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Projective 7-folds with positive defect

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Abstract. New simple proofs are given for the classification theorems of projective k-folds X ($k \leqslant 6$) with defect $\delta > 0$. Moreover 7-folds with $\delta > 1$ and those with $\delta = 1$ and $K_X \otimes \mathcal{O}_X(5)$ spanned are classified. The section of the 10-dimensional spinor variety of $\mathbb{P}^{15}$ by 3 general hyperplanes and Grassmann fibrations over a smooth curve belong to this last class.

0. Introduction

Recently many results on projective manifolds with small dual varieties have been found by [Ein, 1985]. In the first part of this paper (sections 1 and 2) we approach this subject from a topological-adjunction theoretic point of view. The topological basic facts are a formula due to [Landman, 1976] and some results from [Lanteri and Struppa, 1986]. In particular we provide new (and very short) proofs for the classification theorems of projective manifolds with degenerate dual varieties of dimensions 3, 4 and 6, and we partially classify those of dimension 7. In particular we completely classify 7-folds with defect $\delta = 3$: they are scrolls of $\mathbb{P}^5$'s over a smooth surface. An immediate extension of this result to k-folds X ($k \geqslant 7$) is: $\delta > k - 6$ iff X is a scroll of $\mathbb{P}^{(k+\delta)/2}$'s over a $(k-\delta)/2$-fold. This gives an alternate proof of a weaker form of a result of Ein. In the second part of the paper (section 3) we deal with the case $\delta = 1$ and we find a new class of 7-folds with degenerate dual varieties. Actually, under the extra assumption that $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections, we prove that, besides Mukai 7-folds and scrolls of $\mathbb{P}^4$'s over a 3-fold, X can be a fibration of grassmannians $G(1, 4)$ (of lines of $\mathbb{P}^4$) over a smooth curve. All these cases really occur: indeed the section of the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$ by three general hyperplanes is an example of Mukai 7-fold with $\delta = 1$; all scrolls as above have $\delta = 1$ and finally all Grassmann fibrations over a smooth curve have $\delta = 1$. This follows from Proposition 3.5, which we owe to the referee. In an earlier version of the paper we only proved this result for Grassmann bundles; our proof consisted of a detailed topological argument taking advantage of the bundle structure and of the homology of Grassmannians.

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1. Known results (new proofs)

Let $X \subset \mathbb{P}^N$ be a complex connected projective algebraic manifold of dimension dim $X = k$. We always assume that X is not contained in any hyperplane unless X itself is a hyperplane. We are mainly concerned with the class of projective manifolds with degenerate dual varieties:

$$\Delta_k = \{X \subset \mathbb{P}^N : \dim X = k \quad \text{and } \dim X^* < N-1\}.$$

Here $X^* \subset \mathbb{P}^{N^*}$ denotes the dual variety of X. As is known dim $X^* \leqslant N-1$, with equality in the general case. Since the class $\mu(X)$ of X is the number of points that a general line of $\mathbb{P}^{N^*}$ cuts out on X^*, we have $\mu(X) = 0$ iff $X \in \Delta_k$.

Let X_1 be the section of X with a general hyperplane and consider the class

$$\mathscr{L}_k = \{X \subset \mathbb{P}^N : \dim X = k \quad \text{and } b_{k-1}(X_1) = b_{k-1}(X)\},$$

where $b_i(X)$ is the i-th Betti number of X. Many properties of $\mathscr{L}_k$ are discussed in [Lanteri and Struppa, 1986]. In particular we recall that ([Lanteri and Struppa, 1986], Prop. 3.3)

$$\Delta_k \subseteq \mathscr{L}_k \text{ with equality for } k \text{ odd.} \tag{1.0}$$

Finally we denote by $\Sigma(r, s)$ the class of (r, s)-scrolls $(r + s = k)$; we say that $X \subset \mathbb{P}^N$ is a (r, s)-scroll if i) $X = \mathbb{P}(E)$, E a rank-$(r+1)$ holomorphic vector bundle over some projective manifold of dimension s, ii) the fibers of X are linear spaces and iii) r is the maximum integer with these properties.

Many results on Δ_k are known and are mostly due to [Ein, 1985]. Here we reprove some of them using a topological-adjunction theoretic approach. Let $\chi(X)$ be the Euler-Poincaré characteristic of X and let X_i denote the section of X with i general hyperplanes. The *class formula* ([Lamotke, 1981], p. 25)

$$\chi(X) = 2\chi(X_1) - \chi(X_2) + (-1)^k \mu(X)$$

is the main ingredient in the proof of the following unpublished result of [Landman, 1976], see ([Kleiman, 1986] (II.3.18))

$$\mu(X) = (b_k(X) - b_{k-2}(X)) + 2(b_{k-1}(X_1) - b_{k-1}(X)) + (b_{k-2}(X_2) - b_{k-2}(X)). \tag{1.1}$$

The three summands in (1.1) are nonnegative numbers due to the strong and the weak Lefschetz theorems. Hence the characteristic condition for X to have degenerate dual variety is

$$b_k(X) - b_{k-2}(X) = b_{k-1}(X_1) - b_{k-1}(X) = b_{k-2}(X_2) - b_{k-2}(X) = 0. \tag{1.2}$$

This immediately shows that $\Delta_2 = \{\mathbb{P}^2\}$, since for a surface $X \in \Delta_2$ the third equality in (1.2) implies $b_0(X_2) = 1$, i.e. that X has degree one.

The following result was first proved by ([Griffiths and Harris, 1979] (3.26)) using differential geometric techniques. Recently ([Ein, 1985], I. Th. 3.3) gave a different proof. Now we deduce it simply from (1.2).

1.3. PROPOSITION: $\Delta_3 = \{\mathbb{P}^3\} \cup \Sigma(2, 1)$.

Proof. We only prove the inclusion $\subseteq$, the other one being easy. Let $X \in \Delta_3$; then $b_1(X_2) = b_1(X)$, by (1.2), and the assertion follows from ([Lanteri and Palleschi, 1984], Th. 3.2), observing that the quadric threefold does not fulfill $b_2(X_1) = b_2(X)$. ¶

As to dimension 4, the following result has been proved by ([Ein, 1985], I, Th. 3.3) and independently by the authors ([Lanteri and Struppa, 1984], (3.3)). Here we provide a third proof stemming from (1.2).

1.4. PROPOSITION: $\Delta_4 = \{\mathbb{P}^4\} \cup \Sigma(3, 1)$.

Proof. As before, we only prove the inclusion $\subseteq$. Let $X \in \Delta_4$; then $X_1 \in \mathscr{L}_3$ by (1.2) and then X_1 is as in (1.3), in view of (1.0). Then either $X = \mathbb{P}^4$, or $X \in \Sigma(3, 1)$ in view of a known result (e.g. see [Bădescu, 1981], §2). ¶

Unfortunately, due to the lack of knowledge of $\mathscr{L}_4$ [Lanteri and Struppa, 1986], (1.2) is not sufficient to recover the following result of Ein:

1.5. PROPOSITION: ([*Ein*, 1985], *II*, *Th.* 5.1) Δ_5 *consists of* $\mathbb{P}^5$, $\Sigma(4, 1)$, $\Sigma(3, 2)$ *and of any nonsingular hyperplane section of the grassmannian G of lines of* $\mathbb{P}^4$ *embedded in* $\mathbb{P}^9$ *via the Plücker embedding.*

In order to deal with higher dimensions we need the following result essentially contained in a paper of [Sommese, 1976].

1.6. PROPOSITION: *Assume* $X_1 \in \Sigma(r, s)$, $r > 2$. *Then* $X \in \Sigma(r + 1, s)$; *in particular* $r \geqslant s - 1$.

Proof. Let $p : X_1 \to B$ be the projection morphism onto the base B of X_1; since $r > 2$ and by ([Sommese, 1976], Prop. III), p extends to a morphism $\tilde{p} : X \to B$. Let F be a fiber of $\tilde{p}$; then $f = X_1 \cdot F$ is a fiber of p and is an ample divisor in F, since X_1 is ample. But $f \simeq \mathbb{P}^r$ and $\mathcal{O}_{X_1}(1) \otimes \mathcal{O}_f = \mathcal{O}_{\mathbb{P}^r}(1)$. Then $F \simeq \mathbb{P}^{r+1}$ and $\mathcal{O}_X(1) \otimes \mathcal{O}_F = \mathcal{O}_{\mathbb{P}^{r+1}}(1)$ (e.g. see [Sommese, 1976], p. 67). This implies that $X \in \Sigma(r + 1, s)$. Furthermore, since $\tilde{p}$ is a surjection and $p = \tilde{p}_{|X_1}$ makes X_1 into a $\mathbb{P}$-bundle over B, it has to be $r \geqslant s - 1$ ([Sommese, 1976], Prop. V]. ¶

In the context of very ample divisors (1.6) extends the above quoted results of Bădescu on ample divisors which are $\mathbb{P}$-bundles over a smooth curve.

Notice also that (1.6) can be viewed as a converse to Proposition 2.2. in [Lanteri and Struppa, 1986].

First of all we use (1.6) jointly with (1.2) to give an alternate proof of a result of ([Ein, 1985], II, Th. 5.2).

1.7. PROPOSITION: Δ_6 *consists of* $\mathbb{P}^6$, $\Sigma(5, 1)$, $\Sigma(4, 2)$ *and of the grassmannian* G.

Proof. That the above classes of manifolds belong to Δ_6 is easily seen (e.g. see [Lanteri and Struppa, 1986]). Now let $X \in \Delta_6$. Once again by (1.2) this implies that $X_1 \in \mathscr{L}_5$ and therefore X_1 is as in (1.5), in view of (1.0). Firstly assume that X_1 is isomorphic to a hyperplane section of G. Let K_X be the canonical bundle of X; since $K_{X_1} = \mathcal{O}_{X_1}(-4)$, by adjunction we get

$$K_X \otimes \mathcal{O}_{X_1} = \mathcal{O}_{X_1}(-5)$$

and then $K_X = \mathcal{O}_X(-5)$, as $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X_1) \simeq \mathbb{Z}$. So X is a 6-dimensional Del Pezzo manifold in the sense of Fujita and therefore $X = G$ in view of Fujita's classification ([Fujita, 1982], (6.3)). Now, if $X_1 \in \Sigma(4, 1) \cup \Sigma(3, 2)$, then X belongs to $\Sigma(5, 1) \cup \Sigma(4, 2)$, by (1.6). Finally, if $X_1 = \mathbb{P}^5$, then $X = \mathbb{P}^6$, trivially. ¶

2. Dimension 7: defects 3 and 5

Just as for Δ_5, the topological-adjunction theoretic method used before does not yield a complete description of Δ_7.

To study the class Δ_7 we need the notion of defect. Recall that the defect of a nonlinear $X \subset \mathbb{P}^N$ is the integer

$$\delta(X) := N - 1 - \dim X^*.$$

We put also $\delta(\mathbb{P}^k) = k$; this is consistent with our general assumption on X. We will need the following facts.

$$\delta(X_1) = \max\{0, \delta(X) - 1\} \text{ ([Hefez and Kleiman, 1985], (5.9));} \qquad (2.1)$$

$$\text{if } X \in \Sigma(r, s) \text{ with } r \geqslant s, \text{ then } \delta(X) = r - s$$

$$\text{([Lanteri and Struppa, 1986], Prop. 5.2).} \qquad (2.2)$$

An independent proof of (2.2) will follow from (3.5). Moreover (1.1) implies, by induction,

$$b_{k-i}(X_i) = b_{k-i}(X), \quad \text{for } i = 1, \ldots, \delta(X) + 1 \text{ [Landman, 1976].} \qquad (2.3)$$

In view of the parity of $k - \delta$ ([Landman, 1976]; see also [Ein, 1985], I, Th. 2.4), if $X \in \Delta_7$ then either $X = \mathbb{P}^k$, or $\delta(X) = 1, 3, 5$.

The case $\delta(X) = 5$ is settled by the following.

2.4. PROPOSITION: *Let $k \geqslant 3$. Then $\delta(X) = k - 2$ iff $X \in \Sigma(k-1, 1)$.*

Proof. If $X \in \Sigma(k-1, 1)$, then $\delta(X) = k - 2$ (e.g. see [Kleiman, 1977], p. 363). Assume $\delta(X) = k - 2$; then (2.3) gives $b_1(X_{k-1}) = b_1(X)$ and the assertion follows now by ([Lanteri and Palleschi, 1984], Th. 3.2). Notice that quadrics are hypersurfaces, hence $\delta = 0$. ¶

Different proofs of (2.4) have already been given by ([Ein, 1985], I. Th. 3.2 and II. Th. 3.1) and by the authors ([Lanteri and Struppa, 1984], Cor. 3.4). More generally ([Ein, 1985], II, Th. 4.1) has proved that if $\delta(X) \geqslant k/2$, then $X \in \Sigma((k+\delta)/2, (k-\delta)/2)$. Unfortunately for $k = 7$ and $\delta = 3$ this result does not apply; in spite of this we can prove by our method that X belongs indeed to $\Sigma(5, 2)$.

2.5. PROPOSITION: *Let $X \in \Delta_7$ with $\delta(X) = 3$. Then $X \in \Sigma(5, 2)$.*

Proof. We have $\delta(X_1) > 0$, by (2.1), i.e. $X_1 \in \Delta_6$. However it cannot be that $X_1 = G$, since the grassmannian G cannot be an ample divisor ([Fujita, 1981], (5.2)). Then the assertion follows from (1.6), (1.7). ¶

An obvious inductive step based on (1.6), (2.3) and (2.5) shows that:

2.6. PROPOSITION: *Let $k \geqslant 7$; then $\delta(X) = k - 4$ iff $X \in \Sigma(k-2, 2)$.*

For $k \geqslant 8$, (2.6) is absorbed in the more general result of Ein quoted before.

In higher dimensions a new interesting manifold arises: the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$, which parametrizes each one of the two disjoint families of 4-planes lying on a smooth 8-dimensional hyperquadric ([Lazarsfeld and Van de Ven, 1984], p. 16). Such a manifold is known to be self-dual, i.e. $S \simeq S^*$; hence $\delta(S) = 4$. Therefore S_2, the section of S by two general hyperplanes has dimension $k = 8$ and defect $\delta = 2$. Since $S \notin \Sigma(7, 3)$ it follows from (1.6) that $S_2 \notin \Sigma(5, 3)$; this shows that a result like (2.4) or (2.6) cannot hold for $\delta = k - 6$.

3. Dimension 7: defect 1

We finally look at the case $\delta(X) = 1$. We first note that $\Sigma(4, 3)$ does not exhaust the class of 7-folds with $\delta(X) = 1$. Indeed S_3, the section of the spinor

variety S by three general hyperplanes, is such a manifold, by (2.1). In order to extend an argument of Ein, we confine ourselves to the class $\Delta'_7 = \{X \in \Delta_7 : \delta(X) = 1$ and $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections$\}$.
To determine Δ'_7 we need some preliminary discussion. First of all, if $X \in \Delta'_7$, the linear system $|K_X \otimes \mathcal{O}_X(5)|$ defines a morphism $f: X \to f(X)$. Now we use two results of Ein:

through a general point $p \in X$ there passes a 3-dimensional family

of lines $\{\ell\}$, ([Ein, 1985], I, Th. 2.3); (3.1)

$K_{X|\ell} = \mathcal{O}_\ell(-5)$ for every $\ell \in \{\ell\}$, ([Ein, 1985], I, Th. 2.4). (3.2)

Therefore by (3.2) the cone spanned by $\{\ell\}$ is contracted by f and since $\dim(f^{-1}(f(p))) \geqslant 4$ in view of (3.1), we conclude that

$$r = \dim f(X) \leqslant 3.$$

Let $r = 0$; then, since $K_X \otimes \mathcal{O}_X(5)$ is spanned, we have $K_X = \mathcal{O}_X(-5)$, i.e. X is a Mukai 7-fold [Mukai, 1985].

Assume now that $r > 0$ and consider the Stein factorization

$$X \overset{g}{\to} B \to f(X)$$

of f. The general fibre D of g is a (7-r)-fold, by generic smoothness and its normal bundle $N_{D|X}$ is trivial. Hence

$$K_D = K_{X|D'}$$

by adjunction; moreover, since f is constant on D by (3.2), this implies

$$K_D = \mathcal{O}_D(-5).$$

Hence D is a Fano (7-r)-fold of index 5 for $r = 2, 3$ and a Del Pezzo 6-fold in the sense of Fujita, for $r = 1$. Let $\Lambda = \langle D \rangle$ be the linear space spanned by D in $\mathbb{P}^N$. Then we have only the following possibilities for $D \subset \Lambda$, according to the values of r.

i) Let $r = 3$; then $D = \Lambda = \mathbb{P}^4$, in view of [Ochiai and Kobayashi, 1973]; thus X is a $\mathbb{P}^4$-bundle over B and $X \in \Sigma(4, 3)$, since the fibres are embedded linearly.
ii) Let $r = 2$; then $D \subset \Lambda$ is a quadric hypersurface of $\mathbb{P}^6$, by [Ochiai and Kobayashi, 1973].
iii) Let $r = 1$; then the Fujita classification of Del Pezzo manifolds ([Fujita, 1982] (6.3)) implies that $D \subset \Lambda$ is either
   a) a cubic hypersurface of $\mathbb{P}^7$,

b) a complete intersection of type (2,2) of $\mathbb{P}^8$,
c) the grassmannian G embedded in $\mathbb{P}^9$ via the Plücker embedding.

We can now state the main result of this section:

3.3. THEOREM: *Let $X \in \Delta'_7$. Then, either X is a Mukai 7-fold, $X \in \Sigma(4, 3)$, or there exists a morphism $g: X \to B$ over a smooth curve B, whose general fibre is the grassmannian G, and $\mathcal{O}_X(1)$ embeds it into a $\mathbb{P}^9$ via the Plücker embedding.*

This latter case will be referred to as a G-fibration.

Proof. In view of the previous discussion, it clearly suffices to show that cases ii) and iii) a), b) cannot occur. To deal with cases iii), take a general point p of D and a general hyperplane Π tangent to X at p. As $\delta(X) = 1$, we know from [Kleiman, 1986] that Π is tangent to X along a line ℓ_0 on which g is constant by (3.2); On the other hand, $\Lambda = \langle D \rangle$ cannot be contained in Π since otherwise one would have $D \subset \Pi \cap X$: this would imply that D is a component of $\Pi \cdot X$; then, since Π is general, D would coincide with $\Pi \cdot X$ and hence D would be singular at p, contradiction. So $\Lambda \not\subset \Pi$, and, by restricting to Λ, we conclude that $\Pi \cap \Lambda$ is a hyperplane of Λ tangent to D along ℓ_0; but this excludes a) and b) since in those cases any tangent hyperplane is tangent at a single point. As far as case ii) is concerned, the proof runs as above if we know that $\Lambda \not\subset \Pi$; this however cannot be proven with the argument used before, since now codim $D = 2$. So we have only to consider the following case.

3.4. ASSUMPTION: Every hyperplane tangent to X at a general point $x \in X$ contains the linear span $\langle D \rangle$ of the fibre D of g through x.

We show that this leads to a contradiction. To do this consider the correspondence

$$\begin{array}{ccc} & \mathscr{I} = \{(\Pi, L) \in \mathbb{P}^{N*} \times G(6, N) : \Pi \supset L\} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbb{P}^{N*} & & G(6, N). \end{array}$$

The second projection gives $\mathscr{I}$ the structure of a $\mathbb{P}^{N-7}$-bundle over the grass mannian $G(6, N)$ of 6-planes of $\mathbb{P}^N$. Of course we have $\dim\langle D_b \rangle = 6$ for every $b \in B$. So there is an injection $j: B \to G(6, N)$, defined by $j(b) = \langle D_b \rangle$. Let $\mathscr{I}_B$ be the pull-back of $\mathscr{I}$ via j and identify

$$\mathscr{I}_B = \{(\Pi, L, b) \in \mathbb{P}^{N*} \times G(6, N) \times B : \Pi \supset L = \langle D_b \rangle\}$$

with its image projected isomorphically into $\mathbb{P}^{N*} \times B$,

$$\mathscr{I}'_B = \{(\Pi, b) \in \mathbb{P}^{N*} \times B : \Pi \supset \langle D_b \rangle\}.$$

Now let Π be a hyperplane tangent to X at a general point x. As before, since $\delta(X) = 1$, Π is tangent to X along a line $\ell_0 \subset X$ which, by (3.2), is contained in a single fibre D_b of g; moreover, $\Pi \supset \langle D_b \rangle$, by (3.4). Then letting $\varphi(\Pi) = (\Pi, b)$ one defines a rational map $\varphi : X^* \dashrightarrow \mathscr{I}_B^*$, which is birational between X^* and $\overline{\varphi(X^*)}$. Hence

$$\dim X^* \leqslant \dim \mathscr{I}'_B = N - 7 + 2 = N - 5.$$

But this implies $\delta(X) \geqslant 4$, contradiction. ¶

Manifolds as in (3.3) really occur in Δ'_7. To prove it we recall that a complete classification of Mukai manifolds is not yet known; anyway, for $k = 7$, in addition to the quartic hypersurfaces and to the complete intersections of type (2, 3) and (2, 2, 2), which however are not in Δ_7, this class contains the section S_3 of the spinor variety $S \subset \mathbb{P}^{15}$ by three general hyperplanes. Actually, since $K_S = \mathcal{O}_S(-8)$, we have, by adjunction, $K_{S_3} = \mathcal{O}_{S_3}(-5)$. Moreover $\delta(S_3) = 1$, by (2.1). As to the class $\Sigma(4, 3)$ there is nothing to say in view of (2.2).

We conclude the paper by showing that all G-fibrations over a smooth curve are in Δ'_7. Let $g : X \to B$ be such a fibration. First of all notice that $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections. This follows from the fact that the rational map Φ associated with $|K_X \otimes \mathcal{O}_X(5)|$ factors through g and $\dim B = 1$; indeed, by adjunction, Φ is constant along the fibres of g.

Now, in view of (2.4), (2.5) it is enough to show that $\delta(X) \geqslant 1$. This follows immediately from the following general proposition, which we owe to the referee.

3.5. PROPOSITION: *Let $X \subset \mathbb{P}^N$ be a projective k-fold such that through its general point there passes a submanifold Y of dimension h and defect θ. Then $\delta(X) \geqslant \theta - k + h$ (i.e.* $\dim X + \delta(X) \geqslant \dim Y + \delta(Y)$).

Proof. Let Π be a hyperplane tangent to X at a general point $x \in X$. Then, since the defect is the dimension of the contact locus, Π is tangent to Y along a subvariety Z containing x and of dimension θ. Let $f = 0$ be a local equation for Π at x. In a neighbourhood U of x in X, the differential df annihilates the tangent spaces $T_{X,x}$ and $T_{Y,z}$ for every $z \in Z \cap U$. Hence df defines on Z a 'co-section' of the rank-$(k-h)$ bundle $(T_X/T_Y)_Z$, vanishing at x. But then a local computation shows that on $(T_X/T_Y)_{Z \cap U}$ this co-section vanishes on the zero locus of $k - h$ functions and therefore Π is tangent to X along a subvariety of Z of codimension less than or equal to $k - h$. This means that $\delta(X) \geqslant \theta - (k - h)$. ¶

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