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Let $K$ be a finite algebraic extension of $Q$, $R$ the ring of integers of $K$ and $\{v\}$ the set of finite places of $K$. For $v \in \{v\}$ let $|\ |_v$ be the non-archimedian absolute value on $K$ and $K_v$ the completion of $K$ with respect to this absolute value. Let $R_v$ be the ring of integers of $K_v$, $P_v$ the unique maximal ideal of $R_v$ and $k_v = R_v/P_v$. Then $k_v$ is a finite field and we let $q_v = \text{card } k_v$. Let $f(x) = f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $m$. Then for any $v$ we can consider

$$Z(t) = \int_{R_v^{(s)}} |f(x)|_v^s \, dx \big|_v$$

where $s \in C$, $\Re(s) > 0$ and $t = q_v^{-s}$. This has been shown to be a rational function of $t$ by Igusa in [Igusa, 1977]. Writing $Z(t) = P(t)/Q(t)$ we define $\deg Z(t) = \deg P(t) - \deg Q(t)$. Igusa has conjectured in [Igusa, 1984], p. 1027, and [Igusa, 1986], that for almost all $v$, i.e. except for a finite number of $v$, one has $\deg Z(t) = -m$. In this paper Igusa gives many examples where $f$ satisfies the additional property that it is the single invariant polynomial for a connected irreducible simple linear algebraic group.

In this paper we show this conjecture is true if $f$ is non-degenerate with respect to its Newton Polyhedron. This establishes the conjecture for "generic" homogeneous polynomials in a sense to be described below.

§1. The Newton polyhedron of $f$ and its associated toroidal modification

We first recall some of the terminology and basic properties of the Newton polyhedron of an arbitrary polynomial. Other references for this include [Danilov, 1978; Kouchnirenko, 1976; Lichtin, 1981; Varchenko, 1977].

Let $f \in K[x_1, \ldots, x_n]$. We write $f = \sum a_I x^I$, where $I = (i_1, \ldots, i_n)$ and $x^I = x_1^{i_1} \cdots x_n^{i_n}$. Let $\text{Supp}(f) = \{I \in N^n \mid a_I \neq 0\}$. Let $S(f)$ denote the convex hull of $\bigcup_{I \in \text{Supp}(f)} (I + R^n_+)$. Let $\Gamma_+(f)$ be the union of all faces of $S(f)$. Let $\Gamma(f)$ be the union of compact faces only. $\Gamma_+(f)$ is called the Newton polyhedron of $f$ and $\Gamma(f)$ is called the Newton diagram. We will denote a
fixed Newton polyhedron and diagram by $\Gamma_+$ and $\Gamma$ respectively. Given a Newton polyhedron $\Gamma_+$ and its associated Newton diagram $\Gamma$ we define $\Omega_{\Gamma_+} = \{ g \in K[x_1, \ldots, x_n] | \Gamma_+(g) = \Gamma_+ \}$. If $g \in \Omega_{\Gamma_+}$ and $\gamma$ is a face of $\Gamma$, we define $g_\gamma$ to be $\sum_{f \in \gamma} b_f x^f$ if $g = \sum_{f \in \gamma} b_f x^f$. Then we define non-degeneracy as in [Kouchnirenko, 1976].

**Definition:** $f$ is non-degenerate with respect to its Newton polyhedron if for any face $\gamma$ of $\Gamma_+(f)$ the functions $(x_i \cdot \partial f / \partial x_i)$, have no common zero in $(\overline{K} - \{0\})^n$, where $\overline{K}$ denotes the algebraic closure of $K$.

Fix $m$ and $n$. Identify homogeneous polynomials of degree $m$ in $n$ variables with $P_K^N$, where $N = \binom{m+n-1}{m} - 1$. For $\Gamma_+$, a fixed Newton polyhedron $X_{\Gamma_+} = \{ f | \Gamma_+(f) = \Gamma_+ \}$ is a Zariski subset of $P_K^N$. Let $Y_{\Gamma_+} = \{ f \ | f \text{ is non-degenerate with respect to } \Gamma_+ \}$.

Then in a completely analogous manner to the proof of Theorem 6.1 in [Kouchnirenko, 1976] we have the following result which shows the non-degeneracy condition is generic.

**Proposition 1:** $Y_{\Gamma_+}$ is a Zariski open, dense subset of $X_{\Gamma_+}$.

Let $K$ be a finite algebraic extension of $Q$, $\{ v \}$ the finite places of $K$, and $K_v$, $R_v$, $P_v$ and $k_v$ as defined in the introduction. Let $U_v = R_v - P_v$ be the units of $R_v$. We first recall some definitions concerning the reduction of varieties modulo $P_v$.

For $g \in R[x_1, \ldots, x_n]$, $v$ a finite place of $K$, let $\overline{g}_v$ denote the polynomial in $k_v[x_1, \ldots, x_n]$ obtained by reducing the coefficients of $g$ modulo $P_v$. We shall abbreviate this to $\overline{g}$ when $v$ is understood and use the same notation when $g$ is a constant in $R$. Let $V$ be an algebraic set defined over $K$, i.e., $V = \{ x \in \overline{K}^n | f_i(x) = 0, 1 \leq i \leq r \}$, where $f_i(x) \in K[x_1, \ldots, x_n]$.

Let $I(V)$ be the ideal of $V$, i.e., $I(V) = \{ f \in \overline{K}[x_1, \ldots, x_n] | f(x) = 0 \ \forall x \in V \}$. Then we define the reduction of $V$ modulo $P_v$, denoted $V_v$ by

$$V_v = \{ x \in \overline{K}_v^n | \overline{f}_v(x) = 0 \ \forall f \in I(V) \cap R_v[x_1, \ldots, x_n] \}.$$

If $f \in R[x_1, \ldots, x_n]$ then for any finite place $v$ of $K$ we can consider the non-degeneracy of $\overline{f}_v$. We have:

**Proposition 2:** Let $f \in R[x_1, \ldots, x_n]$ be non-degenerate with respect to its Newton polyhedron. Then for almost all $v$

a) $\Gamma_+(\overline{f}_v) = \Gamma_+(f)$

b) $\overline{f}_v$ is non-degenerate with respect to its Newton polyhedron.

**Proof.** Let $S = \{ v | \text{all coefficients of } f \text{ are in } U_v \}$. Then for $v \in S$, $\Gamma_+(\overline{f}_v) = \Gamma_+(f)$ and a) follows since almost all $v$ are in $S$. 

Let \( \tau \) be a face of \( \Gamma(f) \), and write \( f = f_{\tau,1}, \ldots, f_{\tau,t} \) where each \( f_{\tau,i} \) is absolutely irreducible. Let \( V_{\tau,i} \) be the variety defined by \( f_{\tau,i} = 0 \), \( Y \) the hyperplane defined by \( x_i = 0 \), and \( Y = \bigcup_{i=1}^{t} Y_i \). The condition that \( f \) is non-degenerate is equivalent to the condition that for any face \( \tau \) of \( \Gamma(f) \), and \( V_{\tau,i} \) as above, the singular points of each \( V_{\tau,i} \) are contained in \( Y \) and for any \( i, j, i \neq j \) we have \( V_{\tau,i} \cap V_{\tau,j} \subset Y \).

Let \( L \) be a finite extension of \( K \) such that the coefficients of \( f_{\tau,i} \) for any \( \tau, i \) are in \( L \). To each place of \( v \) of \( K \) let \( v' \) be any place of \( L \) dividing \( v \). As a straightforward consequence of Hilbert's Nullstellensatz, for any \( \tau, i, j \) we have \( (V_{\tau,i})_{v'} \supseteq (V_{\tau,j})_{v'} \supseteq Y \), for all \( v, v' \). As a consequence of Proposition 30 in [Shimura, 1955], \( (V_{\tau,i})_{v'} \) is absolutely irreducible and its singularities are contained in \( Y_{v'} \) for almost all places \( v' \) of \( L \). Let \( \tilde{S} \) be the set of \( v \in S \) satisfying the above property for all \( \tau, i \) and all \( v' | v \). Then almost every place of \( K \) is in \( \tilde{S} \) and \( f_v \) is non-degenerate for all \( v \in \tilde{S} \). Q.E.D.

We next describe a toroidal modification of \( K^n \) that we shall use to prove the conjecture for homogeneous \( f \) that are non-degenerate with respect to their Newton polyhedron. The modification we use is not the one utilized in [Lichtin, 1981] or [Lichtin and Meuser, 1985], which gives a nonsingular variety \( Y \), and a morphism \( h: \tilde{Y} \rightarrow K^n \) such that \( f \circ h = 0 \) is a divisor with normal crossings, but a weaker modification that has also been used by Denef in [Denef, not yet published].

Let \( (R^n)\ast = R^n - 0 \). Let \( a^1, \ldots, a^l \) be vectors in \( R^n \) and \( \sigma = \{ \alpha_i a^1 + \cdots + \alpha_i a^l | \alpha_i \in R, 1 \leq i \leq l \} \). \( \sigma \) is called a closed cone which we denote by \( \langle a^1, \ldots, a^l \rangle \). \( \sigma' = \{ \alpha_i a^1 + \cdots + \alpha_i a^l | \alpha_i > 0, 1 \leq i \leq l \} \) is called an open cone. The dimension of any cone is the dimension of the smallest vector subspace of \( R^n \) containing it. \( \sigma \) or \( \sigma' \) is called a simplicial cone if \( a^1, \ldots, a^l \) are linearly independent over \( R \). If \( \sigma \) is a closed cone spanned by integral vectors, then we have the following well-known result on \( \sigma \cap \mathbb{Z}_+^n \) which we shall later use.

**Lemma 1.** Let \( \sigma = \langle a^1, \ldots, a^l \rangle \) be a closed cone in \( R^n \), where each \( a^i, 1 \leq i \leq l \), is an integral vector. There are a finite number of integral vectors \( w^1, \ldots, w^r \) such that

\[
\sigma \cap \mathbb{Z}_+^n = \bigsqcup_{i=1}^r \left( w^i + \sum_{j=1}^l \alpha_j a^j | \alpha_j \in \mathbb{Z}_+ \right).
\]

**Proof:** It is well known that \( \sigma \) has a partition into closed simplicial cones where each such cone is spanned by a subset of \( \{ a^1, \ldots, a^l \} \). Thus we can assume \( \sigma \) is simplicial. We form the parallelootope \( P_\sigma = \left\{ \sum_{j=1}^l \alpha_j a^j | 0 \leq \alpha_j < 1 \right\} \).

Let \( w^1, \ldots, w^r \) be the points in \( P_\sigma \cap \mathbb{Z}_+^n \). Then these \( w^i \) satisfy the statement of the lemma. Q.E.D.
Associated to any Newton polyhedron $\Gamma_+$ we have a partition of $(\mathbb{R}^n_+)^*$ into open cones. For $a \in (\mathbb{R}^n_+)^*$ we let $m(a) = \inf \{ a \cdot y \}$ and $\tau_a = \{ y \in \Gamma_+ \mid y \cdot a = m(a) \}$. $\tau_a$ is called the meet locus of $a$. We define an equivalence relation $\sim$ by $a^1 \sim a^2$ if $\tau_{a^1} = \tau_{a^2}$. This equivalence relation satisfies the following properties:

i) If $a \in (\mathbb{R}^n_+)^*$, $\tau_a$ is a face of $\Gamma_+$.

ii) Let $\tau$ be a face of $\Gamma_+$. Let $F_1, \ldots, F_r$ be the facets of $\Gamma_+$ containing $\tau$. Let $a^i$ denote a vector dual to $F_i$, $1 \leq i \leq r$. Then

$$\{ a \in (\mathbb{R}^n_+)^* \mid \tau_a = \tau \} = \{ \alpha_1 a^1 + \cdots + \alpha_r a^r \mid \alpha_i > 0 \}.$$ 

We denote the cone in the above formula by $\sigma_\tau$. Then its closure $\sigma_\tau$ satisfies $\sigma_\tau = \{ a \in (\mathbb{R}^n_+)^* \mid \tau_a \supseteq \tau \}$. A vector $a = (a_1, \ldots, a_n)$ in $\mathbb{Z}^n_+ - \theta$ is called primitive if the greatest common divisor of the $a_j$, $1 \leq j \leq n$, is one. For each facet of $\Gamma_+$ there is a unique primitive integral vector dual to that facet. The above properties imply each equivalence class under $\sim$ is an open cone spanned by a subset of primitive integral vectors dual to facets.

If $f$ is a homogeneous polynomial of degree $m$ in $n$ variables note that all $I \in \text{Supp}(f)$ lie on the hyperplane $I \cdot x = m$, where $I = (1, \ldots, 1)$. Let $F$ be a face of $\Gamma(f)$. It is straightforward to see that if $P$ is an exposed point of $F$ then $P = I$ for some $I \in \text{Supp}(f)$. Hence $\Gamma(f)$ is a single face with supporting hyperplane $I \cdot x = m$. Let $E(\Gamma_+)$ be the exposed points of $\Gamma_+$. Every $P \in E(\Gamma_+)$ lies in $\Gamma$ hence $I \in \sigma P$. We can partition $\sigma P$ into simplicial cones of the form $\{ \alpha_1 a^1 + \cdots + \alpha_n a^n \mid \alpha_i \in \mathbb{R}, \alpha_i > 0 \}$ where we may assume $a^1 = I$, and $a^2, \ldots, a^n$ are primitive integral vectors dual to noncompact facets of $\Gamma_+$ containing $P$.

Let $\sigma = \langle a^1, \ldots, a^n \rangle$ be the closure of one of the maximum dimension cones corresponding to $P \in E(\Gamma_+)$. Write $a^i = (a_{i1}, \ldots, a_{in})$ and let $M = [a_{ij}]$. Then $M$ determines a morphism $\theta: K^n_+ \to K^n_+$ defined by $\theta(y_1, \ldots, y_n) = (x_1, \ldots, x_n)$ where

$$x_j = y_1^{a_{j1}} \cdots y_n^{a_{jn}}. \quad (1)$$

Let $dx$ be the differential $dx_1 \ldots dx_n$ and $\theta^*(dx)$ its pullback under $\theta$. Then for $f \in R[x_1, \ldots, x_n]$, $\Gamma_+$, and $\theta$ as above we have the following result.

**Proposition 3:**

a) $(f \circ \theta)(y) = y_1^m \prod_{i=2}^n y_i^{m(a^i)} f_\theta(y)$ where $f_\theta(y) \in R[y_2, \ldots, y_n]$, $f_\theta(\theta) \neq 0$.

b) $\theta^*(dx) = (\det M) y_1^{m-1} \prod_{i=2}^n y_i^{a^i} \text{d}y$ where $|a^i| = \sum_{j=1}^n a_{ij}$.

c) Let $S = \{ v \mid \Gamma_+(\tilde{f}_v) = \Gamma_+(f) \text{, } \tilde{f}_v \text{ non-degenerate with respect of } \Gamma_+, \text{ and }$
Proof. a) and b) are just specializations of Varchenko's result, Lemma 10.2 in [Varchenko, 1977]. We write $f = a_p x^p + \sum_I P$ with $P$ as in the discussion above and $x_1^i = x_1^{i_1} \cdots x_n^{i_n}$. Then under the map $\theta$ the monomial $x_1^i$ is transformed to $\sum y_1^i - a'$. For a) we denote that $I \in \Gamma(f)$ implies $I \cdot a_1 = m$ and $I \cdot a_i \geq m(a_i)$ for $2 \leq i \leq n$. Furthermore $P \cdot a_i = m(a_i)$ for all $i$, and $P$ is the only point of $\Gamma(f)$ having this property, so this gives the above factorization of $(f \circ \theta)(y)$. The formula $\theta^*(dx)$ is a straightforward consequence of (1).

For c), we first observe that for $v \in S$ we have $(\tilde{a}_p)_v \neq 0$, hence $(\tilde{f}_\theta)_v(0) \neq 0$. The proof of the rest of c) is identical to Lichtin's proof of Proposition 2.3 in [Lichtin, 1981]. Q.E.D.

Let $K_v$ be the completion of $K$ corresponding to any finite place $v$ of $K$. Using the same notation as in the introduction, for every such place we fix $\pi_v \in P - P^2$. Let $U_v = R_v - P_v$. For $x \in K_v^*$ we can write $x = \pi_v \cdot x_u$ where $u \in U_v$. Let $R_v^{(n)} = R_v \times \cdots \times R_v$ (n times) with a similar meaning for $U(v), P(v)^n$.

Let $\sigma = \langle a^1, \ldots, a^l \rangle$ be the closure of a cone in the partition corresponding to $\Gamma^+$. To each such cone we associate a maximal dimension closed cone $\sigma$ containing $\sigma$, and note that it is not unique. For any place $v$, associated to $\sigma$ we consider the subset of $R_v^{(n)}$ defined by

$$X_\sigma = \{ x \in R_v^{(n)} | (\text{ord } x_1, \ldots, \text{ord } x_n) \in \sigma \}.$$ 

Let $Y_\sigma = R_v^{(l)} \times U_v^{(n-l)}$ and consider the morphism $\theta \mid Y_\sigma : Y_\sigma \to R_v^{(n)}$ where $\theta$ is the morphism associated to $\sigma$ defined by (1). We observe that $(\text{ord } x_1, \ldots, \text{ord } x_n) = \sum_{i=1}^n (\text{ord } y_i) a_i^i$, hence $\theta(Y_\sigma) \subseteq X_\sigma$. The next Lemma gives the properties of $\theta \mid Y_\sigma$ and the decomposition of $X_\sigma$ that were established by Denef, Lemma 3 in [Denef, not yet published]. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in K_v^n$, and $T$ any subset of $K_v^n$, denote by $\gamma T$ the set ${((\gamma_1 x_1, \ldots, \gamma_n x_n)) | (x_1, \ldots, x_n) \in T}$.}

**Lemma 2.** a) The map $\theta \mid Y_\sigma : Y_\sigma \to \theta(Y_\sigma)$ is locally bianaalytic and each fiber has cardinality $K_v(\gamma) = \text{card ker } \theta \mid U_v^{(n)}$. b) If $w^i = (w_{i1}, \ldots, w_{in})$, $1 \leq i \leq r$, are the vectors in $\sigma \cap Z_+^n$ given by Lemma 1, let $\pi^w$ denote $(\pi^{w_i1}, \ldots, \pi^{w_{in}})$. Let $u_1, \ldots, u_{s(v)}$ be the cost representatives for $U_v^{(n)} / \theta(U_v^{(n)})$. Then

$$X_\sigma = \prod_{1 \leq i \leq s(v)} u_i \pi^{w_i}(\theta(Y_\sigma)).$$


§2. The degree of $Z(t)$

Let $v$ be a finite place of $K$. Using the same notation as in the preceding sections, we define an absolute value on $K_v^*$ by $|x|_v = q_v^{-\text{ord}_v x}$. We let $|dx|_v$ be the Haar measure on $K_v$ normalized so that the measure of $R_v$ is one. Then the measure of $a + P_v$ for any $a \in K_v$ is $q_v^{-1}$. If $a \in R_v^{(n)}$, $a + P_v^{(n)}$ will denote a coset modulo $P_v^{(n)}$, i.e. $(a_1 + P_v) \times \cdots \times (a_n + P_v)$ where $a = (a_1, \ldots, a_n)$.

We shall also use $|dx|_v$ defined above for $n = 1$, to be the measure $\prod_{i=1}^n |dx_i|_v$ on $R_v^{(n)}$. When $v$ is fixed we denote $\pi_v$, $|dx|_v$ and $q_v$ by $\pi$, $|dx|$ and $q$ respectively. Letting $t = q^{-s}$ we have the following basic formulas for $N$, $n \in \mathbb{Z}$; $N$, $n \geq 0$.

\[
\begin{align*}
\int_R |x|^{N-s+n-1} |dx| &= \frac{q^n(1-q^{-1})}{q^n-t^N} \\
\int_P |x|^{N-s+n-1} |dx| &= \frac{(1-q^{-1})t^N}{q^n-t^N}.
\end{align*}
\]

(2)

For $f \in K[x_1, \ldots, x_n]$, and any finite place $v$, we can consider the zeta function $Z(t)$ associated to $f$ as defined in the introduction. We then have the following result.

**Theorem.** Let $f(x) = f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial that is non-degenerate with respect to its Newton polyhedron. Then for almost every place $v$ of $K$, $\deg Z(t) = -\deg f(x)$.

**Proof:** Let $\deg f(x) = m$, and $\Gamma_+$ be the Newton polyhedron of $f$. As explained in the previous section, associated to this Newton polyhedron we have a partition of $R^n_+$ into open cones. For $P$ an exposed point of $\Gamma_+$, let $\delta_p$ be the associated maximal dimension open cone. As previously observed we can partition $\delta_p$ into simplicial cones of the form $\{a_1 a^1 + \cdots + a_n a^n | a_i > 0\}$ where $a^1 = I$, if $\delta_p$ is not already in this form. The $a^i$, $2 \leq i \leq n$, are dual to noncompact facets of $\Gamma_+$. Repeating this process for all points of $E(\Gamma_+)$ let $\delta_1, \ldots, \delta_K$ denote the resulting simplicial cones, and let $\sigma_1, \ldots, \sigma_K$ denote the corresponding closed cones. $R^n_+ \subseteq \bigcup_{i=1}^K \sigma_i$ and if $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots K\}$ then

\[
\bigcap_{j=1}^k \sigma_{i_j}\end{align*}

is a closed cone, which is a face of each $\sigma_{i_j}$, hence is a simplicial cone. Furthermore the closed cone $\{a I | a \geq 0\}$ is contained in every such cone.
Consider
\[ \bigcup_{i=1}^{K} \sigma_i - \bigcup_{1 \leq i_1 < i_2 \leq K} (\sigma_{i_1} \cap \sigma_{i_2}) + \cdots + (-1)^{j-1} \times \bigcup_{1 \leq i_1 < \cdots < i_j \leq K} (\sigma_{i_1} \cap \cdots \cap \sigma_{i_j}) + \cdots + (-1)^{K-1}(\sigma_1 \cap \cdots \cap \sigma_k). \]

Since every \((k_1, \ldots, k_n) \in \mathbb{Z}_+^n\) occurs exactly once in (3) we can write \(Z(t)\) as the sum and difference of integrals of the form
\[
\int_{X_\sigma} |f(x)|^s |dx|_v
\]
where \(\sigma = \langle I, a^2, \ldots, a^l \rangle\) for some \(1 \leq l \leq n\), where the \(l = 1\) case is \(\sigma = \langle I \rangle\).

For each maximal dimension cone \(\sigma_k = \langle I, a^2, \ldots, a^n \rangle\) write \(a^l = (a_{i_1}, \ldots, a_{i_l})\), let \(M_k = [a_{ij}]\), and let \(\theta_k\) be the morphism defined by (1) in §1. Let \(S\) be the set of places satisfying the conditions in Proposition 3 c) for \(M_k, 1 \leq k \leq K\).

We now fix \(v \in S\), and \(\sigma = \langle I, a^2, \ldots, a^l \rangle\). Choose a maximal dimension cone \(\sigma_k, 1 \leq k \leq K\), such that \(\sigma_k\) contains \(\sigma\). We denote this choice by \(\bar{\sigma} = \langle I, a^l, a^{l+1}, \ldots, a^n \rangle\) and let \(M, \theta\) be the matrix and morphism associated to \(\bar{\sigma}\). Referring to the decomposition of \(X_\sigma\) in Lemma 2 b) we can write (4) as a sum of integrals of the form
\[
\int_{\mu \sigma(v) \theta(Y_\sigma)} |f(x)|^s |dx|
\]
for some \(u = u_i, 1 \leq i \leq s(v)\), and \(w = w_j, 1 \leq j \leq r\), where \(Y_\sigma = R^{(l)}_o \times U^{(n-l)}_o\).

Write \(f = \sum a_I x^I\), then \(f(u \pi^w x) = \sum a_I u^I \pi^{w \cdot I} x^I\). We have \(w \cdot I \geq m(w)\) for all \(I \in \Gamma_+\), so we let
\[
f_{u,w}(x) = \sum a_I u^I \pi^{w \cdot I - m(w)} x^I. \]

Then the integral in (5) equals
\[
q^{-|w|} \int_{\theta(Y_\sigma)} |f_{u,w}(x)|^s |dx|.
\]

By applying a) and b) in Proposition 3, in addition to the above observations,
we have that the integral in the above is
\[
\frac{1}{\kappa_\theta(v)} \int_{Y'_0} |y_1|^{m + n - 1} |y_1| \cdot \prod_{i=2}^{n} |y_i|^{m(a'_i) + |a'_i| - 1} |g_{u,w}(y)|^s |dy|
\]
where \( Y'_0 = R^{(n)} \times U^{(n)} \) and \( g_{u,w}(y) \in R[y_2, \ldots, y_n] \). Applying (2) to the first integral we have that the contribution to \( Z(t) \) from (5) is
\[
q^n(1 - q^{-1})(q^n - t^m)^{-1}
\]
times
\[
\frac{q^{-|w|}}{\kappa_\theta(v)} t^{m(w)} \prod_{i=2}^{n} |y_i|^{m(a'_i) + |a'_i| - 1} |g_{u,w}(y)|^s |dy_2 \cdots dy_n|.
\]
By our observations above the factor (7) occurs for any integral of the form (5), so we can write
\[
Z(t) = \frac{q^n(1 - q^{-1})}{q^n - t^m} \tilde{Z}(t)
\]
where \( \tilde{Z}(t) \) is the sum and difference of expressions in the form of (8) for all possible \( \sigma, u, w \). We shall show that (8) can be written in the form \( P_{\sigma,u,w}(t)/Q(t) \) where \( Q(t) = (q - t)\prod(q^{a'_i} - t^{m(a'_i)}) \) and the product is over all \( a'_i \) dual to a noncompact facet of \( \Gamma_+ \). We then write \( \tilde{Z}(t) = P(t)/Q(t) \) and

\[
P(t) = \sum_{\sigma} (\text{sign } \sigma) \sum_{u,w} P_{\sigma,u,w}(t)
\]
where \( \text{sign } \sigma = \pm 1 \) is the coefficient of \( \sigma \) in the decomposition (3). Let \( D = 1 + \sum m(a'_i) = \deg Q(t) \). We shall show that after possibly excluding an additional finite set of places in \( S \), that \( \deg P(t) = D \), in which case the theorem follows.

Now consider \( g_{u,w}(y) \). Referring back to \( f_{u,w}(x) \) as given in (6) we see that
\[
\hat{f}_{u,w}(x) = \sum_{j \in \tau_w} \hat{a}_j x^j,
\]
where \( \tau_w \) is a face of \( \Gamma_+ \). We have that \( \hat{f}_{u,w} \) is nondegenerate with respect to its Newton polyhedron since if \( \tau' \) is a face of \( \tau_{w'} \) and \( b \in (k_\theta(0))^n \) is a solution to
\[
\left( x_j \frac{\partial \hat{f}_{u,w}}{\partial x_j} \right)_{\tau'} = 0 \quad 1 \leq j \leq n
\]
then \( \hat{ub} \) would be a solution to
\[
\left( x_j \frac{\partial \hat{f}}{\partial x_j} \right)_{\tau'} = 0 \quad 1 \leq j \leq n.
\]
But $\tau'$ is a face of $\Gamma_+$, hence this contradicts the non-degeneracy of $f$. Thus by applying Proposition 3 we have $\tilde{g}_{u,w}(b) = 0$ implies

$$
\left( y_j \frac{\partial \tilde{g}_{u,w}}{\partial y_j} \right)(b) \neq 0
$$

for some $j$, $2 \leq j \leq n$.

First consider the case where $a = \langle 1, a^2, \ldots, a^l \rangle$ with $l \geq 2$. We shall show that $\deg P_{\sigma, u, w}(t) \leq D$. Then writing the coefficient $c_{\sigma, u, w}$ of $t^D$ in $P_{\sigma, u, w}(t)$ as $(\kappa_\theta(v))^{-1}\tilde{c}_{\sigma, u, w}$ we show $q^{n-1}\tilde{c}_{\sigma, u, w} \equiv 0 \mod q$.

If $w \neq 0$, since $w \in \sigma \cap Z(n)$ by permuting the vectors $\{a^2, \ldots, a^l\}$ we may suppose $w = \alpha_1 + \alpha_2 a^2 + \cdots + \alpha_k a^k$ where $0 < \alpha_i < 1$, $2 \leq i \leq k$, $0 \leq \alpha_1 < 1$ and $k \leq l$. When $w = 0$ set $k = 1$. Then we write the integral in (8) as

$$
\int_{R^{(l-k)}_v \times U^{(n-l)}_v} \prod_{i=2}^l |y_i|^m(a^i)s + |a^i|^{-1} |g_{u,w}(y)| \cdot |dy|.
$$

We have

$$
g_{u,w}(y) = \sum_I a_I u^I \pi^{w \cdot I - m(w)} y_2^{a_2 - m(a^2)} \cdots y_n^{a_n - m(a^n)}.
$$

Observing that $I \in \tau_w$ implies $I \cdot a^i = m(a^i)$, $2 \leq i \leq k$, we have $\tilde{g}_{u,w} \in k_v[y_{k+1}, \ldots, y_n]$. Thus in this case (10) specializes to $\tilde{g}_{u,w}(b) = 0$, $b \in k_v^n$ implies $(y_j \frac{\partial \tilde{g}_{u,w}}{\partial y_j})(b) \neq 0$ for some $j$, $k < j \leq n$; which implies the system of congruences

$$
g_{u,w}(y) \equiv 0 \mod P_v
$$

$$
\left( y_j \frac{\partial g_{u,w}}{\partial y_j} \right)(y) \equiv 0 \mod P_v, \ k < j \leq n
$$

has no solution in $R_v^{(n)}$.

For any subset $J \subseteq \{k+1, \ldots, l\}$ consider cosets $(c_{k+1}, \ldots, c_n) + P_v^{(n-k)}$ of $R_v^{(l-k)} \times U_v^{(n-l)}$ satisfying

$$
c_i \equiv 0 \mod P_v \quad i \in J
$$

$$
c_i \not\equiv 0 \mod P_v \quad i \not\in J
$$

and call these cosets of type $J$. We distinguish the cosets of type $J$ further by saying a coset is of type $J_1$ if it satisfies $g_{u,w} \not\equiv 0 \mod P_v$ in addition to the above conditions and say it is of type $J_2$ if it satisfies $g_{u,w} \equiv 0 \mod P_v$ in addition to the above conditions. We then write (11) as a sum over varying $J$.
of integrals of type
\[ \int_{C_J} \int_{R^{(l-3)}} \prod_{i=2}^l |y_i|^{m(a^i)+|a^i|-1} |g_{u,w}(y)|^s \, dy \]
(14)
where \( C_J \) is a coset of type \( J \).
If \( C_J \) is a coset of type \( J_1 \), by applying the formulas (2), we have that the integral in (14) is of the form \( P_1(t)/Q_1(t) \), where \( \deg P_1(t) = \sum_{i \in J} m(a^i) \) and \( \deg Q_1 = \sum_{i=2}^k m(a^i) + \sum_{i \in J} m(a^i) \). If \( C_J \) is of type \( J_2 \) by (12) we can choose \( j, k < j \leq n \), such that \( y_j \partial g_{u,w}/\partial y_j \not\equiv 0 \mod P_\nu \). We then make the change of variables \( y_j = g_{u,w} \), \( \tilde{y}_i = y_i, \ i \neq j \). Then the integral (14) is of the form \( P_2(t)/Q_2(t) \), where \( \deg P_2(t) = 1 + \sum_{i \in J} m(a^i) \) and \( \deg Q_2(t) = 1 + \sum_{i=2}^k m(a^i) + \sum_{i \in J} m(a^i) \). In either case we have \( P_i(t)/Q_i(t) = R_i(t)/Q(t) \) where \( \deg R_i(t) = D - \sum_{i=2}^k m(a^i) \). Thus (11) is the sum of rational functions with this property, hence referring to (8) we see that for \( w \neq 0 \)
\[ \deg P_{a,u,w}(t) \leq D + m(w) - \sum_{i=2}^k m(a^i). \]
Moreover the coefficient of the highest degree term in \( P_{a,u,w}(t) \) is
\[ \pm \kappa_\theta(v)^{-1} q^{-|w|+\sum_{l=2}^k |a^l|} \sum_{J} (-1)^{|J|} (1-q^{-1})^{|J|+k-1} q^{-(n-k-|J|-1)} \times \left[ N_J q^{-1} + N_J (1-q^{-1}) \right], \]
where \( N_J \) is the number of cosets of type \( J_i \).
If \( w = 0 \), we have \( \deg P_{a,u,w}(t) \leq D \). If \( w \neq 0 \) in order to show this we must show \( m(w) \leq \sum_{i=2}^k m(a^i) \). We have \( w_j = \alpha_i + \sum_{i=2}^k \alpha_i a_{ij} \) where \( \alpha_i < 1, 1 \leq i \leq k \), hence \( w_j < 1 + \sum_{i=2}^k a_{ij} \), and \( w_j \in \mathbb{Z} \) implies \( w_j \leq \sum_{i=2}^k a_{ij} \). Now let \( P \in E(\Gamma_+) \) be such that \( \bar{\sigma} \) is obtained from the partition of \( \bar{\sigma}_P \) into simplicial cones. Write \( P = (P_1, \ldots, P_n) \). We have \( m(w) = P \cdot w \) and \( m(a^i) = P \cdot a^i, 2 \leq i \leq k \). Hence
\[ m(w) \leq \sum_{j=1}^n P_j \left( \sum_{i=2}^k a_{ij} \right) = \sum_{i=2}^k P \cdot a^i. \]
Thus \( m(w) \leq \sum_{i=2}^k m(a^i) \), which implies \( \deg P_{a,u,w}(t) \leq D \).
Now consider \( c_{\sigma,u,w} \). If \( \deg P_{\sigma,u,w}(t) < D \) then \( c_{\sigma,u,w} = 0 \). If \( \deg P_{\sigma,u,w}(t) = D \) then by observing that \( N_{J_1} + N_{J_2} = (q-1)^{n-k-|J|} \) we have

\[
q^{n-1}c_{\sigma,u,w} = \pm q^{-\sum_{i=2}^{k} a_i^i + \sum_{J} |J|} \sum_{J} (-1)^{|J|} (q - 1)^{|J| + k - 1} \times [(q - 1)^{n-k-|J|} - N_J - N_J(q - 1)],
\]

where we let \( N_J = N_{J_2} \). If \( |w| < \sum_{i=2}^{k} a_i^i \) then \( q^{n-1}c_{\sigma,u,w} \) is clearly congruent to zero mod \( q \), but if \( |w| = \sum_{i=2}^{k} a_i^i \) then

\[
q^{n-1}c_{\sigma,u,w} = \pm \sum_{J} (-1)^{|J|} \mod q \equiv 0 \mod q. \tag{15}
\]

This proves our assertion about the case \( \sigma = \langle 1, a^2, \ldots, a^l \rangle \) with \( l \geq 2 \).

The only remaining cases to consider are those where \( u \) varies and \( \sigma = \langle 1 \rangle \): In this case we show that (8) can be written as \( P_{1,u}(t)/Q(t) \) where \( \deg P_{1,u}(t) \leq D \). Denoting the coefficient of \( t^D \) by \( c_{1,u} \) and defining \( \tilde{c}_{1,u} \) as in the previous case we show \( q^{n-1}c_{1,u} \) is an integer and \( q^{n-1}c_{1,u} \not\equiv 0 \mod q \).

In this case the integral in (8) is

\[
\int_{U_v^{(n-1)}} |g_u(y)|^s \, \mathrm{d}y.
\]

Consider the cosets mod \( P_v^{(n-1)} \) of \( U_v^{(n-1)} \). Letting \( N \) be the number of cosets satisfying \( g_u \equiv 0 \mod P_v \) and applying entirely similar reasoning as before we have that the above integral equals

\[
(q^{-1})^{(n-2)} \left[ ((q - 1)^{n-1} - N)q^{-1} + \frac{N(1 - q^{-1})t}{(q - t)} \right].
\]

Then examination of the above shows \( \deg P_{1,u}(t) \leq D \) and

\[
q^{n-1}c_{1,u} = (-1)^D \left[ (q - 1)^{n-1} - N - N(q - 1) \right].
\]

Hence \( q^{n-1}c_{1,u} \) is an integer and

\[
q^{n-1}c_{1,u} \equiv \pm 1 \mod q. \tag{16}
\]

Furthermore we note that the value on the right of the congruence is independent of \( u \).

Let \( c_v \) denote the coefficient of \( t^D \) in \( P(t) \), which we wish to show is nonzero for almost all \( v \). We have \( c_v = \sum_{\sigma} \text{sign } \sigma \sum_{u,w} c_{\sigma,u,w} \). Recalling that the
morphisms associated to the maximal dimension cones were denoted $\theta_1, \ldots, \theta_K$ we define $\kappa(v) = \prod_{i=1}^{K} \kappa_{\theta_i}(v)$. If $\sigma$ is a cone, and the morphism associated to the maximal dimension cone $\sigma$ is $\theta_j$, define $\kappa_{\sigma} = \prod_{i \neq j}^{K} \kappa_{\theta_i}(v)$. We assume $\theta_1$ is the morphism associated to $\langle I \rangle$, and let $\kappa_1(v) = \prod_{i=2}^{K} \kappa_{\theta_i}(v)$. Then

$$q_{\nu}^{n-1}\kappa(v)c_{\nu} = \sum_{\sigma \neq \langle I \rangle} \text{sign } \sigma \sum_{u,w} q_{\nu}^{n-1}\kappa_\sigma(v)\bar{c}_{\sigma,u,w} + \sum_{u} q_{\nu}^{n-1}\kappa_1(v)\bar{c}_{1,u}.$$ 

Let $s_1(v)$ denote the number of coset representatives in $U_{\nu}/\theta_1(U_{\nu}^*)$. Then the congruences in (15) and (16) give

$$q_{\nu}^{n-1}\kappa(v)c_{\nu} \equiv \pm \kappa_1(v)s_1(v) \mod q_{\nu}.$$ 

Now

$$\kappa_1(v) \leq \prod_{i=2}^{K} \text{card } W_{\nu,|M_i|} \leq n^{K-1} \prod_{i=2}^{K} |M_i|$$

where $|M_i|$ is the determinant of the matrix $M_i$ associated to $\theta_i$ and $W_{\nu,|M_i|}$ is the $|M_i|$-th roots of unity in $U_{\nu}$. We also have

$$s_1(v) \leq n \cdot [U_{\nu} : U_{\nu}^{[M_i]}]$$

where $[U_{\nu} : U_{\nu}^{[M_i]}] = \text{card } W_{\nu,|M_i|}$ for almost all $v$. Hence for almost all $v$

$$\kappa_1(v)s_1(v) \leq n^K \prod_{i=1}^{K} |M_i|$$

which implies $\kappa_1(v)s_1(v) \not\equiv 0 \mod q_{\nu}$ for almost all $v$. Therefore $c_{\nu} \not\equiv 0$ for almost all $v$, which concludes the proof. Q.E.D.

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References


On the degree of a local zeta function


**Addendum.** J. Denef has recently given a proof of Igusa’s conjecture in the general case.