i \leq l)$ to $\beta_{i^*} (1 \leq i^* \leq l)$ which has the same position as α_i in the diagram. In the above example AII, we get $\phi(\alpha_{2i}) = \beta_i, 1 \leq i \leq q$. Then ϕ can be extended to an automorphism of \mathfrak{g} due to Theorem 1 in [Seminaire “Sophus Lie” 1954/55, Ex. 11-04]. Under the identification of \mathfrak{g} and \mathfrak{g}^* with respect to the inner product (\cdot, \cdot) , the automorphism ϕ of \mathfrak{g} and \mathfrak{g}^* preserves the inner products (\cdot, \cdot) of \mathfrak{g} and \mathfrak{g}^* , $\phi(\tilde{\omega}_i) = \mu_{i^*}$ if $\phi(\alpha_i) = \beta_{i^*}$, and $\phi(\rho) = \delta$ by definition of $\tilde{\omega}_i, \mu_i, \rho$ and δ . In the above example AII, we get

$$M_i = \mu_i = \phi(\tilde{\omega}_{2i}), \quad \text{and} \quad (M_i + 2\delta, M_i) = (\tilde{\omega}_{2i} + 2\rho, \tilde{\omega}_{2i}),$$

for $i = 1, \dots, q$,

Therefore, we have a list of the first eigenvalues $\lambda_1(G/K, h)$ and the M_i 's of all simply connected irreducible Riemannian symmetric spaces $(G/K, h)$ of compact type:

Here in the Table A.2, \tilde{N} means the universal covering of N and X means that the identity map of $(G/K, h)$ is unstable.

As an application we can discuss the stability or unstability of the identity map of all compact simply connected irreducible Riemannian symmetric spaces. The identity map of a compact Riemannian manifold (M, g) onto itself is stable as a harmonic map (cf. [Sm]), if all the eigenvalues of the Jacobi operator coming from the second variation of a one parameter family of harmonic maps are non-negative. In case of an Einstein manifold (M, g) , i.e., $\text{Ric}_g = cg$, where Ric_g is the Ricci tensor of (M, g) , (M, g) is stable if and only if its first eigenvalue $\lambda_1(M, g)$ of the Laplacian on $C^\infty(M)$ satisfies $\lambda_1(M, g) \geq 2c$ (cf. [Sm, Proposition 2.1]).

Since a compact simply connected Lie group (G, g) whose metric g is induced from the inner product (A.1) satisfies (cf. [K.N, p. 204]) $\text{Ric}_g = \frac{1}{4}g$, we have:

the identity map of (G, g) is stable if and only if $\lambda_1(G, g) \geq \frac{1}{2}$.

TABLE A.2. The first eigenvalue of the Laplacian of a simple connected irreducible Riemannian symmetric space of compact type.

type of G/K	G/K	$M_i, 1 \leq i \leq q$	$\lambda_1(G/K, h)$
A I, $q \geq 2$	$SU(q+1)/SO(q+1)$	$M_i = 2\phi(\tilde{\omega}_1), 1 \leq i \leq q$	$\frac{(q+3)q}{(q+1)^2}$
A II, $q \geq 1$	$SU(2q+2)/Sp(q+1)$	$M_i = \phi(\tilde{\omega}_2), 1 \leq i \leq q$	$\frac{(2q+3)q}{2(q+1)^2}$ X
A III, $\frac{l}{2} \geq q \geq 2$	$SU(l+1)/S(U(l+1-q) \times U(q))$	$M_i = \phi(\tilde{\omega}_1) + \phi(\tilde{\omega}_{l-i+1}), 1 \leq i \leq q$	1
$q \geq 2$	$SU(2q)/S(U(q) \times U(q))$	$\begin{cases} M_i = \phi(\tilde{\omega}_1) + \phi(\tilde{\omega}_{l-i+1}), 1 \leq i \leq q-1 \\ M_q = 2\phi(\tilde{\omega}_q) \end{cases}$	1
A IV, $l \geq 1$	$SU(l+1)/S(U(l) \times U(1))$	$M_i = \phi(\tilde{\omega}_1) + \phi(\tilde{\omega}_l)$	1
B I, $l, \geq q \geq 2$	$SO(2l+1)/SO(2l+1-q) \times SO(q)$	$\begin{cases} M_i = 2\phi(\tilde{\omega}_1), 1 \leq i \leq q-1, \\ M_q = \phi(\tilde{\omega}_q) \\ M_i = 2\phi(\tilde{\omega}_1), 1 \leq i \leq q \end{cases}$	$\min \left\{ \frac{2l+1}{2l-1}, \frac{-q^2 + (2l+1)q}{4l-2} \right\}$ $= \begin{cases} 1, q=2, l \geq 2, \\ \frac{6}{5}, q=3, l=3, \\ \frac{2l+1}{2l-1}, \text{ otherwise} \end{cases}$
B II, $l \geq 2$	$SO(2l+1)/SO(2l)$	$M_i = \phi(\tilde{\omega}_1)$	$\frac{l}{2l-1}$ X
C I, $q \geq 3$	$Sp(q)/U(q)$	$M_i = 2\phi(\tilde{\omega}_1), 1 \leq i \leq q$	1
C II, $\frac{l-1}{2} \geq q \geq 1$	$Sp(l)/Sp(l-q) \times Sp(q)$	$M_i = \phi(\omega_2), 1 \leq i \leq q$	$\frac{l}{l+1}$ X
$q \geq 2$	$Sp(2q)/Sp(q)Sp(q)$		$\frac{2q}{2q+1}$ X

TABLE A.2. (continued)

type of G/K	G/K	$M_i, 1 \leq i \leq q$	$\lambda_1(G/K, h)$
D I, $l-2 \geq q \geq 2$	$SO(2l)/SO(2l-q) \times SO(q)$	$\begin{cases} M_i = 2\phi(\omega_i), 1 \leq i \leq q-1, \\ M_q = \phi(\omega_q) \end{cases}$	$\min \left\{ \frac{l}{l-1}, \frac{-q^2+2lq}{4l-4} \right\}$ $= \begin{cases} 1, q=2, \\ \frac{l}{l-1}, q \geq 3, \end{cases}$
$q \geq 2$	$SO(2q+2)/SO(q+2) \times SO(q)$	$\begin{cases} M_i = 2\phi(\tilde{\omega}_i), 1 \leq i \leq q-1, \\ M_q = \phi(\tilde{\omega}_q) + \phi(\tilde{\omega}_{q+1}) \end{cases}$	$\min \left\{ \frac{q+1}{q}, \frac{q+2}{4} \right\}$ $= \begin{cases} 1, q=2, \\ \frac{5}{4}, q=3, \\ \frac{q+1}{q}, q \geq 4, \end{cases}$
$q \geq 2$	$SO(2q)/SO(q) \times SO(q)$	$M_i = 2\phi(\tilde{\omega}_i), 1 \leq i \leq q$	$\min \left\{ \frac{q}{q-1}, \frac{q^2}{4q-4} \right\}$ $= \begin{cases} 1, q=2, \\ \frac{9}{8}, q=3, \\ \frac{q}{q-1}, q \geq 4, \end{cases}$
D II, $l \geq 2$	$SO(2l)/SO(2l-1)$	$M_1 = \phi(\tilde{\omega}_1)$	$\frac{2l-1}{4l-4}$ X
D III, $q \geq 2$	$SO(4q)/U(2q)$	$\begin{cases} M_i = \phi(\tilde{\omega}_i), 1 \leq i \leq q-1, \\ M_1 = 2\phi(\tilde{\omega}_{2q}) \end{cases}$	1
$q \geq 2$	$SO(4q+2)/U(2q+1)$	$\begin{cases} M_i = \phi(\tilde{\omega}_i), 1 \leq i \leq q-1, \\ M_q = \phi(\tilde{\omega}_{2q}) + \phi(\tilde{\omega}_{2q+1}) \end{cases}$	1
E I	$\widetilde{E_6/Sp(4)}$	$M_i = 2\phi(\tilde{\omega}_i), 1 \leq i \leq 6$	$\frac{14}{9}$
E II	$\widetilde{E_6/SU(2) \cdot SU(6)}$	$\begin{aligned} M_1 &= 2\phi(\tilde{\omega}_2), M_2 = 2\phi(\tilde{\omega}_4), \\ M_3 &= \phi(\tilde{\omega}_3) + \phi(\tilde{\omega}_5), \\ M_4 &= \phi(\omega_1) + \phi(\tilde{\omega}_6) \end{aligned}$	$\frac{3}{2}$

E IV	E_6/F_4	$M_1 = \phi(\tilde{\omega}_1), M_2 = \phi(\tilde{\omega}_6)$	$\frac{13}{18}$	X
E V	$\overbrace{E_7/SU(8)}$	$M_i = 2\Phi(\tilde{\omega}_i), 1 \leq i \leq 7$	$\frac{5}{3}$	\cdot
E VI	$\overbrace{E_7/SO(12) \cdot SU(2)}$	$M_1 = 2\phi(\tilde{\omega}_1), M_2 = 2\phi(\tilde{\omega}_3),$ $M_3 = \phi(\tilde{\omega}_4), M_4 = \phi(\tilde{\omega}_6)$	$\frac{14}{9}$	
E VII	$\overbrace{E_7/E_6 \cdot SO(2)}$	$M_1 = 2\phi(\tilde{\omega}_7),$ $M_2 = \phi(\tilde{\omega}_6), M_3 = \phi(\tilde{\omega}_1)$	1	
E VIII	$E_8/SO(16)$	$M_i = 2\phi(\tilde{\omega}_i), 1 \leq i \leq 8$	$\frac{31}{15}$	
E IX	$\overbrace{E_8/E_7 \cdot SU(2)}$	$M_1 = 2\phi(\tilde{\omega}_8), M_2 = 2\phi(\tilde{\omega}_7),$ $M_3 = \phi(\tilde{\omega}_6), M_4 = \phi(\tilde{\omega}_1)$	$\frac{8}{5}$	
F I	$\overbrace{F_4/Sp(3) \cdot SU(2)}$	$M_i = 2\phi(\tilde{\omega}_i), 1 \leq i \leq 4$	$\frac{13}{9}$	
F II	$F_4/Spin(9)$	$M_1 = \phi(\tilde{\omega}_4)$	$\frac{2}{3}$	X
G	$G_2/SU(2) \times SU(2)$	$M_i = 2\phi(\tilde{\omega}_i), i = 1, 2$	$\frac{7}{6}$	

Moreover we know the Ricci tensor Ric_h of a simply connected irreducible Riemannian symmetric space $(G/K, h)$ of compact type satisfies $\text{Ric}_h = \frac{1}{2} h$ (cf. [T.K, p. 213]), so we have:

the identity map of $(G/K, h)$ is stable if and only if $\lambda_1(G/K, h) \geq 1$.

Together with the Tables A.1, A.2, we obtain:

THEOREM A.1: (1) *Let G be a compact simply connected simple Lie group, g a bi-invariant Riemannian metric on G . Then, the identity map of (G, g) is unstable if and only if the type of G is one of the following: $A_l, l \geq 1, B_2, C_l, l \geq 2$ and D_3 .* (2) *Let $(G/K, h)$ be a simply connected irreducible Riemannian symmetric space of compact type. Then, the identity map is unstable if and only if the type of G/K is one of the following: AII, BII, CII, DII, EIV and FII, that is, $SU(2q+2)/Sp(q+1), q \geq 1$, the unit sphere $S^n, n \geq 3$, the quaternion Grassmann manifolds $Sp(l)/Sp(l-q) \times Sp(q), l-q \geq q \geq 1, E_6/F_4$, and the Cayley projective space $F_4/Spin(9)$.*

REMARK: The classical irreducible Riemannian symmetric spaces with stable or unstable identity map have been known in [Sm, Proposition 2.13], and also see [N2]. However it should be noticed that the statement (3.1) in [N2] is false.

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