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Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces  


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Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces

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0. In this paper we generalize the classical isoperimetric inequality on $S^n$ to non-invariant measures and prove as a corollary the concentration of measure on spheres $S(X)$ of uniformly convex Banach spaces $X$. Our argument avoids symmetrization and (or) the calculus of variations by a direct appeal to Cavaliery's principle similar to that used in Hadwiger's proof of the Brunn-Minkowski theorem [H]. In fact, we use the localized Brunn's theorem at the final stage of our proof, though a slight rearrangement of our argument would imply this theorem. (In the Appendix, we give, for the completeness sake, a short proof of Brunn's theorem). One of the applications is the lower exponential bound on the dimension of $l_\infty$ admitting a symmetric map $S(X) \to S(l_\infty)$ with a fixed Lipschitz constant.

In order to keep the presentation transparent we did not attempt to state the most general isoperimetric inequality serving all possible applications. This has unavoidably led to repetitions of some arguments at different places in the paper as some readers may notice.

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1. Let $\mu$ be some measure on the Euclidean sphere $S^n$ and let $A$ and $B$ be two disjoint subsets in $S^n$. We seek an upper bound on $\text{dist} (A, B)$ in terms of the measures $\mu(A)$ and $\mu(B)$, where "\text{dist}" is some metric on $S^n$. If $B$ is the complement of the $\varepsilon$-neighbourhood of $A$, then for $\varepsilon \to 0$ our question reduces to the isoperimetric problem.

2. To formulate our main results we have to introduce some notions.

2.1. Consider an open arc $\sigma \subset S^n$ between two opposite points $t_+$ and $t_-$ in $S^n$ and call a subset $\Sigma$ $\sigma$-admissible if it is a union of open arcs between $t_+$ and $t_-$ and if every point $t$ in $\sigma$ lies in the interior of $\Sigma$. Next divide $\sigma$ into three subintervals, say $\sigma = (t_+, a) \cup (a, b) \cup (b, t_-)$, called $\alpha_1$, $\alpha_2$ and $\alpha_3$.
respectively, and let $A_i, i = 1, 2, 3$ be open subsets in $S^n$ such that $A_i \cap \sigma = \alpha_i$. Finally, take a Borel measure $\mu$ on $S^n$, and define the relative canonical measure $\mu_\sigma(x_2/\alpha_i)$ for $i = 1, 3$ by

$$\mu_\sigma(x_2/\alpha_i) = \inf \liminf_{\Sigma \to \sigma} \frac{\mu(A_i \cap \Sigma)}{\mu(A_i \cap \Sigma)}$$

(2.1)

where $\inf$ is taken over all above triples $(A_1, A_2, A_3)$ and where we assume $\infty/\infty = \infty$ and $0/0 = 0$, and where the convergence $\Sigma \to \sigma$ is understood for the Hausdorff topology for subsets in the sphere.

2.2. In the case of a “good” measure $\mu$ the definition (2.1) simplifies as follows.

Consider the family of all non-negative measures on $S^n$ with continuous density functions. We will call such measures regular. So, for every regular measure $\mu$ there exists $f_\mu(t) \in C(S^n)$ such that for any Borel set $A \subset S^n$, $\mu(A) = \int_A f_\mu(t) \, dt$. Take two opposite points $t$ and $-t$ on $S^n$, and consider all maximal arcs $\sigma$ between $t$ and $-t$. This gives a partition $\mathcal{H}_t$ of $S^n \backslash \{t, -t\}$ and hence every regular measure $\mu$ induces a measure (defined up to a constant) on every $\sigma \in \mathcal{H}_t$, called $\mu_\sigma$. We call such partitions canonical partitions. Clearly, in this case (2.1) may be rewritten as

$$\mu_\sigma(x_2/\alpha_i) = \frac{\mu_\sigma(x_2)}{\mu_\sigma(\alpha_i)}.$$

**Examples 2.3:**

a. If $\mu$ is the standard measure on $S^n$, then, obviously $\mu_\sigma = \text{Const.} (\sin \theta)^{n-1} \, d\theta$ for $\theta$ — the angle from $[0, \pi]$ parametrizing $\sigma$.

b. Let $S^n_+ \subset S^n$ be a hemisphere, and let $S^n_+ \simeq \mathbb{R}^n$ be a projective isomorphism. Then, in case $t$ and $-t$ lie on $\partial S^n_+$, arcs $\sigma$ are straight lines in $\mathbb{R}^n$, and so the $\mathbb{R}^n$-invariant measure $\mu$ on $\mathbb{R}^n \simeq S^n_+$ gives the Lebesgue measure $dt$ on $c$’s.

2.4. Next, let $A_1$ and $A_3$ be closed subsets in $S^n$, let $A_2^+$ be the union of all arcs in $S^n$ between $A_1$ and $A_3$ (e.g. $A_2^+ = \text{Conv} A_1 \cup A_3$ in the case of convex sets $A_1$ and $A_3$), and let $A_2 = A_1 \ast A_3$ be defined as

$$A_2 = A_1 \ast A_3 = A_2^+ \setminus (A_1 \cup A_3).$$

(2.2)

Assume the $\mu$-measures of $A_1$ and $A_3$ to be in $(0, \infty)$ and let $\lambda = \mu(A_1)/\mu(A_3)$. Call a pair of points $a \in A_1$ and $b \in A_3$ extremal if there is a maximal arc $\sigma$
in $S^n$ which contains $a$ and $b$ such that the open interval $(a, b) \subset \sigma$ misses $A_1$ and $A_3$. The define

$$\lambda\text{-dist}_\mu(a, b)m = \inf_{\sigma} \max \lambda(\mu_\sigma(\alpha_2/\alpha_1), \mu_\sigma(\alpha_2/\alpha_3))$$

where a maximal arc $\sigma = (t_-, t_+)$ is divided into intervals $\alpha_1 = (t_-, a)$, $\alpha_2 = [a, b]$ $\alpha_3 = (b, t_+)$, and

$$\lambda\text{-dist}_\mu(A_1, A_3) = \inf (\lambda\text{-dist}_\mu(a, b))$$

over all extremal pairs $(a, b)$ and all $\sigma$. In what follows we abbreviate: $\lambda\text{-dist} = \text{dist}$.

**Theorem 3.** $\mu(A_2) \geq \lambda\text{-dist}_\mu(A_1, A_3) \mu(A_3) = \lambda^{-1} \lambda\text{-dist}_\mu(A_1, A_3) \mu(A_1)$.

**Example 3.1:** If $\mu$ is the $O(n)$-invariant measure, then the explicit formula for the canonical measure (see Example 2.3.a) gives a sharp lower bound for the spherical distance between $A_1$ and $A_3$ with equality for balls around opposite points in $S^n$. Thus, we recapture the classical isoperimetric inequality for $S^n$.

3.2. The proof of the theorem involves a few constructions which we consider to be of independent interest. By an obvious approximation argument and the definition (2.1), we may (and shall) assume the measure $\mu$ is positive regular which means, in addition to the regularity condition, that $\mu(A) > 0$ for every open subset $A$ in $S^n$.

**Non important Remark.** One could eliminate $\lambda$ from the story by multiplying the measure by $\lambda$ on $A_3$, and thus reducing the problem to the case $\lambda = 1$. However we prefer to keep $\lambda$.

Take an open hemisphere $S^n_+$, and fix a projective isomorphism $S^n_+ \leftrightarrow \mathbb{R}^n$ sending every straight line of $\mathbb{R}^n$ to a maximal arc on $S^n_+$ (and conversely). This provides a one-to-one correspondence between positive regular measures on $\mathbb{R}^n$ and $S^n_+$, thus identifying measures on $S^n_+$ and $\mathbb{R}^n$ which we denote by the same $\mu$.

4. Convex restrictions of measures. Take an affine (i.e., a translate of a linear) subspace $E \subset \mathbb{R}^n$, fix a projection $p: \mathbb{R}^n \rightarrow E$, and consider decreasing sequences of convex subsets $K_i, i \in \mathbb{N}$ in $\mathbb{R}^n$, such that $M = \cap K_i \subset E$. By restricting a given measure $\mu$ to each $K_i$, and then by projecting to $E$, we obtain a sequence of measures $\mu_i$ on $E$. Call a non identically zero measure
v on $E$ a convex restriction of $\mu$ to $M$ if for some sequence of above $K_i$ and some sequence of real numbers $\lambda_i$,

$$v = \lim \lambda_i \mu_i,$$

for the weak limit of measures.

If a measure $v'$ on a convex subset $M'$ in an affinit subspace $E' \subset E$ is a convex restriction of $v$, then obviously, $v'$ also is a convex restriction of the original $\mu$.

5. First step. Use of Brunn's Theorem. Take a $k$-dimensional subspace $E \subset \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$. Observe that the $(n - k)$-dimensional symmetrization (see [H] or [B.2] $S_E K$ is convex by Brunn’s Theorem (see Appendix).

**Lemma 5.1:** Let a decreasing family of convex sets $\{k_i\}$ define a convex restriction measure $\mu_M(M = \cap K_i \subset E)$ of $\mu$. Then the family $\{S_E K_i\}$ defines the same measure $\mu_M$.

**Proof.** This follows from the definition of $S_E K_i$ and the uniqueness of the Radon–Nicodym derivative $f_\mu$ which is a continuous function in our case and therefore well defined on $M$.

**Remark 5.2:** If $\mu$ is absolutely continuous (rather than regular) with respect to Lebesgue measure, then the Lemma only holds true for almost every $k$-dimensional subspace $E \subset \mathbb{R}^n$.

6. Convex partitions of $S^n$ and $\mathbb{R}^n$. A set $A \subset S^n_+$ is called convex if it contains the arc (in $S^n_+$) between $a$ and $b$ for all $a$ and $b$ in $A$. Clearly $A$ is convex iff the set corresponding to it by the projective isomorphism in $\mathbb{R}^n$ is a convex set in $\mathbb{R}^n$.

**Definition 6.1:** We say that $\alpha$ is a $k$-dimensional convex partition of $\mathbb{R}^n$ if

i) every $A \in \alpha$ is convex and $k$-dimensional, i.e., there exists a $k$-dimensional affine subspace $E$ such that $A \subset E$ and the interior $\hat{A}$ of $A$ in $E$ is not empty.

ii) there exists a family of convex open neighbourhood $K_i$ of $\hat{A}$ such that $\cap K_i = \hat{A}$ and every $K_i = \cup \hat{A}_s$ for some $A_s \in \alpha$.

The image of $\alpha$ on $S^n_+$ by a projective isomorphism is called the $k$-dimensional convex partition of $S^n_+$.
6.2. Consider a 1-dimensional convex partition $\alpha$ of $\mathbb{R}^n$. By Rohlin’s measure decomposition theory (see [R]), there is a unique induced measure $\mu_\alpha$ on almost every (for the quotient measure on $\mathbb{R}^n/\alpha I$). In fact, this $\mu_\alpha$ equals the convex restriction of $\mu$ defined by some sequence of convex bodies $\{K_i\}$ such that $\bigcap K_i = \hat{I}$ and every $K_i = \bigcup \hat{I}_x$ for some $I_x \in \alpha$ (such family exists by 6.1, ii). Therefore, (use 5.1) $\mu_\alpha$ is the convex restriction of $\mu$ defined by the family $\{S_i, K_i\}$ obtained from $\{K_i\}$ by the symmetrization around $I$.

7. Second step. Construction of 1-dimensional partitions of $S^n$ use of the Borsuk-Ulam Theorem. We consider a regular positive measure $\mu$ on $S^n$. Let $A_1$, $A_3$ and $A_2 = A_1 \ast A_3 \subset S^n$ be subsets from 2.4. Let

$$
\mu(A_1) = \lambda \mu(A_3).
$$

Define $H_x^+ = \{y \in S^n : (y, x) \geq 0\}$ and $H_x^- = -H_x^+$. Note that $\hat{H}_x^+ = S^+_x$. We consider a map $\varphi : S^n \to \mathbb{R}^2$ such that

$$
\varphi(x) = (\mu(A_1 \cap H_x^+), \mu(A_3 \cap H_x^+)).
$$

By the Borsuk–Ulam Theorem there exists $x_0$ such that $\varphi(x_0) = \varphi(-x_0)$ which means that for $i = 1$ and 3

$$
\mu(A_i) = 2\mu(A_i \cap H_{x_0}^+)
$$

and as a consequence

$$
\lambda = \mu(A_1 \cap H_{x_0}^+)/\mu(A_3 \cap H_{x_0}^+).
$$

First, we fix one such $x_0$ and let $S^n_+ = \hat{H}_{x_0}^+$ and $A^+_i = S^+_x \cap A_i$. Next, we define a convex partition of $S^+_x$ by induction as follows. If $M \subset S^+_x$ is one of the convex sets from the preceding inductive step, then, by assumption

$$
\mu(A^+_i \cap M)/\mu(A^+_3 \cap M) = \lambda;
$$

next we construct a map $\varphi : S^n \to \mathbb{R}^2$ using $A^+_i \cap M$ as above (instead of $A_i$). The map $\varphi$ determines how to divide $M$ into two convex pieces $M^+$ and $M^-$ by a hyperplane in such a way that $\mu(A^+_i \cap M^+)/\mu(A^+_3 \cap M^+) = \lambda$ again. The same holds for the intersections $A^+_i \cap M^-$. We continue to refine our partitions, and obtain in the limit a partition $\alpha_{n-1}$ whose elements have a
strictly smaller dimension than \( n \) (using that \( \mu \) is positive). By Rohlin's theory [R] (compare 6.2) our measure \( \mu \) defines (up to factor) a (convex restriction) measure \( \mu_\alpha \) on almost every \( M_\alpha \in a_{n-1} \). The construction implies that \( \mu_\alpha(A_1 \cap M_\alpha)/\mu_\alpha(A_3 \cap M_\alpha) = \lambda \).

Then we may continue the same procedure with every \( M_\alpha \) if \( \dim M_\alpha \geq 2 \). The last condition is important when the Borsuk–Ulam Theorem is used.

Finally, we construct a partition \( \alpha \) of \( S^n_\alpha \) such that for almost every \( I \in \alpha \)

- \( i) \dim I = 1 \)
- \( ii) \mu_i(A_1 \cap I)/\mu_i(A_3 \cap I) = \lambda \)

where

- \( iii) \mu_i is a convex restriction measure of \( \mu \) induced by the partition \( \alpha \). \)

(The last property follows from 4).

- \( iv) \) If \( \mu_i(A_2 \cap I) \geq \alpha_i\mu_i(A_i \cap I) \) for almost every \( I \in \alpha \) then \( \mu(A_2) \geq \alpha_i\mu(A_i) \) (for \( i = 1, 3 \)).

8. Conclusion of the proof. We regard \( \alpha \) constructed in section 7 as a partition of \( \mathbb{R}^n \sim S^n_+ \) into straight intervals \( I \subset \mathbb{R}^n \). Then, by the property iii) of \( \mu_i \) (see 7) and Remark 6.2, the measure \( \mu_i \) on \( I \) is a convex restriction of \( \mu \) defined by a family \( \{K_i\}_{i \in \mathbb{N}} \) where \( K_i \) are convex sets having the \((n - 1)\)-dimensional symmetry in the direction perpendicular to \( I \) centered at \( I \). Therefore, \( \mu_i \) is defined by some family of shrinking convex sets \( \{T_i\}_{i \in \mathbb{N}} \) in \( \mathbb{R}^2 \). Let us first consider a case when \( A_1 \) and \( A_3 \) are convex sets. Then \( A_2 \cap I \) is a single interval for every \( I \in \alpha \). In this case (see Fig. 1 with \( A_2 \cap I \) for \([a, b]\)) we may replace \( T_i \) by a cone of rotation \( C_i \) around \( I \) (or in the degenerate case by a cylinder) such that the convex restriction measure \( \mu_c \) defined on \( I \) by \( \mu \) and the family \( C = \{C_i\}_{i \in \mathbb{N}} \) satisfies for \( i = 1 \) and 3

- \( i) \) \( \mu_c(A_2 \cap I)/\mu_c(A_1 \cap I) \leq \mu(A_2 \cap I)/\mu(A_1 \cap I) \).

Also it is clear that a family of symmetric cones centered on the straight line containing \( I \) is defined by a canonical partition (see 2.2). Therefore,

- \( ii) \mu_c = \mu_o \) up to a constant factor for some \( \sigma \) containing \( I \).
Let now $q = \text{dist}_\sigma (A_1, A_3)$. Then, for extremal points $a$ and $b$ on $I$ and any $\sigma, I \subset \sigma$, we have either

$$q \leq \lambda \mu_\sigma (\alpha_2) / \mu_\sigma (\alpha_1)$$

or

$$q \leq \mu_\sigma (\alpha_2) / \mu_\sigma (\alpha_3),$$

where $\alpha_1 = (-t, a)$, $\alpha_3 = [b, t)$ and $\alpha_2 = A_2 \cap I$ and the points $\pm t$ are joined by the arc $\sigma$.

So, 8 i) and 7 iv) imply Theorem 3 in this special case (we use again that $\mu_\sigma (A_1 \cap I) / \lambda = \mu_\sigma (A_3 \cap I)$).

In the general case we have again to prove that

$$\mu_\sigma (A_2 \cap I) \geq \frac{q}{\lambda} \min \{ \mu_\sigma (A_1 \cap I); \lambda \mu_\sigma (A_3 \cap I) \} \quad (8.1)$$

By an approximation argument we may assume that $A_2 \cap I$ contains a finite number of intervals. Let $A'_2$ be one of such intervals and $A_2'$ be the union of all intervals from $A_2 \cap I$ on the one side (say on the left) of $A'_2$. Call also $A_i'$ the part of $A_i \cap I$ on the same left part of $A'_2$ for $i = 1$ and 3. Let, for example, $A'_2$ be joined from the right by an interval from $A_1 \cap I$, called $A''_1$.

We will assume that

$$\mu_\sigma (A'_2) \geq \frac{q}{\lambda} \min \{ \mu_\sigma (A'_i); \lambda \mu_\sigma (A'_i) \} \quad (8.2)$$

(i.e. (8.1) is satisfied for the sets $A'_i$), and we prove (8.1) for the sets $A'_1' \cup A''_1$.

(We leave for the reader to check the starting point of such induction which will be finished after a finite number of steps and will prove (8.1)). Let $\alpha_3 = A'_2 \subset I$. We choose a maximal arc $\sigma \supset I$ (= a straight line under the projective isomorphism $S^n_+ \sim \mathbb{R}^n$) in the same way (see Fig. 1 with $\alpha_2$ for $[a, b]$) as we did earlier for the convex case where $\alpha_2 = A'_2$ played the role of the entire $A_2$. Let $\alpha_1$ be the right hand (corresponding to the right hand
side position of $A'_i$ with respect to $A'_2$) component of the complement $\sigma \setminus \alpha_2$
and let $\alpha_3$ be the left hand component. By the definition of the $\lambda$-dist. $q$

$$\mu_\sigma(A'_2) \geq \frac{q}{\lambda} \min \{\mu_\sigma(\alpha_1); \lambda \mu_\sigma(\alpha_3)\}.$$ 

By the construction of $\sigma$ (see again Fig. 1 and the explanation), we have

$$\mu_\sigma(A'_2) \geq \mu_\sigma(A'_3) \quad \text{and} \quad \mu_\sigma(A'_1) \leq \mu_\sigma(A'_3) \leq \mu_\sigma(\alpha_3),$$

$$\mu_\sigma(A'_3) \leq \mu_\sigma(A'_3) \leq \mu_\sigma(\alpha_3).$$

Therefore

$$\mu_\sigma(A'_2) \geq \frac{q}{\lambda} \{\min \mu_\sigma(A'_3); \lambda \mu_\sigma(A'_3)\}$$

(8.3)

Adding (8.2) and (8.3), we have

$$\mu_\sigma(A'_2 \cup A'_3) \geq \frac{q}{\lambda} \min \{\mu_\sigma(A'_1 \cup A'_3); \mu_\sigma(A'_1) + \lambda \mu_\sigma(A'_3);$$

$$\lambda \mu_\sigma(A'_3) + \mu_\sigma(A'_3); 2\lambda \mu_\sigma(A'_3)\} \geq \frac{q}{\lambda} \{\min \mu_\sigma(A'_1 \cup A'_3); \lambda \mu_\sigma(A'_3)\}.$$ 

So, as we wanted, (8.1) is proved for the sets $A'_i \cup A''_i$ (note that $A''_3$ is empty in our case before).

**Remark 9:** The 1-dimensional partition constructed in Section 7 is not necessarily a convex partition (we passed through intermediate dimensions). We indicate below how to modify this construction to obtain a 1-dimensional convex partition satisfying property ii) from 7.
Let $M \subset S^n$ be a convex n-dimensional body from the intermediate step of construction. Then $\mu(A_i^+ \cap M)/\mu(A_i^+ \cap M) = \lambda$. Take a triple of points $x_1, x_2, x_3 \subset M$ which maximize the determinant $|(x_i, x_j)_{i=1,2,3}|$. Define the map $\varphi: S^2 \to \mathbb{R}^2$ using $A_i^+ \cap M$ as above for $S^2 \subset \mathbb{R}^3 = \text{span} \{x_1, x_2, x_3\}$. Using this map, subdivide $M$ into two pieces $M^+$ and $M^-$ by a hyperplane such that the number $\lambda$ coincide with the above. Thus, one can obtain a 1-dimensional convex partition $\alpha$ satisfying ii) from 7.

10. The preceeding construction can be adjusted to various results related to the isoperimetric inequalities.

**Theorem 10.1.** Let $A$ and $B$ be closed subsets of $S^n$, $\mu$ a regular positive measure on $S^n$ and $f(x, y)$ any continuous function on $(S^n \times S^n) - \{(t, -t), t \in S^n\}$. There exists a maximal open arc $\sigma$ and disjoined sets $\alpha_i \subset \sigma$, $i = 1, 2, 3$ (where $\alpha_2 = \alpha_1 \cdot \alpha_3$) such that

$$\inf \{f(a, b): a \in \alpha_i, b \in \alpha_3\} \geq \inf \{f(x, y): x \in A, y \in B, x \neq y\} \quad (10.1)$$

and for $C = A \ast B$

$$\frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_1)} \leq \frac{\mu(C)}{\mu(A)}; \quad \frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_3)} \leq \frac{\mu(C)}{\mu(B)} \quad (10.2).$$

**Proof** of the theorem follows from Section 8.

**Remark 10.2.** An important case is when $f(x, y)$ is a distance function on $S^n$.

**Remark 10.3.** We say that $f(x, y)$ is monotone if for any maximal arc $\sigma$ and for every $x, y, z \subset \sigma$, $y \in (x, z) \subset \sigma$,

$$\max \{f(x, y), f(y, z)\} \leq f(x, z)$$

Now, if in the theorem a function $f(x, y)$ is monotone, then there exists a maximal open arc $\sigma$ (joined some points $t \in S^n$) and a partition $\sigma$ on three intervals: $\alpha_1 = (-t, a], \alpha_2 = (a, b), \alpha_3 = [b, t)$ such that $f(a, b) \geq \inf \{f(x, y): x \in A, y \in B, x \neq -y\}$ and (10.2) is satisfied for the above $\sigma$ and $\alpha_i \subset \sigma$. □
Theorem 10.4. Let \( A \) and \( B \) be closed subsets of \( S^n \), \( \mu \) a regular positive measure on \( S^n \) and \( C = A \ast B \).

i) There exist maximal open arcs \( \sigma_i \), intervals \( I_i \subset \sigma_i \) and convex restriction measures \( v_i \) on \( I_i \) (\( i = 1, 2 \)) such that

\[
v_1(C \cap I_i)/v_1(A \cap I_i) \geq \mu(C)/\mu(A), \quad v_1(C \cap I_i)/v_1(B \cap I_i) \geq \mu(C)/\mu(B)
\]

and

\[
v_2(C \cap I_i)/v_2(B \cap I_i) \leq \mu(C)/\mu(B), \quad v_2(C \cap I_i)/v_2(B \cap I_i) = \mu(C)/\mu(B).
\]

ii) If, in addition, \( \mu \) is a probability measure then there exists a maximal open arc \( \sigma \), an interval \( I \subset \sigma \) and a probability convex restriction measure \( v \) on \( I \) such that

\[
v(A \cap I) = \mu(A) \quad \text{and} \quad v(B \cap I) = \mu(B).
\]

11. Concentration of measure on the unit sphere of a uniformly convex normed spaced. Let a normed space \( X = (\mathbb{R}^{n+1}, \| \cdot \|) \) have for fixed \( \varepsilon > 0 \) the modulus of convexity at least \( \delta(\varepsilon) > 0 \). It means that for every two points \( x, y \) in \( X \), \( \| x \| = \| y \| = 1 \) and \( \| x - y \| \geq \varepsilon \),

\[
1 - \frac{\| x + y \|}{2} \geq \delta(\varepsilon).
\]

Also let \( \delta(\varepsilon) \) be a monotone (increasing) function.

11.1. Linear functionals. Take a vector \( f \in X^* \), \( \| f \| = 1 \). Define \( K = \{ x \in X, \| x \| \leq 1 \} \), \( S(X) = \partial K = \{ x \in X, \| x \| = 1 \} \) and \( K_\lambda = K \cap \{ x : f(x) = \lambda \} \). Clearly, \( \text{Vol}_n K_\lambda = \text{Vol}_n K_{-\lambda} \) and \( (K_\lambda + K_{-\lambda})/2 = A \subset K_0 \). By the Brunn–Minkowski inequality,

\[
\text{Vol}_n(K_\lambda + K_{-\lambda})^{1/n} \geq (\text{Vol}_n K_\lambda)^{1/n} + (\text{Vol}_n K_{-\lambda})^{1/n},
\]

and therefore \( \text{Vol}_n K_\lambda \leq \text{Vol}_n A \). Note also that for any \( x \in K_\lambda \) and \( y \in K_{-\lambda} \) we have \( \| x - y \| \geq 2\lambda \) and, consequently, \( \| x + y \|/2 \leq 1 - \delta(2\lambda) \). Therefore (see [GM2])
LEMMA. \( \text{Vol}_n K_x \leq (1 - \delta(2\lambda))^n \text{Vol}_n K_0. \)

So, we see that the volume of the levels of a linear functional are exponentially concentrated at the zero level. We continued this direction in [GM2] to show that (see [GM2], Theorem 3.2).

THEOREM. For every \( f \in X^* \), \( \|f\|^* = 1, \)

\[
\text{Vol}_{n+1} \{ x \in K, |f(x)| \geq \varepsilon \} \leq (n + 1)e^{-n(2\lambda)/2} \text{Vol}_{n+1} K.
\]

We will extend the results from 11.1 to an arbitrary 1-Lipschitz function on \( S(X) \).

11.2. A measure on \( S(X) \). A standard \((n + 1)\)-dimensional volume on \( \mathbb{R}^{n+1} \) induces the probability measure \( \mu \) on \( S(X) \): for any Borel set \( A \subset S(X) \),

\[
\mu(A) \overset{\text{def}}{=} \frac{\text{Vol}_{n+1} \{ \cup tA, 0 \leq t \leq 1 \}}{\text{Vol} K}. \quad (11.1)
\]

To apply Theorem 3 to this measure \( \mu \) on \( S(X) \) we have to estimate \( \mu_\sigma \) for any maximal arc \( \sigma \). However our estimate will also work for any convex restriction measure, and the application of Theorem 10.4 will be easier.

Note that Theorem 3 and Theorem 10.4 concern measures on \( S^n \) in the projective sense and are applicable for \( \mu \) on \( S(X) \) (the reader who feels uncomfortable at this point, may choose any euclidean sphere \( S^n \) and, using the radial projection of \( S(X) \) to \( S^n \), transport all constructions and results from \( S(X) \) to \( S^n \) and vice versa).

Fix \( z \in S(X) \). Let \( f \in X^* \), \( \|f\| = 1 \), be the support functional at \( z \), i.e. \( f(z) = 1 \). Consider \( \text{Ker} f \cap S(X) = S_0 \). Take any \( x \in S_0 \). We study \( \sigma \) which joins \( \pm z \) and pass through \( x \) (i.e. a “half” of the two-dimensional sphere \( S(X) \cap \text{span} \{ x; z \} \)). Choose a parametrization of the arc \( \sigma \). Take \( x_\theta \in \theta \), such that \( \theta = \varrho(x_\theta, -z) = \int_{-z}^{x_\theta} \|dx_t\| \), i.e., \( \theta \) is the length of the arc \((-z, x_\theta) \). It is known [S] that \( \varrho = \varrho(z, -z) \) changes between \( 3 \leq \alpha \leq 4 \) (\( \alpha \) is the “\( \pi \)” of a normed space \( E = \text{span} \{ z; x \} \)) and for \( C < t, \theta < a \)

\[
\|x_t - x_\theta\| \leq \varrho(x_t, x_\theta) \leq 2\|x_t - x_\theta\| \quad (11.2)
\]

So for any \( t \in (0, a) \) we have the unique \( x_t \in \sigma \).
PROPOSITION 11.3. Let $\delta(\omega)$ be as in 11. If $v$ is a convex restriction probability measure on $\sigma$ then there exists $t_0 \in [0, a]$ such that for any $\theta > 0$

$$
\begin{align*}
\frac{v[-z, x_{t_0-\theta}]}{v[x_{t_0}, x_{t_0}]} &\leq \frac{[1 - \delta(\theta/4)]^{n-1}}{1 - [1 - \delta(\theta/4)]^{n-1}} v[x_{t_0}, x_{t_0}] \quad \text{(if $t_0 > \theta$)} \\
\text{and} \\
\frac{v[x_{t_0+\theta}, z]}{v[x_{t_0}, x_{t_0}]} &\leq \frac{[1 - \delta(\theta/4)]^{n-1}}{1 - [1 - \delta(\theta/4)]^{n-1}} v[x_{t_0}, x_{t_0}] \quad \text{(if $t_0 + \theta < a$)}
\end{align*}
$$

(11.3)

where one may choose $t_0$ as the (unique) maximum point of the density $f_\omega(t)$ of the measure $v$. The function $\psi(t) = [f_\omega(t)]^{1/(n-1)}$ satisfies the following “weak concavity” condition: there exists a number $a, 0 < a < 1/2$, such that for any $0 < t_1 < t_2 < a$ and $\theta = t_2 - t_1$

$$
\frac{\psi(t_1) + \psi(t_2)}{2} \leq \max_{\tau\in(t_1+2a,t_2-a)} \psi(\tau)
$$

It follows from (11.3) that (if $\theta < t_0 < a - \theta$)

$$
v[(z, x_{t_0-\theta}) \cup (x_{t_0+\theta}, z)] \leq (1 - \delta(\theta/4))^{n-1} \approx e^{-\delta(\theta/4)(n-1)}
$$

(11.4)

Proof. We use an argument similar to that of 11.1 where a concentration property of linear functionals was proved. Define $\Delta v(x_i)$ to be an infinitesimal $(n-1)$-dimensional volume of infinitesimal convex neighborhood $\Delta_i$ of $x_i$ in $S(X) \cap \{x : f(x) = f(x_i)\}$ which induces the density $f_\omega(t)$ of the probability convex restriction measure $v$ on $\sigma$ at the point $x_i$. Then, by the Brunn–Minkowski inequality, for any $0 < t_1 < t_2 < a$

$$
\text{Vol} \left( \frac{\Delta_{t_1} + \Delta_{t_2}}{2} \right)^{1/(n-1)} \geq \frac{[\Delta v(x_i)]^{1/(n-1)} + [\Delta v(x_i)]^{1/(n-1)}}{2} \\
\geq \min_{i=1,2} \{\Delta v(x_i)\}^{1/(n-1)}.
$$

Also, for any $y_i \in \Delta_{t_1}$ and $y_{i_2} \in \Delta_{t_2}$, we have

$$
\frac{y_{t_1} + y_{t_2}}{2} = \lambda y_{t_3}
$$

(11.5)
for some \(0 < \lambda < 1 - \delta\) (\(\|x_{t_2} - x_{t_1}\|\)) where \(y_{t_3} \in S(X)\) and \(y_{t_3}\) belongs to the arc joining \(y_{t_1}\) and \(y_{t_2}\), i.e., \(y_{t_3}\) belongs to any convex neighborhood of the arc \(\{x_{t_1}, x_{t_2}\}\) which is contained \(\Delta_{t_1}\) and \(\Delta_{t_2}\). Therefore,

\[
\text{Vol} \left( \frac{\Delta_{t_1} + \Delta_{t_2}}{2} \right) \leq \left[ 1 - \delta(\|x_{t_2} - x_{t_1}\|) \right]^{n-1} \max_{t_1 < t < t_2} \text{Vol} \Delta_t
\]

(11.6)

(we use 11.2: \(\|x_{t_2} - x_{t_1}\| \geq (t_2 - t_1)/2\)). So we have the following inequalities for the density function \(f(t)\):

\[
\frac{[f_{i_1}(t_2)]^{1/(n-1)} + [f_{i_1}(t_1)]^{1/(n-1)}}{2} \leq \left( 1 - \delta \left( \frac{t_2 - t_1}{2} \right) \right)^{n-1} \max_{t_1 < \tau < t_2} f_{i_1}(\tau)^{1/(n-1)}
\]

(11.7')

and

\[
\min_{i=1,2} f_{i_1}(t_i) < \left( 1 - \delta \left( \frac{t_2 - t_1}{2} \right) \right)^{n-1} \max_{t_1 < \tau < t_2} f_{i_1}(\tau).
\]

(11.7'')

It follows from (11.7'') that \(f(t)\) has no local minima.

Note that for a euclidean space the point \(y_{t_3}\) in (11.5) is in the middle of the arc joining \(y_{t_1}\) and \(y_{t_2}\) and \(t_3 = (t_2 + t_1)/2\). Also the (Banach–Mazur) distance between an arbitrary two-dimensional normed space and the euclidean two-dimensional space is at most \(\sqrt{2}\). Therefore, there exists a numerical constant \(\alpha\), \(0 < \alpha < 1/2\), so that \(t_3 \in (t_1 + \alpha(t_2 - t_1), t_2 - \alpha(t_2 - t_1))\). By this reason, we may take the maximum in (11.6), (11.7') and (11.7'') in the interval \((t_1 + \alpha(t_2 - t_1), t_2 - \alpha(t_2 - t_1))\). It follows that \([f_{i_1}(t)]^{1/(n-1)}\) satisfies the “weak concavity” condition. Let \(f_{i_1}\) attain the maximum at \(t_0 \in [0, \alpha]\). If \(t_0 < \alpha\), then for every \(0 < t_2 < t_0\)

\[
f_{i_1}(t_2) \leq \left( 1 - \delta \left( \frac{t_2 - t_0}{2} \right) \right)^{n-1} f_{i_1}(t_0)
\]

(we have a similar inequality for every \(0 < t_1 < t_0\) if \(t_0 > 0\)).

By monotonicity of \(f_{i_1}(t)\) on the intervals \([t_0, \alpha]\) and \([0, t_0]\), we also have

\[
f_{i_1}(t + \theta) \leq (1 - \delta(\theta/2))^{n-1} f_{i_1}(t)
\]

(11.8)
for \( a > t + \theta > t > t_0 \) (if, of course, \( t_0 < a \)) and, similarly, if \( t_0 > 0 \) and \( 0 < t - \theta < t < t_0 \)

\[
f_\varepsilon(t - \theta) < (1 - \delta(\theta/2)^{n-1} f_\varepsilon(t).
\]

Now, integrate (11.8) from \( t_0 + \theta \) to \( a - \theta \) (assuming \( t_0 < a - 2\theta \)) and obtain (to simplify notations we write \( v[t, \tau] \) instead of \( v[x, x_{\tau}] \))

\[
X(\theta) \overset{\text{def}}{=} v[t_0 + 2\theta; a] \leq (1 - \delta(\theta/2))^{n-1} v[t_0 + \theta; a - \theta]
\]

\[
\leq (1 - \delta(\theta/2))^{n-1} \{v[t_0 + \theta; t_0 + 2\theta] + \chi(\theta)\}
\]

Therefore,

\[
X(\theta) \leq \frac{[1 - \delta(\theta/2)]^{n-1}}{1 - [1 - \delta(\theta/2)]^{n-1}} v[t_0 + \theta; t_0 + 2\theta]
\]

for \( \theta > 0 \). Similarly we deal with the comparison of \( v[0; t_0 - 2\theta] \) and \( v[t_0 - 2\theta; t_0 - \theta] \). Then the statement of the proposition follows.

\( \square \)

**REMARK:** For the canonical measure \( \mu_\sigma \) (i.e., for a convex restriction of \( \mu \) corresponding to the canonical partition) the maximum \( t_0 \) of \( f_{\mu_\sigma} \) is strictly inside the interval \( 0 < t_0 < a \), i.e. \( x_{t_0} \in \sigma \setminus \{ \pm z \} \).

**Corollary 11.4:** Let \( I_\varepsilon(x_{t_0}) = \{ x \in \sigma : \| x - x_{t_0} \| \leq 2\varepsilon \} \)

Then

\[
v(\sigma - I_\varepsilon) \leq \left( 1 - \delta \left( \frac{\varepsilon}{4} \right) \right)^{n-1} \approx e^{-\delta(\varepsilon/(4n-1)).}
\]

**Corollary 11.5:** Let an arc \( [a, b] \subset \sigma \), where \( \sigma \) is a maximal arc joined points \( \pm z \), and \( \| a - b \| \geq \varepsilon > 0 \). Then there exists a number \( \lambda(\varepsilon) > 0 \) depending only on \( \delta_x(\varepsilon) > 0 \) such that for any convex measure \( v \) induced by \( \mu \) either

\[
\frac{v(-z, a)}{v(a, b)} \leq e^{-\lambda(\varepsilon)n}
\]

or

\[
\frac{v(b, z)}{v(a, b)} \leq e^{-\lambda(\varepsilon)n}.
\]
\textit{Proof.} Use (11.4).

\textbf{Remark 11.6:} Proposition 11.3 and Corollaries 11.4 and 11.5 remain valid when the following changes are made in these results: consider any closed interval $I = [\alpha, \beta] \subset \sigma$ instead of $\sigma$ and any convex restriction probability measure $\nu$ on $I$ instead of on $\sigma$. In this case, the maximum point $x_{t_0} \in I$. We also have to replace $-z$ by $\alpha$ and $z$ by $\beta$.

\textbf{Theorem 11.7:} Let $\delta_X(\varepsilon)$ be the modulus of convexity of a normed $(n + 1)$-dimensional space $X$ and $\mu$ be the probability measure (11.1) on $S(X)$. Let $a(\varepsilon) = \delta((\varepsilon/8) - \theta_n)$ and $\delta(\theta_n/4) = 1 - (1/2)^{1/(n-1)} \simeq \ln 2/(n-1)$. Then, for every Borel set $A \subset S(X)$, $\mu(A) \geq 1/2$ and every $\varepsilon > 0$

$$
\mu(A_{\varepsilon}) \geq 1 - e^{-a(\varepsilon)n}
$$

where $A_{\varepsilon} = \{ x \in S(X); \, g(x, A_{\varepsilon}) \leq \varepsilon \}$ and $g(x, y) = \| x - y \|$.

\textit{Proof.} Note that the principal part of the Theorem is the existence of a number $a(\varepsilon) > 0$ depending only on $\delta_X(\varepsilon) > 0$ such that $\mu(A_{\varepsilon}) \geq 1 - e^{-a(\varepsilon)n}$. This follows straightforwardly from Theorem 3 and Corollary 11.5. Indeed, use Theorem 3 with $A$ instead of $A_1$ and $B = (A_{\varepsilon})^c$ instead of $A_3$. By Corollary 11.5 we have that one of the numbers $\mu_n(\alpha_2/\alpha_1)$ or $\mu_n(\alpha_2/\alpha_3)$ is at least $e^{\lambda n}$ (i.e. exponentially large). Therefore, $\lambda$ must be at most $2e^{-\lambda n}$ (because $\mu(A_2) \leq 1$). However, to compute $a(\varepsilon)$ we use Theorem 10.4.ii). By this theorem there exists an interval $I \subset \sigma$ and a probability convex restriction measure $\nu$ on $I$ such that

$$
\nu(A \cap I) = \mu(A) \geq 1/2, \quad \nu(B \cap I) = \mu(B),
$$

where, as above, $B = (A_{\varepsilon})^c$. Let $x_{t_0}$ be the point on $I$ where the maximum of the density function of $\nu$ is attained. Take $\theta_0$ such that $\nu(I_{t_0}(x_{t_0})) = \nu\{ x \in I; \| x - x_{t_0} \| \leq 2\theta_0 \} = 1/2$. Then, by Corollary 11.4 and Remark 11.6

$$
[1 - \delta(\theta_0)]^{n-1} \geq 1/2.
$$

It means that

$$
\delta(\theta_0) \leq 1 - (1/2)^{1/(n-1)} \simeq \frac{\ln 2}{n - 1}.
$$
Let $\theta_n$ be such that $\delta(\theta_n) = 1 - (1/2)^{1/(n-1)} \approx \ln 2/(n - 1)$. Then $v[I_{\theta_n}(x_{i_0})] \geq 1/2$. Therefore, if $v(A \cap I) \geq 1/2$, then there exists $x_i \in A \cap I$ and $\|x_i - x_{i_0}\| \leq 2\theta_n$. Now, take $\varepsilon$-neighborhood of $\{x_i\}$ and let $\varepsilon = 2\theta + 4\theta_n$. Then $\{x_i\}_\varepsilon \supset \{x_{i_0}\}_2$ and $B \cap \{x_i\}_\varepsilon = \emptyset$. Therefore, by Remark 11.6 applied to Corollary 11.4 we have

$$\mu(B) = v(B \cap I) \leq v(I - I_{\theta_n}(x_{i_0})) \leq \left(1 - \delta\left(\frac{\theta}{4}\right)\right)^{n-1} \approx e^{-\delta(\theta/4)(n-1)}$$

$$= e^{-(\varepsilon - \theta_n)(n-1)}.$$ 

**Remark 11.8:** Note that Theorem 11.7 shows that any family of finite dimensional spaces $\{X_n, \text{dim } X_n \to \infty\}$, such that $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$, is a Levy family (see definition and a number of related examples in [GM1], [AM]).

Let $f(x)$ be a continuous function on $S(X)$, dim $X = n + 1$ where $S(X)$, dim $X = n + 1$ where $S(X)$ is the unit sphere of $X$. We call $L_f$ the *median* of $f(x)$ (or Levy mean) if

$$\mu\{x \in S(X): f(x) \geq L_f\} \geq 1/2 \quad \text{and} \quad \mu\{x \in S(X): f \leq L_f\} \geq 1/2.$$ 

Let $\omega(\varepsilon)$ be the modulus of continuity of the function $f(x)$.

It follows from Theorem 11.7 that

$$\mu\{x \in S(X): |f(x) - L_f| \leq \omega(\varepsilon)\} \geq 1 - 2e^{-a(\varepsilon)n}.$$ (11.9)

### 12. Application to a Lipschitz embedding problem.

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a uniformly convex space with the modulus of convexity $\delta(\varepsilon) > 0$ (for $\varepsilon > 0$). Let $S(X) = \{x \in \|x\| = 1\}$ and, similarly, $S(l_\infty^n)$ be the unit sphere of the space $l_\infty^n$ of dimension $N$.

**Theorem:** Fix $1 > \varepsilon > 0$. If $N < \frac{1}{2} e^{a(\varepsilon)n}$ where $a(\varepsilon) > 0$ was defined in Theorem 11.7, then there exists no $1$-Lipschitz antipodal (i.e., $\varphi(-x) = -\varphi(x)$) map

$$\varphi: S(X) \to S(l_\infty^n).$$
Proof. Assume \( \varphi \) exists. Let \( f_i(x), i = 1, \ldots, N \), be the \( i \)-th coordinate in \( l_\infty^N \) of \( \varphi(x) \), i.e., \( \varphi(x) = (f_i(x))_{i=1}^N \in S(l_\infty^N) \). Then,

i) \( \max_i |f_i(x)| = 1 \) for \( x \in S(X) \),

ii) \( f_i(-x) = -f_i(x) \) for any \( i = 1, \ldots, N \) and \( x \in S(X) \)

iii) \( |f_i(x) - f_i(y)| \leq \|x - y\|, i = 1, \ldots, N; x, y \in S(X) \) (because \( \|\varphi(x) - \varphi(y)\| \leq \|x - y\| \) implies \( \max_i |f_i(x) - f_i(y)| \leq \|x - y\| \)).

Define \( A_i = \{x \in S(X): |f_i(x)| \leq \epsilon\} \). By ii), 0 is the median of \( f_i \) for every \( i \in \{1, \ldots, N\} \) (see 11.8 for the definition). Also, by iii), and (11.9),

\[
\mu(A_i) \geq 1 - 2e^{-a\epsilon n}.
\]

Then,

\[
\mu \left( \bigcap_{i=1}^N A_i \right) \geq 1 - 2Ne^{-a\epsilon n}
\]

and, in the case of \( N < \frac{1}{2} e^{a\epsilon n} \), there exists

\[
x \in \bigcap_{i=1}^N A_i.
\]

Hence, \( |f_i(x)| \leq \epsilon \) for every \( i = 1, \ldots, N \) which contradicts i).

Note that in the case of a linear embedding \( \varphi: X \to l_\infty^N, \dim X = n \), the above Theorem was proved by Pisier [P].

Appendix

Let \( P: \mathbb{R}^n \to \mathbb{R}^{n-k} \) be a linear projection, and let \( K \) be a convex subset in \( \mathbb{R}^n \). We study the \( k \)-dimensional volume of the intersections \( K \cap P^{-1}(x) \) for \( x \in \mathbb{R}^{n-k} \).

**BRUNN'S THEOREM:** Let \( \varphi(x) = \text{Vol}_k K \cap P^{-1}(x) \). Then the function \( \varphi^{1/k} \) is concave on the image \( P(K) \subset \mathbb{R}^{n-k} \).

We will prove a more general statement.

**DEFINITION:** We say that a function \( f: K \to \mathbb{R} \) is \( \alpha \)-concave (\( \alpha > 0 \)) if

i) \( K \) is a convex set in \( \mathbb{R}^n \),

ii) \( f(x) \geq 0 \) for \( x \in K \)
iii) \( f^{\frac{1}{\alpha}} \) is concave on \( K \), i.e.

\[
f^{\frac{1}{\alpha}} \left( \frac{x_1 + x_2}{2} \right) \geq \frac{f^{\frac{1}{\alpha}}(x_1) + f^{\frac{1}{\alpha}}(x_2)}{2}
\]

for any \( x_1, x_2 \in K \).

**Lemma 1:** Let \( f \) be \( \alpha \)-concave, \( g \) be \( \beta \)-concave and let \( \text{Dom } f = \text{Dom } g = K \subset \mathbb{R}^n \). Then the product \( fg \) is \((\alpha + \beta)\)-concave.

**Proof.** Let \( x_1, x_2 \in K \). Then

\[
\frac{[f(x_1)g(x_1)]^{1/(\alpha + \beta)} + [f(x_2)g(x_2)]^{1/(\alpha + \beta)}}{2} \leq \left[ \frac{f(x_1)^{\frac{1}{\alpha}} + f(x_2)^{\frac{1}{\alpha}}}{2} \right]^{\alpha/(\alpha + \beta)} \times \left[ \frac{g(x_1)^{\frac{1}{\beta}} + g(x_2)^{\frac{1}{\beta}}}{2} \right]^{\beta/(\alpha + \beta)}
\]

(by the Hölder inequality for \( p = (\alpha + \beta)/\alpha \) and \( q = (\alpha + \beta)/\beta \))

\[
\leq \left[ f \left( \frac{x_1 + x_2}{2} \right) \right]^{1/(\alpha + \beta)} \left[ g \left( \frac{x_1 + x_2}{2} \right) \right]^{1/(\alpha + \beta)}
\]

(by \( \alpha \)- and \( \beta \)-concavity of the functions \( f \) and \( g \)).

Consider a linear projection \( P: \mathbb{R}^m \overset{\text{onto}}{\longrightarrow} \mathbb{R}^{m-1} \). Then \( \mathbb{R}^m = \mathbb{R}^{m-1} + \text{Ker } P \). Let \( K \) be a convex set \( K \subset \mathbb{R}^m \). We have for every \( x \in K \): \( x = y + t \), where \( y \in PK \subset \mathbb{R}^{m-1} \) and \( t \in I_y = \{ x \in K: Px = y \} \) is an interval in \( y + \text{Ker } P \). Let \( f \) be a function, \( \text{Dom } f = K \). We will write \( f(x) = f(y; t) \) where \( x = y + t, y \in PK \) and \( t \in I_y \). Define a projection \( Pf \) of a function \( f \) as the function with \( \text{Dom } Pf = P \text{ Dom } f (= PK) \) and

\[
(Pf)(y) \overset{\text{def}}{=} \int_{t \in I_y} f(y; t) \, dt.
\]

**Lemma 2:** If \( f \) is \( \alpha \)-concave, then \( Pf \) is \((1 + \alpha)\)-concave.
Proof. It is sufficient, by the definition of concavity, to consider the case
\( m = 2 \). Then \( PK \) is an interval. Let \( x_1, x_2 \in PK \) and \( x = (x_1 + x_2)/2 \) and let \( I_{x_i} = [a_i, b_i], i = 1, 2 \). We may assume that
\[
I_x = \left[ \frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2} \right].
\]
Let \( c_i \in I_{x_i} (i = 1, 2) \) satisfy
\[
\int_{a_i}^{b_i} f(t, x_i)dt = 2 \int_{a_i}^{c_i} f(t, x_i)dt = 2 \int_{a_i}^{b_i} f(t, x_i)dt.
\]
Let \( K_{up} \) be the convex hull of the two intervals (for \( i = 1, 2 \)) \([(c_i; x_i), (b_i; x_i)] \subset K \) and \( K_{down} \) be the convex hull of the intervals (again for \( i = 1, 2 \)) \([(a_i; x_i), (c_i; x_i)] \subset K \). Let \( f_{i} = f|_{K_{up}} \) and \( f_{2} = f|_{K_{down}} \). It is easy to check (we leave it to the reader) that if \( \varphi_i = Pf_i (i = 1, 2) \) are \((1 + \alpha)\)-concave, then the same is true for the original projection \( Pf \). Therefore, our problem is reduced to the \((1 + \alpha)\)-concavity of the functions \( \varphi_1 \) and \( \varphi_2 \). We continue this procedure, and build the partitions of the intervals \([a_i, b_i] \) for \( i = 1, 2 \):
\[
t_0,i = a_i < t_{1,i} < \cdots < t_{n-1,i} < b_i = t_{n,i},
\]
such that
\[
\int_{a_i}^{b_i} f(t, x_i)dt = \sum_{p=1}^{n} \int_{t_{p-1,i}}^{t_{i}} f(t, x_i)dt
\]
for every \( p = 1, \ldots, n \) and \( i = 1, 2 \). Let \( K_p \subset K \) be a trapez which is the convex hull of the two intervals \([(t_{p-1,i}; x_i), (t_{p,i}; x_i)] (i = 1, 2) \). Then, by the above remark, one only needs to check that the functions \( Pf|_{K_p} \) are \((1 + \alpha)\)-concave for every \( p = 1, \ldots, n \). By the obvious approximation argument, the problem is reduced now to the following observation:

Let \( t_i \in I_{x_i} \) and \( \Delta_i > 0 \) \((i = 1, 2)\); for \( x = \lambda x_1 + (1 - \lambda)x_2 \). Set \( t(x) = \lambda t_1 + (1 - \lambda)t_2 \) and \( \Delta(x) = \lambda \Delta_1 + (1 - \lambda)\Delta_2 \). Then, by Lemma 1, the function \( f(t(x), x) \). \( \Delta(x) \) is \((1 + \alpha)\)-concave because \( f \) is \( \alpha \)-concave and the linear function \( \Delta(x) \) is \( 1 \)-concave.

Now, we prove Brunn’s theorem as follows. We start from the characteristic function \( \chi_K(x) \) of the set \( K \) which is \( \alpha \)-concave for energy \( \alpha > 0 \). After \( k \) consequent projections we come to the function \( \varphi(x) \) on \( \tilde{K} \) which is, by Lemma 2, \((k + \alpha)\)-concave for every \( \alpha > 0 \) and, therefore, \( k \)-concave.

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References


