E. Ballico  
Ph. Ellia  

On the hypersurfaces containing a general projective curve  

*Compositio Mathematica*, tome 60, no 1 (1986), p. 85-95  
<http://www.numdam.org/item?id=CM_1986__60_1_85_0>
ON THE HYPERSURFACES CONTAINING A GENERAL PROJECTIVE CURVE

E. Ballico and Ph. Ellia

If $C$ is a smooth curve in $\mathbb{P}^N$ a natural question to ask is the number of hypersurfaces of degree $k$ containing the curve $C$. This turns out to the study of the natural map of restriction $r_C(k): H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(C, \mathcal{O}_C(k))$. We say that $C$ has maximal rank if for every $k \geq 1$ $r_C(k)$ has maximal rank as a map between vector spaces. In this paper we prove the following theorem.

**THEOREM 1:** Fix integers $N$, $d$, $g$ with $N \geq 3$, $g \geq 0$, $d \geq \max(2g - 1, g + N)$. Then a general non degenerate embedding of degree $d$ in $\mathbb{P}^N$ of a general curve of genus $g$ has maximal rank.

The proof of Theorem 1 gives as a byproduct the following result.

**THEOREM 2:** Fix an integer $N \geq 3$. There exists a function $e_N: \mathbb{N} \to \mathbb{N}$ with $\lim_{g \to +\infty} e_N(g) = +\infty$ and with the following property: for all integers $d$, $g$ with $g \geq 0$, $d \geq 2g - e_N(g)$, a general embedding of degree $d$ in $\mathbb{P}^N$ of a general curve of genus $g$ has maximal rank.

Both theorems are particular cases of the maximal rank conjecture, which states that a general embedding of a curve with general moduli has maximal rank.

Previously we proved stronger results for $N = 4$ ([2]) and $N = 3$ ([3]). We use in an essential way reducible curves and the general methods introduced in [5] and [7]. The smoothing theorems we use were proved in [9] and [6].

**Notations**

We work over an algebraically closed field. Fix a closed subscheme $X$ of a projective space $K$. Let $r_{X,K}(n): H^0(K, \mathcal{O}_K(n)) \to H^0(X, \mathcal{O}_X(n))$ be the restriction map and let $\mathcal{F}_{X,K}$ be the ideal sheaf of $X$ in $K$. If $K = \mathbb{P}^N$, we will write often $r_X(n)$ and $\mathcal{F}_X$ instead of $r_{X,K}(n)$ and $\mathcal{F}_{X,K}$. Fix integers $d$, $g$, $N$ with $N \geq 3$, $g \geq 0$, $d > 0$. Let $Z(d, g; N)$ be the
closure in the Hilbert scheme \( \text{Hilb} \, \mathbb{P}^N \) of the set of smooth, connected curves \( C \) in \( \mathbb{P}^N \) with \( \text{deg} \, C = d \), \( C \) of genus \( g \), \( h^1(C, \mathcal{O}_C(1)) = 0 \), and spanning a linear space of dimension \( \min(N, d - g) \). Obviously \( Z(d, g; N) \) is irreducible.

Fix a curve \( C \) and a line \( L \) in \( \mathbb{P}^N \); \( L \) is a \( k \)-secant to \( C \), \( k = 1, 2 \), if it intersects \( C \) exactly at \( k \) points, all smooth points of \( C \), and quasi-transversally.

§1. Preliminaries

As in [7], [1], [2], [3] we use in an essential way the existence of suitable reducible curves in \( Z(d, g; N) \). Fix a curve \( X \in Z(d, g; N) \) with at most ordinary nodes as singularities and \( h^1(X, N_X) = 0 \), where \( N_X \) is the normal bundle of \( X \) in \( \mathbb{P}^N \) and a line \( L \) which is \( k \)-secant to \( X \) with \( k = 1 \) or \( 2 \). If \( d < g + N \) and \( k = 1 \), assume that \( L \) is not contained in the linear space spanned by \( X \). Then \( X \cup L \) is in \( Z(d + 1, g + k - 1; N) \) ([9] or [6]).

Fix integers \( d, g, N \) with \( g \geq 0 \), \( N \geq 3 \) and \( d \geq g + N \). If \( d = N \), we say that \( (N, 0; N) \) has critical value 1. If \( d > N \), let \( n \) be the first integer \( m \geq 2 \) such that

\[
md + 1 - g \leq \left( \frac{N + m}{N} \right); 
\]  

(1)

in this case we say that \( (d, g; N) \) (or \( (d, g) \) for short) has critical value \( n \). Note that if (1) is satisfied, then

\[
d(m + 1) + 1 - g \leq \left( \frac{N + m + 1}{N} \right),
\]

because (1) implies

\[
d \leq \left( \frac{N + m}{N} \right) / (m - 1)
\]

and the inequality we have to check follows from the inequality:

\[
d < \left( \frac{N + m}{N - 1} \right).
\]

We say that the surjective part of Theorem 1 holds in \( \mathbb{P}^N \) for a datum \( (d, g) \) with critical value \( n \) if for a general \( Y \in Z(d, g; N) \) the restriction map \( r_Y(n) \) is surjective. We say that the injective part of Theorem 1 holds for the datum \( (d, g; N) \) with critical value \( n \) if for a general \( X \in Z(d, g; N) \) the map \( r_X(n - 1) \) is injective. By Castelnuovo’s lemma
Theorem 1 holds if for all data the injective and the surjective parts of Theorem 1 are true. Theorem 1 is trivial for all data with critical value 1. The injective part of Theorem 1 is trivial for all data with critical value 2.

In 1.1 we show in particular that the surjective part of Theorem 1 is true for all data with critical value 2. The next result can be considered as a partial extension to non-complete linear systems of [1].

**PROPOSITION 1.1:** Fix integers \( d, g, N \) with \( N \geq 3, g \geq 0, d \geq g + N \) and \( 2d + 1 - g \leq (N + 1)(N + 2)/2 \). Then a general element of \( Z(d, g; N) \) has maximal rank.

**PROOF:** If \( d = g + N \), the result was proved in [1]. Assume \( d > g + N \) and the result true for \( (d - 1, g; N) \). Fix \( X \in Z(d - 1, g; N) \) with maximal rank, hence with \( r_X(2) \) surjective. It is sufficient to prove that for a general line \( L \) intersecting \( X \), we have \( \dim \ker r_{X \cup L}(2) \leq \dim \ker r_X(2) - 2 \). We may assume \( X \) irreducible. Fix a point \( P \) which is not a base point of \( H^0(P^N, \mathcal{I}_X(2)) \). If \( L \) is a line containing \( P \) we have \( \dim \ker r_{X \cup L}(2) \leq \dim \ker r_X(2) - 1 \). Fix a quadric \( Q \) containing \( X \) and \( P \). If \( L \not\subset Q \), then \( \dim \ker r_{X \cup L}(2) < \dim \ker r_{X \cup \{P\}}(2) \): we won. If \( P' \) is a point of \( Q \), \( P' \) near \( P \), then \( P' \) is not a base point of \( H^0(P^N, \mathcal{I}_X(2)) \). Hence we won if for a fixed \( A \in X \) and a general \( P' \) in \( Q \), the line \( [AP'] \) is not contained in \( Q \). If for all such \( P' \), \( [AP'] \) is contained in \( Q \), then \( Q \) is a cone with vertex \( A \). But since \( X \) is non-degenerate, \( Q \) cannot be a cone with vertex containing \( X \). \( \square \)

**§2. Intersection with a hyperplane**

The following easy lemma is the heart of this paper.

**LEMMA 2.1:** Fix \( N \geq 3, n \geq 1 \). Let \( C \subset \mathbb{P}^N \) be a nondegenerate, irreducible curve and \( H \subset \mathbb{P}^N \) a hyperplane. Fix a vector subspace \( V \) of \( H^0(H, \mathcal{O}_H(n)) \). For a curve \( A \) in \( \mathbb{P}^N \), \( A \) intersecting transversally \( H \), set \( V(A) := \{ f \in V : f(P) = 0 \text{ for each } P \in A \cap H \} \). Then for a general reducible conic \( S \) such that each of the irreducible components of \( S \) intersects \( C \), we have \( \dim V(S) = \max(0, \dim V - 2) \).

**PROOF:** For a general line intersecting \( C \), we have \( \dim V(L) = \max(0, \dim V - 1) \). Hence we may assume \( \dim V \geq 2 \). Suppose that the lemma is false. Then for every line \( R \) intersecting \( C \) and \( L \) but not contained in \( H \), \( V(L) \) has \( R \cap H \) in the base locus. But if \( R \) is near to \( L \), \( R \cap H \) is not in the base locus of \( V \), hence \( L \cap H \) is in the base locus of \( V(R) \) and we have \( V(R) = V(L) \). For a general line \( B \) intersecting \( C \) and \( R \) we have \( V(B) = V(R) \). In a finite number of steps we obtain that
$V(L)$ has $H$ in the base locus, because $C$ is not degenerate: contradiction. □

This lemma is the key difference between this paper and [2]. Now the proofs are easier and shorter, but the result weaker. To show how we will use this lemma we state an immediate Corollary of 2.1.

**Corollary 2.2:** Fix non negative integers $n, d, g, x, n, j$ with $N \geq 3$, $n \geq 1$, $x \leq d$. Fix a hyperplane $H$ in $\mathbb{P}^N$ and a curve $W$ in $H$ with $r_{W,H}(n)$ surjective. Let $j$ be the dimension of the linear space spanned by $W$; if $j \leq N - 2$ assume $x \leq j + 1$ and set $j' = j$; otherwise set $j' = j + 1$. Assume $d \geq 2g + \max(0, x - j' - 1)$. Then there exists $Y \in Z(d, g; N)$, $Y$ intersecting transversally $H$, with $\text{card}(Y \cap W) = x$ and $r_{W \cup (Y \cap H),H}(n)$ of maximal rank.

**Proof:** Note that $\text{Aut}(H)$ acts transitively on the set of $N + 1$ ordered points of $H$ such that any $N$ of them span $H$. Hence the case $d = N$ is trivial and we assume $d > N$. For the same reason there is a curve $C \in Z(N + 1, \min(1, g); N)$ intersecting $H$ transversally with $\text{card}(C \cap W) = \min(N + 1, x)$ and $r_{W \cup (C \cap H),H}(n)$ of maximal rank. Then we take $\max(0, N + 1 - x)$ lines $L_i$, each $L_i$ intersecting both $C$ and $W$. Then we apply 2.1. □

§3. Proof of Theorem 1

In section 1 we proved Theorem 1 for curves with critical value at most 2. Since Theorem 1 is known to be true in $\mathbb{P}^3$ and $\mathbb{P}^4$ ([3],[2]), it is sufficient to prove the following two lemmas.

**Lemma 3.1:** Fix $N \geq 5$, $n \geq 3$. Assume that theorem 1 hold in $\mathbb{P}^s$ for all $s$ with $3 \leq s \leq N - 1$ and that theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $< n$. Then the surjectivity part of theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $n$.

**Lemma 3.2:** Fix $N \geq 5$, $n \geq 3$. Assume that theorem 1 holds in $\mathbb{P}^s$ for all $s$ with $3 \leq s \leq N - 1$ and that theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $< n$. Then the injectivity part of theorem 1 holds in $\mathbb{P}^N$ for all data with critical value $n$.

In this section we prove 3.1 and 3.2, hence Theorem 1. Fix a datum $(d, g)$ with $d \geq \max(g + N, 2g - 1)$ and critical value $n \geq 3$ in $\mathbb{P}^N$, $N \geq 5$. 

PROOF OF LEMMA 3.1: Fix natural numbers $p$, $g'$ with $p \leq g$, $g' \leq g$ and maximal with the following properties

$$
(2p + N)(n - 1) + 1 - p \leq \binom{N + n - 1}{N}
$$

$$
(n - 1)(\max(g' + N, 2g' - 1)) + 1 - g' \leq \binom{N + n - 1}{N}
$$

The integers $p$, $g'$ exist because $(N, 0; N)$ has critical value $1 \leq n - 1$. Define integers $f \geq 2p + N$, $d' = \max(g' + N, 2g' - 1)$ by the relations

$$
\binom{N + n - 1}{N} - n + 2 \leq (n - 1)f + 1 - p \leq \binom{N + n - 1}{N}
$$

(2)

$$
\binom{N + n - 1}{N} - n + 2 \leq (n - 1)d' + 1 - g' \leq \binom{N + n - 1}{N}
$$

(3)

Note that $p \leq g'$ and $f \leq d' < d$ because $(d, g)$ has critical value $n$. Set $d'' = d - d'$, $g'' = g - g'$, $x = \min([(d - f + 1)/2], g - p)$, $j = g - p - x$, $e = \binom{N + n}{N} - nd - 1 + g$,

$$
k = \binom{N + n - 1}{N} - (n - 1)f - 1 + p,
$$

$$
k' = \binom{N + n - 1}{N} - (n - 1)d' - 1 + g'.
$$

By (2) and (3) we have $0 \leq k \leq n - 2$ and $0 \leq k' \leq n - 2$. By the definition of $k$ and $e$ we obtain

$$
(d - f)n + 1 - (g - p) + (f - 1) + (e - k) = \binom{N + n - 1}{N - 1}
$$

(4)

By the maximality of $p$ we have either $p = g$ or $f \leq 2p + N + 1$, hence $d - f \geq 2(g - p) - N - 2$. Hence we have $j \leq (N + 3)/2$. By the maximality of $g'$ we have either $g' = g$ or $d' \leq 2g'$ or $g' + N \geq 2g' - 1$ and $d' \leq g' + N + 1$. Assume $g' + N \geq 2g' - 1$, hence $g' \leq N + 1$. Since $k' \leq n - 2$ we obtain

$$
(n - 1)(g' + N + 1) + 1 - g' + (n - 2) \leq \binom{N + n - 1}{N}
$$

which is false for $N \geq 5$, $n \geq 3$. Hence we have $d'' \geq 2g'' - 1$.

We need two numerical lemmas:

**Sublemma 3.3:** If $N \geq 5$ and $n \geq 3$, we have $f \geq 2n - 4 + N$. 

PROOF: Since
\[(n-1)f \geq \binom{N+n-1}{N} - 1,\]
the lemma is trivial. \(\square\)

**Sublemma 3.4:** Assume \(k > e\). Then
(a) \(d - f \geq 2n - 1 + N\) if \(N > 5\), \(n \geq 4\) or \(N \geq 6\), \(n \geq 3\).
(b) \(d - f \geq 9\) if \(N = 5\), \(n = 3\) and if \(d - f = 9\), then \(g - p \leq 4\).
(c) \(d'' \geq n - 1\); \(d - f \geq 2N - 2\), hence \(d - f \geq x + N - 1\).

**Proof:** (a) By (2) we have
\[f \leq \binom{N+n}{N} / (n - 3/2).\]
Then (4) gives the contradiction if \(N \geq 5\), \(n \geq 6\) or \(N \geq 6\), \(n = 5\) or \(N \geq 7\), \(n = 4\) or \(N \geq 12\), \(n = 3\). The remaining cases for (a) and (b) have to be checked directly. For example assume \(N = 5\), \(n = 3\). By the definitions of \(p\) and \(f\) we obtain \(p = 3\) and \(f = 11\). From (4) we get \(d - f \geq 9\) and if \(d - f = 9\), then \(g - p \leq 4\). Part (c) is easier. \(\square\)

We distinguish 5 cases.

**Case (A):** \(k \leq e\), \(d - f \geq g - p + 1\), \(d - f \geq 6\). Take a hyperplane \(H\). We claim the existence of \(W \subset H\), \(W \in Z(d - f, x; N - 1)\) with \(r_{W,H}(n)\) surjective. Indeed since \(d - f - x \geq 3\), we have \(Z(d - f, x; N - 1) \neq \emptyset\). If a general \(W \in Z(d - f, x; N - 1)\) spans \(H\), the claim follows from the inductive assumption, (4) and the inequality \(f - 1 \geq j\) which holds by 3.3. If a general \(W \in Z(d - f, x; N - 1)\) does not span \(H\), it spans a linear space of dimension \(d - f - x \geq 3\) and we may use the inductive assumption and the inequality
\[n(d - f) + 1 - x \leq \binom{d - f - x + n}{n}\]
which is true if \(n \geq 3\), \(d - f \geq 6\).

We may assume that a curve \(W\) as in the claim contains \(j + 1\) general points of \(H\) because \(d - f - x \geq j + 1\). By the inductive assumption, the inequality \(f - p \geq N + j + 1\) and Corollary 2.2 we may find \(X \in Z(f, p; N)\), \(X\) intersecting transversally \(H\), with \(\text{card}(X \cap W) = j + 1\) and \(r_{W,H}(n)\) surjective. Since \(W\) can be degenerate to a suitable union of lines, \(X \cup W\) is a smooth point of \(\text{Hilb} \mathbb{P}^N\) and \(W \cup W \in Z(d, g; N)\).

Take \(A \subset \mathbb{P}^N \setminus H\), \(B \subset H\), with \(\text{card}(A) = k\), \(\text{card}(B) = e - k\), \(A\) and \(B\) general. It is sufficient to prove that \(r_{X \cup W \cup A \cup B}(n)\) is injective, hence...
bijective. Take \( f \in H^0(\mathbb{P}^N, \mathcal{J}_{X \cup W \cup A \cup B}(n)). \) The restriction of \( h \) to \( H \) vanishes on \( W \cup (X \cap H) \cup B \), hence vanishes identically. Thus \( h \) is divided by the equation \( z \) of \( H \). Since \( h/z \) vanishes on \( X \cup A \), we have \( h = 0 \).

**Case (B):** \( k > e, p \geq k - e \). Assume \( d - f \leq g - p + n - 2 \). Since \( d - f \geq 2(g - p) - N - 2 \), we find \( d - f \leq 2n - 2 + N \), contradicting 3.4. We take a general \( E \in Z(f, p - k + e; N) \) with \( r_E(n - 1) \) surjective, hence \( h^0(\mathbb{P}^N, \mathcal{J}_E(n - 1)) = 1 \). Note that by 3.3 and a degeneration of \( E \) to a union of lines, we may assume that \( E \) contains \( 1 + k - e + j \) general points of a hyperplane \( H \). We may take \( W \in Z(d - f, x; N - 1) \), \( W \subset H \), with \( r_{W,H}(n) \) surjective and \( \text{card}(W \cap E) = 1 + k - e + j \) because \( d - f - x \geq N - 1 \) and \( d - f - x \geq j + k - e + 1 \) by 3.4; in particular \( W \) spans \( H \). By 2.2 we may deform \( E \) to \( E' \), \( W \) to \( W' \) with \( r_{E',n - 1} \) surjective, \( r_{W',E' \cap (E' \cap H),H}(n) \) surjective and \( \text{card}(E' \cap W') = 1 + k - e + j \). Note that \( W' \cup E' \in Z(d, g; N) \). As in case A) we prove the surjectivity of \( r_{E',W'}(n) \).

**Case (C):** \( k > e, p < k - e \). Note that we have \( p = g = g' \) because by 2.3 we cannot have \( f \leq 2p + N + 1 \leq 2n - 5 + N \); hence \( f = d' \). By a particular case of the main result of [4] there exists \( F \subset \mathbb{P}^N \), \( F \) disjoint union of a rational curve \( T \) of degree \( f - (k - e - g) \) and \( (k - e - g) \) lines with \( r_F(n - 1) \) surjective. By 3.4(c) we may find a curve \( W \) contained in a hyperplane \( H \), \( W \) rational and connected, \( \deg W = d'' \), with \( r_{W,H}(n) \) surjective, \( W \) intersecting every connected component of \( F \) and intersecting \( T \) exactly at \( 1 + g \) points. We conclude as in case (A).

**Case (D):** \( k \leq e, d - f \leq g - p \). Since \( d - f \geq 2(g - p) - N - 2 \), we have \( d - f \leq g - p \leq N + 2 \). If \( g'' \neq 0 \), we have \( d' \leq 2g' \) and \( d - f \geq d'' \geq 2g'' - 1 \), hence \( g'' \leq (N + 3)/2 \). First assume \( g'' \geq 2 \). We take \( E \in Z(d', g'; N) \), \( E \) intersecting transversally a hyperplane \( H \), and a connected elliptic curve \( W \subset H \), with \( \deg W = d'' \) and \( \text{card}(E \cap W) = g'' \). This is possible because \( d'' \geq 2g'' - 1 \geq 3 \). It is sufficient to prove that we may find \( E \) and \( W \) as above with \( r_{W \cup (E \cap H),H}(n) \) surjective. Set \( u = \min(N, g') \). By [1] (as used in 1.1) we may find \( C \in Z(u + N, u; N) \) with \( r_C(2) \) surjective. We may assume that \( C \) intersects transversally \( H \). From the linear normality of \( C \) and the exact sequence

\[
0 \to \mathcal{J}_C(1) \to \mathcal{J}_C(2) \to J_{C \cap H,H}(2) \to 0
\]

we obtain that \( r_{C \cap H,H}(2) \) is surjective. Now we take a hyperplane \( A \) of \( H \) containing exactly \( g'' \) points of \( C \cap H \); this is possible because \( g'' \leq N - 1 \) for \( N \geq 5 \). In \( A \) we add an elliptic curve \( W \), \( \deg(W) = d'' \), \( W \) containing \( g'' \) points of \( C \cap H \). We may assume \( r_{W,A}(3) \) surjective (even if \( d'' \geq N \)) by the inductive assumption. As in case A) we find that
$r_{W \cup (C \cap H), H}(3)$ is surjective. By 2.1 we may find $E \supset C$ with the properties we are looking for.

If $g'' \leq 1$ we take as $A$ a hyperplane of $H$ containing $1 + g''$ points of $C \cap H$ and we take in $A$ a connected rational curve of degree $d''$ containing $1 + g''$ points of $C$.

**Case (E):** $d - f \leq 5$. By case (D) we may assume $d - f \geq g - p + 1$. We take a suitable $Y \in Z(f, p; N)$ and we add in a hyperplane $H$ a connected, rational curve of degree $d - f$ containing $g - p + 1$ points of $Y$.

The proof of Lemma 3.1 is over.

**Proof of Lemma 3.2:** Since 1.1 works even in the injective range we may assume $n \geq 4$. Let $s$, $s'$ be the maximal integers with $0 \leq s \leq g$, $0 \leq s' \leq g$ and

\[(n - 2)(2s + N) + 1 - s \leq \left(\frac{N + n - 2}{N}\right) + n - 3\]

\[(n - 2)(\max(s' + N, 2s' - 1)) \leq \left(\frac{N + n - 2}{N}\right) + n - 3\]

Let $r$, $r'$ be the only integers with $r \geq 2s + N$, $r' \geq \max(s' + N, 2s' - 1)$ and satisfying

\[\left(\frac{N + n - 2}{N}\right) \leq (n - 2)r + 1 - s \leq \left(\frac{N + n - 2}{N}\right) + n - 3 \tag{5}\]

\[\left(\frac{N + n - 2}{N}\right) \leq (n - 2)r' + 1 - s' \leq \left(\frac{N + n - 2}{N}\right) + n - 3 \tag{6}\]

We have $s \leq s'$, $r \leq r' < d$ because $(d, g)$ has critical value $n$. If $s < g$ we have $r \leq 2s + N + 1$ by the maximality of $s$. Hence $d - r \geq 2(g - s) - N - 2$. Set $x' = \min(g - s, [(d - r + 1)/2])$ and $j' = g - s - x'$; we have $j' \leq (N + 3)/2$. From the definitions of $h$ and $i$ we find

\[(n - 1)(d - r) + 1 - (g - s) + r - 1 + h - i = \left(\frac{N + n - 2}{N - 1}\right) \tag{7}\]

We need the following numerical lemmas.

**Sublemma 3.5:** If $N \geq 5$ and $n \geq 4$ we have $r \geq 2n + N - 5$.

**Proof:** We have

\[(n - 2)(2n + N - 5) \leq \left(\frac{N + n - 2}{N}\right). \square\]
**SUBLEMMA 3.6:** Fix $N \geq 5$, $n \geq 4$. We have $d - r \geq g - s + n - 3$ and $d - r \geq 6$.

**PROOF:** Assume $d - r \leq g - s + n - 4$. Then (7) gives a contradiction if $(N, n) \neq (5, 4)$. If $N = 5$, $n = 4$, by definition we find $s \leq 3$ and $r \leq 12$. Hence (7) gives $11 \geq d - r \geq g - s$. We obtain $d > 2g - 1$, contradiction. The last part is similar. □

Let $H$ be a hyperplane of $\mathbb{P}^N$. As in the proof of 3.1 we distinguish a few cases.

**Case (A):** $h \leq i$. We take $X \in \mathbb{Z}(r, s; N)$ with $r_X(n - 2)$ injective. As in the corresponding case of 3.1 we may find $W \in \mathbb{Z}(d - r, x'; N - 1)$, $W \subset H$, with $r_{W, H}(n)$ of maximal rank and $\text{card}(W \cap X) = 1 + j'$ (use 3.5, 3.6). Since $h \leq i$ we may deform $W \cup X$ to $W' \cup X'$ with $r_{W' \cup X'}(n - 1)$ injective.

**Case (B):** $h > i$, $s \geq n - 2 - n + i$. Set $m = r - 1$, $m' = s - (n - 2 - h + i)$. Take $Y \in \mathbb{Z}(m, m'; N)$ with $r_Y(n - 2)$ injective. By 3.6 we may find $W \in \mathbb{Z}(d - m, x'; N - 1)$, $W \subset H$, with $r_{W, H}(n - 1)$ of maximal rank. We may apply to $Y \cup W$ the smoothing theorems for $k$-secants, $k = 1, 2$, because $m - m' \geq N + 1 + (n - 2 - h + i)$.

**Case (C):** $h > 1$, $s < n - 2 - h + i$. If $s < g$, then $r \leq 2s + N + 1$. By 3.5 we have $s = g$. By [4] we may find a curve $Y$, $\deg Y = r - 1$, $Y$ disjoint union of a rational curve $T$, $\deg T = r - 1 - (n - 2 - s - h + i)$, and $n - 2 - s - h + i$ disjoint lines, with $r_Y(n - 2)$ injective. By 3.6 we may find $W \in \mathbb{Z}(d - r + 1, 0; N - 1)$, $W \subset H$, with $r_{W, H}(n - 1)$ of maximal rank, $W$ intersecting every connected component of $Y$ and intersecting $T$ in exactly $1 + g$ points.

The proofs of 3.2 and Theorem 1 are over.

**§4. Proof of theorem 2**

As a byproduct of the proof of Theorem 1, we will obtain a proof of Theorem 2. From this proof it would be possible to obtain an explicit bound for the functions $e_N$; however this bound is too weak in any explicit situation. Since if $d < 2g - 1$, $d \geq g + N$, the genus of a triple $(d, g; N)$ with critical value $n$ goes to infinity as $n$ goes to infinity, we may fix $(d, g; N)$ with $N \geq 5$, $d \leq 2g - 2$, $d \geq g + N$, $d \leq 2g - n + N + 1$ and critical value $n \geq g - N$; it is sufficient to prove that a general element in $\mathbb{Z}(d, g; N)$ has maximal rank.

We use the notations of Section 3, but with these new bounds on $d$. First consider the surjective part as in 3.1. The definitions of $f$, $p$, $d'$, $g'$ make sense even now. Certainly we have $s \leq g' < g$ because $d < 2g - 1$. 

Hence $f \leq 2p + n + 1$. Again we define $k$, $k'$, $e$, $x$, $j$, $d''$, $g''$ with the same formulas. Now we have $d - f \geq 2(g - p) - n$ and $2j \leq (n - 3)$.

First assume $d - f \geq 4n + 1$. In case (A) we do not need the assumptions "$d - f \geq 6"$ and "$d - f \geq g - p + 1"$; hence we do not need cases (D) and (E). Indeed by the assumptions on $d - f$ and $n$, we may take $W \in Z(d - f, x; N - 1)$, $W$ spanning a hyperplane $H$ and containing $n + 1 \geq 1 + j$ general points of $H$. In case (B) we may take $W \in Z(d - f, x; N - 1)$ intersecting $Y$ at $1 + (n - 2 - h + i) - s + g - x \leq 2n + 1$ points, because $d - f - x \geq 2n$. Case (C) cannot occur now because $p < g$.

Now assume $d - f \leq 4n$. Set $D = f - 4n - 2$, $G = p - 2n - 1$. We need two numerical lemmas.

**Lemma 4.1:** Assume $N \geq 5$ and $n \geq 11$. We have $p \geq 3n + 2$, hence $D \geq 2n + 2 + N$.

**Proof:** If $p \leq 3n$, we have $f \leq 6n + N + 1$ and (2) gives a contradiction.

**Lemma 4.2:** Assume $N \geq 5$, $n \geq 11$ and $d - f \leq 4n$. Then $e \geq (4n + 2)(n - 1) + n - 2$.

**Proof:** Use (2) and (4).

We repeat the construction of 3.1 substituting $(f, g)$ with $(D, G)$. By Lemma 4.2 we have $k + (4n + 2)(n - 1) \leq e$ if $n \geq 11$, hence it is sufficient to consider case (A). Now we show what to change in the proof of 3.2 to obtain the injectivity part of Theorem 2. We may define $r$ and $s$ using the same formulas. Now we have $s < g$ because $(2g - 1, g)$ has critical value at least $n$; now we have $d - r \geq 2(g - s) - n$.

If $d - r \geq 4n + 1$, we may copy the proof of 3.2 with the same modifications just given. We conclude using the following lemma.

**Lemma 4.3:** If $N \geq 5$ and $n \geq 11$, we have $d - r \geq 4n + 1$.

**Proof:** Assuming $d - r \leq 4n$. Since

$$r \leq \left(\frac{N + n - 2}{N}\right)/(n - 3),$$

the lemma follows from (6).

**References**


(Oblatum 20-V-1985)

E. Ballico
Scuola Normale Superiore
I-56100 Pisa
Italy

Ph. Ellia
CNRS LA 168
Département de Mathématiques
Université de Nice
Parc Valrose
F-06034 Nice Cedex
France