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B. DWORK

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ON THE TATE CONSTANT

B. Dwork

I. Introduction

1. Our work on the relation between the congruence zeta function and the p-adic analysis began in February, 1958 with the suggestion of J. Tate that his constant C (described below) may be constructed by p-adic analytic methods. (For an alternate description of C see [Dw, p. 1 equation (0.1)].)

Let k be a field of characteristic zero complete with respect to a discrete valuation, with valuation ring $\mathfrak{O}$ and residue class field $\bar{k} = \mathfrak{O}/\mathfrak{p}$. Let A be an elliptic curve defined over k by an equation

$$y^2 + (a_1 x + a_2) y = x^3 + a_3 x^2 + a_4 x + a_5 \tag{1}$$

where the $a_i \in \mathfrak{O}$. Letting $x = ty$ we find

$$y^3 t^3 + y^2(-1 - a_1 t + a_3 t^2) + y(-a_2 + a_4 t) + a_5 = 0$$

and hence there exists a unique solution in $k((t))$ for y with a pole of order 3 at $t = 0$. This solution is of the form

$$y = t^{-3} + B_{-2} t^{-2} + \dots \tag{2}$$

and the coefficients B_i lie in $\mathfrak{O}$. Clearly

$$x = t^{-2} + B_{-2} t^{-1} + \dots. \tag{3}$$

Let

$$\mathrm{d}u = -\mathrm{d}x/(2y + a_1 x + a_2) \tag{4}$$

a differential of the first kind on A. In terms of the uniformizing parameter $t = x/y$ at infinity we have after integration

$$u = t + \tfrac{1}{2} D_1 t^2 + \tfrac{1}{3} D_2 t^3 + \dots \tag{5}$$

where the D_i lie in $\mathfrak{O}$.

THEOREM: (*J. Tate*, 1958, *unpublished*):

1. *If the reduced curve $\bar{A}$ defined over $\bar{k}$ is non-singular and has Hasse invariant not zero then there exists a unit C in the maximal unramified extension, K, of k such that* $\exp Cu$ ($\in K[[t]]$) *has in fact integral coefficients in K.*

2. *If $\bar{k}$ is finite, i.e. k is a p-adic field then the unit root of the zeta function of the reduced curve $\bar{A}$ is $C^{\sigma-1}$ where σ is the Frobenius automorphism of K over k.*

2. Explanation of the Tate's theorem (J. Tate, 1958, unpublished)

Let

$$S = \{(x, y) \in A \| x | > 1\}.$$

Equations (2) and (3) show that S is parameterized by $t \in D(0, 1^-)$. The map $t \mapsto u(t)$ gives a homomorphism of S into k_+. Since k is of characteristic zero,

$$u(t) = 0$$

for each $P \in S$ which is a division point. Since u is a one to one map of $D(0)$, $|\pi|^-)$ onto itself, $u(t_0) = 0$ can only be valid for $t_0 = 0$ if $t_0 \in D(0, |\pi|^-)$. On the other hand it is shown by Lutz [L] that for $P \in S$

$$\operatorname{ord} t(pP) \geqslant \operatorname{Min}(1 + \operatorname{ord} t(P), 4 \operatorname{ord} t(P))$$

and hence if $u(t(P_0)) = 0$ then $t(p^\nu P_0) \in D(0, |\pi|^-)$ for suitable ν and so P_0 is a p^{th} power division point.

Since $1 + \mathfrak{p}(\mathfrak{p} = D(0, 1^-))$ does have points of finite order, Tate sought an isomorphism of S into $1 + \mathfrak{p}$ such as $t \mapsto \exp \theta \cdot u(t)$. If one exists with integral coefficients then it is invertible and gives an isomorphism of S_K with $1 + \mathfrak{p}_K$ where K is a complete field containing $k(\theta)$. The exact sequence

$$0 \to S \overset{\text{inj}}{\to} A \overset{\text{red}}{\to} \bar{A} \to 0.$$

together with the fact that for $(l, p) = 1$ both A and $\bar{A}$ have l^2 points of order l shows again that the only division points in S are of p power order. If there are p points of order p in $\bar{A}$ then there are only p in S (as there are in $1 + \mathfrak{p}$) and so the isomorphism of Tate could (and in fact does) exist. If $\bar{A}$ has no points of order p then there are p^2 such points in S and then the suggested isomorphism is impossible. This explains the role of the Hasse invariant.

3. In 1958 (unpublished) we evaluated the Tate constant, C, for the Legendre model ($p \neq 2$)

$$y^2 = x(1-x)(\lambda - x).$$

Using $t = 1/\sqrt{x}$ as parameter at ∞, we may write

$$\mathrm{d}u = \sum_{n=0}^{\infty} t^{2n} D_{2n}(\lambda)\, \mathrm{d}t \tag{3.1}$$

where

$$D_{2n} = (-1)^n \sum_{i=0}^{n} \binom{-1/2}{n-i} \binom{-1/2}{i} \lambda^i. \tag{3.2}$$

If λ lies in an unramified extension of $\mathbb{Q}_p$ then the existence of C (again in an unramified extension) is equivalent (by the Dieudonne criterion [D]) to congruences

$$C^{\sigma-1} D_{mp^s-1}(\sigma\lambda) \equiv D_{mp^{s+1}-1}(\lambda) \bmod p^{s+1} \tag{3.3}$$

for all $s \in \mathbb{N}$, $(m, p) = 1$, where σ is the absolute Frobenius. The consistency of these conditions was demonstrated by means of the formal congruences

$$D_{mp^{s+1}-1}(\lambda) \equiv (-1)^{(p-1)/2} D_{mp^s-1}(\lambda) F(\lambda) \\ /F(\lambda^p) \bmod p^{s+1}\mathbb{Z}_p[[\lambda]] \tag{3.4}$$

where

$$F(\lambda) = F(\tfrac{1}{2}, \tfrac{1}{2}; 1, \lambda) = \sum \left((\tfrac{1}{2})_j / j! \right)^2 \lambda^j.$$

By means of these congruences we showed that $F(\lambda)/F(\lambda^p)$ extends to an analytic element f on the Hasse domain

$$H = \left\{ \lambda \mid |D_{p-1}(\lambda)| \geqslant 1 \right\}. \tag{3.5}$$

(This may have been the first application of Krasner's theory of p-adic analytic continuation and in particular of his theorem of unicity (C.R. 1954).) Congruences similar to (3.4) are treated elsewhere [Dw 1969, Dw 1973]. Thus if $\sigma\lambda_0 = \lambda_0^p$, i.e. λ_0 is a Teichmuller representative of its residue class then $C(\lambda_0) \in K$ is to be chosen so that

$$C(\lambda_0)^{\sigma-1} = f(\lambda_0). \tag{3.6}$$

More generally if $\lambda = \lambda_0 + \lambda_1$, $|\lambda_1| < 1$ then we must put

$$C(\lambda) = C(\lambda_0)\Big/\left[1 + \sum_{s=1}^{\infty} \eta_s(\lambda_0)\lambda_1^s\right] \tag{3.7}$$

where

$$\eta_s = s!^{-1}F^{(s)}/F$$

the point being that η_s is an analytic element on H whose restriction to $D(0, 1^-)$ is as indicated. This may also be expressed by the condition

$$C(\lambda) = C(\lambda_0)/v(\lambda) \tag{3.8}$$

where v is the unique branch of $F(\frac{1}{2}, \frac{1}{2}, 1, \lambda)$ at λ_0 (i.e. the unique solution of the corresponding second order differential equation) which is bounded on $D(\lambda_0, 1^-)$ and such that

$$v(\lambda_0) = 1.$$

The object of this note is to explain how the theorem of Tate may be treated by means of normalized solution matrices.

4. Heuristics

We assume the reduced curve is ordinary.

If ω_1, ω_2 are 'eigenvectors' of Frobenius i.e.

$$\omega_{1,\lambda^p}^{\sigma\Phi} = p\omega_{1,\lambda} + d\xi_1$$

$$\omega_{2,\lambda^p}^{\sigma\Phi} = \omega_{2,\lambda} + d\xi_2$$

where ξ_1, ξ_2 are daggerized algebraic functions on A and we think of σ as operating on coefficients of the differential forms while Φ represents $x \to x^p$, then upon integration, setting $I_{j,\lambda} = \int \omega_{j,\lambda}$, a local abelian integral, we obtain

$$I_{1,\lambda^p}^{\sigma\Phi} = pI_{1,\lambda} + \xi_1 \tag{4.1}$$

$$I_{2,\lambda^p}^{\sigma\Phi} = I_{2,\lambda} + \xi_2 \tag{4.2}$$

We are tempted to deduce Tate's theorem by applying Dieudonne's criterion to (4.1). There are two questions:

Is ξ_1 bounded by p on a generic disk? (4.3)

$I_{1,\lambda}$ need not be an integral of the first kind. (4.4)

Our purpose is to show how these objections may be met by means of the theory of normalized solution matrices of the hypergeometric differential equation as explained in Chapter 9 of [Dw]. The present treatment is based upon the relation between abelian integrals and the dual space K_f [Dw, Chapter 2]. This relation was brought to our attention by A. Adolphson and S. Sperber.

The results given here go beyond the results indicated above for the Legendre model but more general results have been formulated for varieties with ordinary reduction.

5. Notation

L_λ = analytic functions on the complement of sets of the type $D(0, \epsilon_0) \cup D(1, \epsilon_1) \cup D(\lambda^{-1}, \epsilon_{\lambda^{-1}}) \cup D(\infty, \epsilon_\infty)$ (where ϵ_i is less than the distance from i to the remaining elements of $\{0, 1, \lambda^{-1}, \infty\}$, distance from ∞ to be computed in terms of $1/x$).

$$f = x^{b-1}(1-x)^{c-b}(1-\lambda x)^{-a}$$

$$G = xf(\lambda, x)/x^p f(\lambda^p, x^p)$$

$$E = x\frac{\partial}{\partial x}$$

$$D_{f,\lambda} = (xf)^{-1} \circ E \circ xf$$

$\Phi x = x^p$ (The notation distinguishes between the space variable and the $\phi\lambda = \lambda^p$ parameter)

$$\omega_{a,b,c} = f_{a,b,c}(x)\,dx/(1-x)$$

$u, \hat{u}, \eta, \tau$ [DW 9.6]

σ = absolute Frobenius

$I_{a,b,c,\nu}$(equation 1.7)

$$S = \{0, 1, 1/\lambda, \infty\}$$

$$(\mu_a, \mu_b, \mu_c) = ((p-1)a, (p-1)b, (p-1)c)$$

$$(\tilde{a}, \tilde{b}, \tilde{c}) = (1-a, 1-b, 1-c)$$

$$\tilde{f} = f_{\tilde{a}, \tilde{b}, \tilde{c}}$$

$T_\nu = x, 1-x, 1-\lambda x, 1/x$ = local parameter at ν

P_ν = principal part at ν [Dw 4.1]

$K_f = \ker \gamma_- \circ D_{1/xf}$ [Dw Chapter 2]

$\alpha_f^* = \gamma_- \circ x^{p-1}G \circ \Phi$

γ_+, γ_- [Dw 2.1.1, 4.1]

$1^*, \left(\frac{1}{1-x}\right)^*$ [Dw 2.4.4]

$\pi = (-p)^{1/(p-1)}$

Hasse domain = set of all residue classes where $(1.1)_{a,b,c}$ has a bounded solution.

For $\xi \in L_\lambda$, $|\xi|_{\text{gauss}} = |\xi(t)|$ where t is a generic unit.

II. Action of Frobenius on abelian integrals

1. We shall consider the hypergeometric differential equation

$$\frac{d}{d\lambda}(u_1, u_2) = (u_1, u_2)\begin{pmatrix} -\frac{c}{\lambda} & \frac{c-a}{1-\lambda} \\ \frac{c-b}{\lambda} & \frac{a+b-c}{1-\lambda} \end{pmatrix} \tag{1.1$_{a,b,c}$}$$

in a split case of period one. More precisely we assume

$$c-a,\ c-b,\ a,\ b \text{ are all non-integral} \tag{1.2}$$

$$(p-1)(a, b, c) \in \mathbb{Z}^3 \tag{1.3}$$

$$a, b, c \in \mathbb{Q} \cap [0, 1] \tag{1.4}$$

and that either

$$0 \leqslant c < \operatorname{Min}(a, b) \leqslant \operatorname{Max}(a, b) < 1 \text{ (Type I)} \tag{1.4.1}$$

or

$$0 < \operatorname{Max}(a, b) < c \leqslant 1 \text{ (Type II)} \tag{1.4.2}$$

We shall restrict λ to the region

$$|\lambda| = |\lambda - 1| = 1 \tag{1.5}$$

We start by observing that the dual space K_f [Dw Chapter 2] consists of 4-tuples

$$\xi = (\xi_0, \xi_1, \xi_{1/\lambda}, \xi_\infty)$$

which by [Dw 2.3./5.7] are essentially a set of local expansions of abelian integrals. More precisely for $\nu \in S = \langle 0, 1, 1/\lambda, \infty \rangle$ we have

$$f^{-1} \cdot \xi_\nu = \int_\nu^x \frac{1}{xf}\left(\frac{A}{1-x} + \frac{B}{1-\lambda x}\right)^{dx} \tag{1.6}$$

where A, B are independent of ν, x but do depend upon ξ. Under the identifications of [Dw Chapter 14] these abelian integrals are associated with $(\tilde{a}, \tilde{b}, \tilde{c}) = (1-a, 1-b, 1-c)$. Indeed the meaning of [Dw 2.5.2] is that

$$\begin{aligned} f^{-1} \cdot (1^*)_\nu &= (a-c)\lambda \int_\nu^x \omega_{\tilde{a},\tilde{b},\tilde{c}+1} \overset{\text{def}}{=} I_{\tilde{a},\tilde{b},\tilde{c}+1,\lambda,\nu}(x) \\ f^{-1} \cdot \left(\left(\frac{1}{1-x}\right)^*\right)_\nu &= (b-c)(1-\lambda) \int_\nu^x \omega_{\tilde{a},\tilde{b},\tilde{c}} \overset{\text{def}}{=} I_{\tilde{a},\tilde{b},\tilde{c},\lambda,\nu}(x) \end{aligned} \tag{1.7}$$

where for x close to ν $\int_\nu^x \omega$ has the conventional meaning if ω is holomorphic at ν, while otherwise it is defined (uniquely by virtue of (1.2)) by the condition that its product with f be holomorphic at ν.

By hypothesis (1.3) (cf[Dw 4.4.3])

$$\alpha_f^* = \gamma_- \circ x^{p-1}G \circ \Phi \tag{1.8.1}$$

maps K_{f,λ^p} into $K_{f,\lambda}$ with matrix, B, [Dw 4.5.1] so that (using the subscript to denote the value of λ),

$$\alpha_f^* \begin{pmatrix} 1^* \\ \left(\frac{1}{1-x}\right)^* \end{pmatrix}_{\lambda^p} = B(\lambda) \begin{pmatrix} 1^* \\ \left(\frac{1}{1-x}\right)^* \end{pmatrix}_\lambda . \tag{1.8.2}$$

It follows from the definitions that

$$x^{p-1}G \begin{pmatrix} 1^* \\ \left(\frac{1}{1-x}\right)^* \end{pmatrix}_{\lambda^p}^{\Phi} = B(\lambda) \begin{pmatrix} 1^* \\ \left(\frac{1}{1-x}\right)^* \end{pmatrix}_\lambda + \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \tag{1.9.1}$$

where

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = -\gamma_+ x^{p-1}G \begin{pmatrix} 1^* \\ \left(\frac{1}{1-x}\right)^* \end{pmatrix}_{\lambda^p}^{\Phi} \in L_\lambda \times L_\lambda . \tag{1.9.2}$$

The key point is the existence of estimates for the gauss norms of z_0, z_1

1.10 LEMMA:
Subject to 1.4.1

$$|z_0|_{\text{gauss}} \leqslant 1$$

$$|z_1|_{\text{gauss}} \leqslant |p|.$$

Subject to 1.4.2

$$|z_0|_{\text{gauss}} \leqslant |p|$$

$$|z_1|_{\text{gauss}} \leqslant 1.$$

The proof parallels an earlier calculation (Dw 1983, Lemma 3.18]. The full details are given in section 2 below.

In the remainder of this section we assume the validity of these estimates.

Using (1.7) we rewrite (1.9.1) in the form

$$\begin{pmatrix} I_{\tilde{a},\tilde{b},\tilde{c}+1,\nu^p,\lambda^p}(x^p) \\ I_{\tilde{a},\tilde{b},\tilde{c},\nu^p,\lambda^p}(x^p) \end{pmatrix} = B(\lambda)\begin{pmatrix} I_{\tilde{a},\tilde{b},\tilde{c}+1,\nu,\lambda}(x) \\ I_{\tilde{a},\tilde{b},\tilde{c},\nu,\lambda}(x) \end{pmatrix} + \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \cdot f^{-1} \tag{1.11}$$

Let now $D(\lambda_0, 1^-)$ represent a residue class in the Hasse domain. We will use the results of [Dw Chapter 9]. Conditions 9.0.1 – 9.0.4 of the reference are cetainly satisfied. Condition 9.0.5 may be disregarded by virtue of [Dw 1983, Theorem 4].

Letting Y be the normalized solution matrix of (1.1) we have

$$Y^{\sigma\phi}B = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} Y \\ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} Y \end{cases} \text{subject to } \begin{matrix} 1.4.1 \\ 1.4.2 \end{matrix} \tag{1.12}$$

(Note that Y is defined over a maximal unramified extension of $\mathbb{Q}_p(\lambda_0)$).

We now assume 1.4.1. The main facts are

$$Y = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & \hat{u} \end{pmatrix}\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \tag{1.13}$$

where [Dw 9.6] u, $\hat{u}$, η, τ satisfy the conditions

η is an analytic element bounded by unity

on the Hasse domain of (1.1) (1.13.1)

$u(1, \eta)$ is the unique bounded solution of (1.1) on $D(\lambda_0, 1^-)$

$$(1.13.2)$$

$u(\lambda)/u^\sigma(\lambda^p)$ extends to an analytic element on the Hasse domain.

$$(1.13.3)$$

$$\tau^\sigma(\lambda^p) \equiv p\tau(\lambda) \bmod p \quad (\forall \lambda \in D(\lambda_0, 1^-)) \tag{1.13.4}$$

$$u\hat{u} = \text{Wronskian of (1.1)} \tag{1.13.5}$$

$$u, \hat{u} \text{ assume unit values on } D(\lambda_0, 1^-). \tag{1.13.6}$$

$$(u, u\eta) \text{ are the normalized periods of } (\omega_{\mathrm{a,b,c+1}}, \omega_{\mathrm{a,b,c}}) \tag{1.13.7}$$

Multiplying (1.11) by $Y^\sigma(\lambda^p)$, setting

$$I_{1,\nu}(\lambda, x) = u(\lambda)(I_{\tilde{a},\tilde{b},\tilde{c}+1,\nu,\lambda}(x) + \eta(\lambda)(I_{\tilde{a},\tilde{b},\tilde{c},\nu,\lambda}(x)) \tag{1.14.1}$$

$$I_{2,\nu}(\lambda, x) = \hat{u}(\lambda) I_{\tilde{a},\tilde{b},\tilde{c},\nu,\lambda}(x) + \tau(\lambda) I_{1,\nu}(\lambda, x), \tag{1.14.2}$$

and defining

$$w_1(\lambda, x) = u^\sigma(\lambda^p)(z_0 + \eta^\phi z_1) \cdot f^{-1} \tag{1.15.1}$$

$$w_2(\lambda, x) = \hat{u}^\sigma(\lambda^p) z_1 \cdot f^{-1} + \tau^\sigma(\lambda^p) w_1 \tag{1.15.2}$$

we have by virtue of (1.12),

$$I^\sigma_{1,\nu^p}(\lambda^p, x^p) = I_{1,\nu}(\lambda, x) + w_1(\lambda, x) \tag{1.16.1}$$

$$I^\sigma_{2,\nu^p}(\lambda^p, x^p) = pI_{2,\nu}(\lambda, x) + w_2(\lambda, x) \tag{1.16.2}$$

Lemma 1.10 and equation 1.15.1 shows that

$$|w_1(\lambda, x)|_{\text{gauss}} \leqslant 1 \tag{1.17}$$

and hence by (1.16.1),

$$|I_{1,\nu}(\lambda, x)| \leqslant 1 \text{ for } x \in D(\nu, 1^-). \tag{1.18}$$

We now put

$$J(\lambda, x) = \hat{u}(\lambda) I_{\tilde{a},\tilde{b},\tilde{c},\nu,\lambda}(x) = (b-c)(1-\lambda)\hat{u}(\lambda)\int_\nu^x \omega_{\tilde{a},\tilde{b},\tilde{c}} \tag{1.19}$$

and so by 1.14.2

$$J(\lambda, x) = I_{2,\nu}(\lambda, x) - \tau(\lambda) I_{1,\nu}(\lambda, x) \tag{1.19.1}$$

We now compute with the aid of 1.16

$$J^{\sigma}(\lambda^p, x^p) = pJ(\lambda, x) + H(\lambda, x) \tag{1.20}$$

where

$$H(\lambda, x) = \hat{u}^{\sigma}(\lambda^p) z_1 f^{-1} + (p\tau(\lambda) - \tau^{\sigma}(\lambda^p)) I_{1,\nu} \tag{1.21.1}$$

For $\lambda \in D(\lambda_0, 1^-)$, $x \in D(\nu, 1^-)$, we have by virtue of Lemma 1.10, and 1.13.4

$$|H(\lambda, x)| \leqslant |p|. \tag{1.21.2}$$

It now follows from the criterion of Dieudonné that $\exp J(\lambda, x)$ is a power series in a fractional power of T_ν with integral coefficients.

Since (a, b, c) is of Type I, $(\tilde{a}, \tilde{b}, \tilde{c})$ is of Type II and hence $\omega_{\tilde{a},\tilde{b},\tilde{c}}$ is the unique differential of the first kind in

$$\omega_{\tilde{a},\tilde{b},\tilde{c}} \mathbf{Q}\left[x, x^{-1}, (1-x)^{-1}, (1-x\lambda)^{-1}\right].$$

We now explain the coefficient

$$C(\lambda) = (b-c)(1-\lambda)\hat{u}(\lambda) \tag{1.22}$$

in the right hand side of 1.19.

Letting $\tilde{B}$ denote the matrix to be used in (1.8.2) if (a, b, c) were replaced by $(\tilde{a}, \tilde{b}, \tilde{c})$, we deduce from [Dw 4.7, 2.5.2]

$$\begin{aligned} B = p &\begin{pmatrix} (a-c)\lambda & 0 \\ 0 & (b-c)(1-\lambda) \end{pmatrix}^{\phi} \\ &\times \tilde{B}^* \begin{pmatrix} (a-c)\lambda & 0 \\ 0 & (b-c)(1-\lambda) \end{pmatrix}^{-1}, \end{aligned} \tag{1.23}$$

where $\tilde{B}^*$ is the inverse of the transpose of $\tilde{B}$. From equation 1.12 we deduce (subject to 1.4.1)

$$\tilde{Y}^{\sigma\phi}\tilde{B} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \tilde{Y} \tag{1.24}$$

where

$$\tilde{Y} = Y^* \begin{pmatrix} (a-c)\lambda & 0 \\ 0 & (b-c)(1-\lambda) \end{pmatrix}^{-1}. \tag{1.25}$$

Thus $\tilde{Y}$ is the normalized solution of $(1.1)_{\tilde{a},\tilde{b},\tilde{c}}$. Using (1.13) we find the bounded normalized solution to be

$$\left(-\eta\frac{b-c}{a-c}\frac{1-\lambda}{\lambda}, 1\right)\frac{1}{(b-c)(1-\lambda)}\hat{u}^{-1} \tag{1.26}$$

Thus the coefficient, $C(\lambda)$, is the reciprocal of the normalized period of the differential $\omega_{\tilde{a},\tilde{b},\tilde{c},\lambda}$. To complete our treatment of the first part of Tate's theorem we put $(\tilde{a}, \tilde{b}, \tilde{c}) = (\frac{1}{2}, \frac{1}{2}, 1)$.

The second part of Tate's theorem is also demonstrated since (1.26) gives an 'eigenvector' of a semi-linear transformation with matrix $\tilde{B}$ corresponding to the 'eigenvalue' 1. Using Adolphson's explanation in the appendix of [Dw] we may deduce the connection between $C(\lambda_0)^{1-\sigma}$ and the unit root of the corresponding L-function. In the next section we complete the treatment by verifying 1.10.

2. *Detailed estimates*

To verify II Lemma 1.10 we use $\xi^{(i)}$ $(i = 0, 1)$ to denote the two elements of our basis of K_f, i.e.

$$\begin{aligned} \xi^{(0)} &= 1^* \\ \xi^{(1)} &= \left(\frac{1}{1-x}\right)^* \end{aligned} \tag{2.1}$$

and we assert the following bounds:

$$|P_\nu x^{p-1} G \xi^{(i)}_{\nu^p} \Phi|_{\text{gauss}} \leqslant \epsilon^{(i)} \tag{2.2}$$

where

$$\epsilon^{(0)} = \begin{matrix} 1 & \text{if } 1.4.1(\text{II}) \\ |p| & \text{if } 1.4.2(\text{II}). \end{matrix}$$

$$\epsilon^{(1)} = \begin{matrix} |p| & \text{if } 1.4.1(\text{II}) \\ 1 & \text{if } 1.4.2(\text{II}). \end{matrix}$$

Our treatment is similar to that of the proof of Lemma 3.18 of [Dw 1983] but in that calculation only residues were considered. We consider the four values of ν seperately.

At $\nu = 0$, $x^{p-1}G$ is analytic and hence the principal parts at $\nu = 0$ all vanish.

At $\nu = \infty$ we use [Dw 1983, p. 132 line 1*]

$$x^{p-1}G\xi_{\infty}^{(i)}\phi \equiv \pm \frac{A + B/\lambda^p}{c-a}\lambda^{\mu_a}\left(1 - \frac{1}{x}\right)^{c-b}\left(1 - \frac{1}{\lambda x}\right)^{-a} \cdot x^{\mu_a - \mu_c - 1} \bmod \frac{1}{x^p}x^{\mu_a-\mu_c-1} \tag{2.3}$$

where A, B are given by Table 2.4

TABLE 2.4

i	$i=0$	$i=1$
A	0	$b-c$
B	$\lambda^p(a-c)$	$-\lambda^p(b-c)$

The principal part at infinity vanishes if either $A + B/\lambda^p = 0$ (i.e. $i = 1$) or if $\mu_a \leqslant \mu_c$, i.e. if condition 1.4.2 II holds. This leaves only the case in which both $i = 0$ and 1.4.1 II holds. It is clear from (2.3) that in this case the coefficients of the principal part are integral. This completes the treatment of $\nu = \infty$.

At $\nu = 1$ we first compute

$$x^{p-1}G = T_1^{\mu_b - \mu_c} \cdot H \cdot \sum_{s=0}^{\infty} \theta_s T_1^{-s} \tag{2.4}$$

where

$$\Lambda\left(\frac{1}{1-x}\right) = T_1^p/(1-x^p)$$

$$H = (1-\lambda x)^{\mu_a} x^{-\mu_b}\Lambda\left(\frac{1}{1-\lambda x}\right)^{-a} x^{p-1} \tag{2.5.1}$$

$$\operatorname{ord} \theta_s \geqslant s/(p-1) \qquad s \geqslant 0 \tag{2.5.2}$$

$$\sum \theta_s T_1^{-s} = \Lambda\left(\frac{1}{1-x}\right)^{c-b} \tag{2.5.3}$$

Now H is analytic and bounded by unity on $D(0, 1^-)$ and so $P_1(x^{p-1}G)$ coincides with P_1 of

$$\pi^{\mu_b - \mu_c} \sum_{s=0}^{\infty} \pi^s / T_1^s \qquad \text{if } b > c \tag{2.6.1}$$

$$\sum_{j=0}^{\mu_c - \mu_b} 1 \cdot T_1^{-j} + \sum_{s=\mu_c-\mu_b+1}^{\infty} \pi^{s-(\mu_c-\mu_b)}/T_1^s \qquad \text{if } c > b \tag{2.6.2}$$

where in 2.6 the symbolic coefficients 1, π^s serve only to indicate upper bounds for the magnitudes of the actual coefficients.

An upper bound for the coefficient A_s of T_1^s in $\xi_1^{(i)}(x^p)$ may be deduced [Dw p. 25] from that of T_1^s in all polynomials of the form

$$\left\{\left(1-(1-T_1)^p\right)^m \frac{1}{b-c+l}\right\}_{m \geqslant l \geqslant 0},$$

the restriction being that $m \geqslant 1$ unless $i=1$ and in that case the term involving $m=0$ is the constant term, -1. Thus

$$\begin{aligned} |A_0| &= 0 \qquad \text{if } i=0 \\ &= 1 \qquad \text{if } i=1 \end{aligned} \tag{2.7.1}$$

$$|A_s| \leqslant \sup\left|\frac{p^{j_1+j_2+\cdots+j_{p-1}}}{b-c+m'}\right| \qquad \text{if } s \geqslant 1 \tag{2.7.2}$$

the supremum being over all $j_1, \ldots, j_p$, m, m' such that

$$\begin{cases} j_1+\cdots+j_p=m \geqslant 1 \\ j_1+2j_2+\cdots+pj_p=s \\ m \geqslant m' \geqslant 0 \end{cases} \tag{2.7.3}$$

For $b>c$, the coefficients of the principal part of $x^{p-1}G\xi_1^{(i)\phi}$ are by (2.6.1) bounded by

$$\operatorname{Sup}|\pi^{\mu_b-\mu_c}\pi^t A_s|, \tag{2.8}$$

the supremum being over all s, t such that.

$$t \geqslant s+1, \qquad s \geqslant 0. \tag{2.8.1.}$$

The term in (2.8) is clearly bounded from above by $|\pi|$ if $s=0$. Furthermore for $s \geqslant 1$ we have by 2.8.1,

$$|\pi^t A_s| \leqslant |\pi^{s+1}A_s| \leqslant \sup\left|\pi^{1+j_1+2j_2+\ldots+pj_p}p^{j_1+\ldots+j_{p-1}}\frac{1}{b-c+m'}\right| \tag{2.9.1}$$

the supremum being again over 2.7.3. Now

$$\begin{aligned} j_1+2j_2+\ldots+pj_p+(p-1)(j_1+\ldots+j_{p-1}) &\geqslant p(j_1+\ldots+j_p) \\ &= pm \end{aligned} \tag{2.9.2}$$

and so

$$|\pi^t A_s| \leqslant |\pi| \operatorname{Sup} |\pi^{pm}/(b-c+m')| \tag{2.9.3}$$

We assert

$$|\pi^{\mu_b-\mu_c}\pi^t A_s| \leqslant |\pi| \tag{2.10}$$

This is certainly the case if

$$|\pi^{\mu_b-\mu_c+pm}/(b-c+m)| \leqslant 1 \tag{2.11}$$

for all $m \geqslant 0$. This is clearly valid if $b-c+m$ is a unit. Let then

$$\operatorname{ord}(b-c+m) = r \geqslant 1. \tag{2.12}$$

Since

$$b-c = \frac{\mu_b-\mu_c}{p-1} \equiv -(\mu_b-\mu_c)(1+p+\ldots+p^{r-1}) \bmod p^r, \tag{2.13}$$

we conclude that

$$m \equiv (\mu_b-\mu_c)(1+p+\ldots+p^{r-1}) \bmod p^r. \tag{2.13.1}$$

Since $p-1 \geqslant \mu_b > \mu_c \geqslant 0$ the right side of 2.13.1 is a minimal representative of m mod p^r and so

$$m \geqslant (\mu_b-\mu_c)(1+p+\ldots+p^{r-1}) \tag{2.14}$$

Assertion 2.11 is therefore valid since for $r \geqslant 0$

$$(\mu_b-\mu_c)(p+\cdots+p^r) \geqslant r(p-1).$$

This shows that

$$|P_1 x^{p-1} G \xi_1^{(i)\phi}|_{\text{gauss}} \leqslant |\pi| \qquad i = 0, 1 \tag{2.15}$$

if $b > c$. The estimate $|\pi|$ may be replaced by $|p|$ on ramification theoretic grounds (but there is no need to assume that λ is restricted to an unramified extension of $\mathbb{Q}_p$).

We now consider the case in which $c > b$. By 2.6.2 the coefficients of $P_1 x^{p-1} G \xi_1^{(i)\phi}$ are bounded by

$$\operatorname{Max}\left(\operatorname*{Sup}_{s<\mu_c-\mu_b} |A_s|, \operatorname*{Sup}_{s\geqslant\mu_c-\mu_b} |\pi^{1+s-(\mu_c-\mu_b)} A_s| \right) \tag{2.16}$$

We assert

$$\sup_{s<\mu_c-\mu_b} |A_s| \leqslant \begin{matrix} |\pi| \\ 1 \end{matrix} \qquad \text{if } \begin{matrix} i=0 \\ i=1 \end{matrix} \tag{2.17}$$

This estimate is by 2.7.1 valid for $s=0$, while for $1 \leqslant s < \mu_c - \mu_b$ we see from (2.7.2) since $j_p = 0$ and $m \geqslant 1$ that $|A_s| \leqslant |p|$ unless there exists $m' \equiv c-b$ satisfying 2.7.3. Since

$$m' \leqslant m \leqslant s < \mu_c - \mu_b < 2p - (p-1)(c-b) \tag{2.18}$$

the only possible exceptional value of m' is $1+(p-1)(1+b-c)$ and for this value of m' we have

$$\operatorname{ord}\left(\frac{p^m}{b-c+m'}\right) = m-1 \geqslant m'-1 = (p-1)(1+b-c) \geqslant 1. \tag{2.19}$$

This completes the verification of 2.17.

We assert

$$|\pi^{s-(\mu_c-\mu_b)}A_s| \leqslant 1 \tag{2.20}$$

for $s \geqslant \mu_c - \mu_b$, $i=0, 1$. It is enough to show that subject to 2.7.3 we have

$$\begin{aligned} &j_1 + 2j_2 + \ldots + pj_p + (p-1)(j_1 + \ldots + j_{p-1}) \\ &\quad \geqslant \mu_c - \mu_b + (p-1)\operatorname{ord}(b-c+m') \end{aligned} \tag{2.21}$$

This is certainly valid if

$$pm \geqslant (\mu_c - \mu_b) + (p-1)\operatorname{ord}(b-c+m') \tag{2.22}$$

whenever $m \geqslant 1$, $m \geqslant m' \geqslant 0$. To verify this we may assume

$$\operatorname{ord}(b-c+m') = r \geqslant 1.$$

The minimal representative of $c-b$ mod p^r is $(p^r-1)(1+b-c)+1$ and so

$$m \geqslant m' \geqslant (p^r-1)(1+b-c)+1 = p^r(1+b-c)+(c-b) \tag{2.23}$$

and since

$$p(1-(c-b)) \geqslant 1$$

we have

$$pm \geqslant p^r + p(c-b) \geqslant (p-1)r + (p-1)(c-b)$$

which is the assertion 2.22

This completes the proof of (2.20) which together with (2.17) and (2.16) shows that for $b < c$

$$|P_1 x^{p-1} G \xi_1^{(i)\phi}|_{\text{gauss}} \leqslant \begin{matrix} |p| & \text{if } i=0 \\ 1 & \text{if } i=1 \end{matrix}$$

This completes the treatment of the principal part at $\nu = 1$. The treatment of $\nu = 1/\lambda$ is similar. We have

$$P_{1/\lambda} x^{p-1} G = P_{1/\lambda} T_{1/\lambda}^{\mu_a} \sum_{s=0}^{\infty} \pi^s / T_{1/\lambda}^s$$

where again the coefficients on the right are symbolic.

The coefficient, A_s, of $T_{1/\lambda}^s$ in ξ_{1/λ^p}^{ϕ} is bounded by that of $T_{1/\lambda}^s$ in all polynomials of the form

$$\left\{\left(1-\left(1-T_{1/\lambda}\right)^p\right)^m / (a+m')\right\}_{m \geqslant m' \geqslant 0}$$

We deduce

$$|P_{1/\lambda} x^{p-1} G \xi_{1/\lambda^p}^{(i)\phi}| \leqslant |p| \qquad (i=0,1)$$

by verifying that subject to 2.7.3 we have

$$|p^{j_1+j_2+\dots+j_{p-1}} \pi^{1+\mu_a+j_1+2j_2+\dots+pj_p} / (a+m')| < 1$$

i.e.

$$pm + \mu_a \geqslant (p-1)\operatorname{ord}(a+m')$$

for $0 \leqslant m' \leqslant m$. The proof coincides with that of 2.11.

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