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§1. Introduction

This paper is the first of a series devoted to the explication of the general theory of metaplectic forms [6] for number theory. The objective of this paper is the definition and study of certain Dirichlet series which give expression to this arithmetic content. The coefficients of the simplest class of these Dirichlet series are Gauss sums. The residues of these Dirichlet series at the principal pole give the coefficients of the next class, and so on.

One of the fundamental concepts of the theory of exceptional metaplectic forms is the ‘distribution’ c introduced in [6], Theorem II.2.2. This is associated, over an A-field k containing the nth roots of unity, to a split torus of the n-fold metaplectic cover of GLr. More generally one can associate to a split torus of an n-fold cover of GLr×⋯×GLr (⊂ GLr1+⋯+rs) an analogous distribution. We shall construct Dirichlet series associated with the n-fold cover of GLr1×GLr2 which are essentially Fourier coefficients of the Eisenstein series obtained by inducing an exceptional automorphic representation of GLr1×GLr2 (over kA) to an n-fold cover of GLr1+r2. The coefficients of these Dirichlet series involve the c-distribution for GLr1×GLr2, or, what turns out to be the same, the product of the c-distributions for n-fold covers of GLr1 and GLr2. The residues at the principal poles are the given by the values of the c-distribution for an n-fold cover of GLr1×GLr2.

The precise statement is given in Theorem 5.1. These results should be understood in conjunction with the formal properties of the c-distribution given in §3.

There are two weaknesses in Theorem 5.1 as it is formulated here. Firstly, rather than being defined by a series the ‘Dirichlet series’ is given as a rather inconvenient limit. This is merely a technical point since the convergence of the Dirichlet series can be shown to be convergent; this requires estimates on the values of the c-distributions which will be the subject of the next paper in this series. The other weakness is that Theorem 5.1 still has to be ‘deciphered’ in any special case. The reason for this is that the function
\(\tau\) (or \(\tilde{\tau}\)) which can be computed by a local analysis at a finite number of places does not have a simple form. This problem does not arise in the case when \(k\) is a function field and this case will be discussed in detail in a later paper.

The notations of this paper have been chosen to agree with those of [6] and although most are recalled here there are some that are not.

\section*{§2. Eisenstein series}

Let \(k\) be a global field such that \(\text{Card} \, \mu_k(k) = n\). Let \(G\) be an algebraic group of the form \(GL_{r_1} \times \cdots \times GL_{r_s}\) and let \(G_k = GL_{r_1}(k) \times \cdots \times GL_{r_s}(k)\), \(G_\mathbb{A} = GL_{r_1}(\mathbb{A}) \times \cdots \times GL_{r_s}(\mathbb{A})\). As in [6] §0.2 we form a metaplectic extension

\[1 \rightarrow \mu_n(k) \overset{i}{\rightarrow} \tilde{G}_\mathbb{A} \overset{p}{\rightarrow} G_\mathbb{A} \rightarrow 1;\]

it would suffice to embed \(G\) in \(GL_{r_1 + \cdots + r_s}\) by the obvious representation and to restrict an extension as described in [6] §0.2. Let \(s: G_k \rightarrow \tilde{G}_\mathbb{A}\) be the lift of the standard embedding constructed as in [6] §0.2.

Let \(H\) be the diagonal subgroup of \(G\); let \(Z^0(G)\) be the centre of \(G\) and let \(\tilde{Z}_\mathbb{A}(G)\) be the centre of \(\tilde{G}_\mathbb{A}\). Let \(H_n\) be the subgroup of those elements which are \(n\)th powers. Let \(\tilde{H}_n = p^{-1}(H_n(k_\mathbb{A}))\), \(\tilde{H}_\mathbb{A} = p^{-1}(H(k_\mathbb{A}))\). Let \(N\) be the unipotent subgroup with zero entries under the diagonal; then \(N_\mathbb{A} = N(k_\mathbb{A})\) lifts to a subgroup \(N^*_\mathbb{A}\) of \(\tilde{G}_\mathbb{A}\) and this lift is unique. Let \(B = HN, B_n = H_nN, B_\mathbb{A} = \tilde{H}_\mathbb{A}N_\mathbb{A}, \tilde{B}_n = \tilde{H}_nN_\mathbb{A}\).

For a place \(v\) of \(k\) let \(\tilde{G}_v\) be the corresponding factor of \(\tilde{G}_\mathbb{A}\). If \(|n|_v = 1\) let \(K_v\) be the standard maximal compact subgroup of \(G_v = p(G_v)\); let \(K_v^*\) be the standard lift of \(K_v\) to \(\tilde{G}_v\) ([6] §0.1).

Let \(\Phi\) be the set of roots of \(H\) relative to \(G\) and let \(\Phi^+\) be that set of positive roots defined by \(N\). Let \(\mu_\Phi\) be the modulus quasicharacter of \(\tilde{H}_\mathbb{A}\) defined by \(\frac{1}{2} \Sigma_{\omega \in \Phi^+} \alpha (\text{cf. [6] §1.1})\).

Let \(\varepsilon\) be an injective character of \(\mu_n(k)\). Let \(\Omega_\varepsilon(G)\) be the set of quas-characters \(\omega\) of \(\tilde{H}_{n,\mathbb{A}}\) trivial on \(\tilde{H}_{n,\mathbb{A}} \cap s(G_\mathbb{A})\) and such that \(\omega \cdot i = \varepsilon\). Let \(X_\varepsilon(G)\) be the set of quas-characters \(\chi\) of \(\tilde{Z}_\mathbb{A}(G) \cap \tilde{H}_{n,\mathbb{A}}\), trivial on \(\tilde{Z}_\mathbb{A}(G) \cap \tilde{H}_{n,\mathbb{A}} \cap s(G_\mathbb{A})\) and such that \(\chi \cdot i = \varepsilon\). Let \(\varrho: \Omega_\varepsilon(G) \rightarrow X_\varepsilon(G)\) be the restriction map, and for \(\chi \in X_\varepsilon(G)\) let \(\Omega_{\chi}(G) = \varrho^{-1}(\chi)\). As usual \(\Omega_\varepsilon(G), X_\varepsilon(G)\) and \(\Omega_{\chi}(G)\) have structures of complex manifolds, and \(\varrho\) is analytic.

We now shall define a holomorphic vector bundle \(F\) over \(\Omega_\varepsilon(G)\). The fibre of \(F\) at \(\omega\), denoted by \(F(\omega)\), is the space of functions \(f: \tilde{G}_\mathbb{A} \rightarrow \mathbb{C}\) such that

\[f(hng) = \omega \mu(h) \cdot f(g)\]
if \( h \in \widetilde{H}_{n,A}, n \in N_A^*, g \in \widetilde{G}_A \). Moreover \( f \) is to be a finite linear combination of 'primitive' functions \( \otimes f_v \) where \( f_v \) is smooth (resp. locally constant) if \( v \) is Archimedean (resp. non-Archimedean). For almost all \( v \) such that \( |n|_v = 1 \) we demand finally that \( f_v = f_v^0 \) where \( f_v^0 \) is the function defined by

\[
\begin{align*}
f_v^0(g) &= 0 & g \in \widetilde{G}_v - \widetilde{B}_{n,v} \cdot K_v^* \\
f_v^0(k) &= 1 & k \in K_v^*.
\end{align*}
\]

Note that \( F(\omega) \) is a \( \widetilde{H}_A \times \widetilde{G}_A \)-module by

\[
((\eta, \gamma)f)(g) := f(\eta^{-1} \cdot g \cdot \gamma) \quad (\eta \in \widetilde{H}_A, \gamma, g \in \widetilde{G}_A).
\]

For \( \alpha \in \Phi \) let \( H_\alpha : GL_1 \to GL \), be the corresponding coroot. As in [6] §1.2 we obtain a homomorphism

\[
H_\alpha^\kappa : k_A^\times \to \widetilde{G}_A
\]

such that

\[
p(H_\alpha^\kappa(x)) = H_\alpha(x^\alpha).
\]

For \( \omega \in \Omega_\kappa(G) \) let \( \omega_\alpha^\kappa = \omega \cdot H_\alpha^\kappa \). Recall that one says that \( \omega \) is dominant if \( \omega_\alpha^\kappa \) is of the form \( \| \sigma_\alpha^{(\omega)} \|_A \) where \( \sigma_\alpha^{(\omega)} > 0 \) for \( \alpha \in \Phi_+ \) and \( \| \|_A \) denotes the idele norm.

Let \( H_k^* = s(G_k) \cap \widetilde{H}_A \).

We can now define a \( H_k^* \times s(G_k) \)-invariant linear form \( E(\omega) : F(\omega) \to \mathbb{C} \) when \( \omega \mu^{-1} \) is dominant by

\[
\langle E(\omega), f \rangle := \sum_{\gamma \in \mathbb{G}_A \backslash \mathbb{G}} f(\gamma)
\]

where \( G_k^* = s(G_k), B_{n,k}^* = G_k^* \cap \widetilde{B}_{n,A} \).

We can rephrase this by extending \( \omega \) to \( H_k^* \widetilde{H}_{n,A} \) with \( \omega|H_k^* = 1 \) so that for \( \eta \in H_k^* \widetilde{H}_{n,A} \)

\[
\langle E(\omega), (\eta, 1)f \rangle = \omega \mu_A(\eta)^{-1} \langle E(\omega), f \rangle.
\]

The linear form \( E \) which is defined on an open subset of \( \Omega_\kappa(G) \) has an analytic continuation as a meromorphic function to all of \( \Omega_\kappa(G) \), and its analytic properties can be described ([4], [5]).
Let us call $m \in \Omega(G)$ exceptional if for every positive simple root $\alpha$ one has $\omega_\alpha^m = \|\|_A$. When $\omega$ is exceptional the representation $F(\omega)$ becomes reducible and has a unique irreducible quotient which we denote by $V_0(\omega)$. There is a residue $\Theta(\omega)$ of $E$ at $\omega$ which is a $H^*_k \times G^*_k$-invariant linear form $\Theta(\omega) : V_0(\omega) \rightarrow \mathbb{C}$ with analogous properties to $E$. Details, in the case of $GL_r$ are given in [6] §II.1 and the general case involves no new ideas.

We shall next use the $V_0(\omega)$ to construct further Eisenstein series. Let $r = r_1 + r_2 + \ldots + r_s$ and let $\Pi$ be the parabolic subgroup of $GL_r$ containing $B$ (the standard Borel subgroup of $GL_r$) and with Levi component isomorphic to $G$. Let $U_\Pi$ be the unipotent radical of $\Pi$.

We let $\Omega_{exc}(G)$ be the set of exceptional quasicharacters in $\Omega(G)$. Let $W(G)$ be the Weyl group of $H$ in $G$, that is the normaliser of $H$ modulo the centralizer of $H$. This can be identified with a subgroup of $G^*_k$. If $\omega_1$ and $\omega_2$ are in $\Omega_{exc}(G)$ then $\omega_1 \cdot \omega_2^{-1}$ is $W(G)$-invariant and trivial on $p^{-1}(H_{n,k})$. The set of such quasicharacters form a complex manifold $\tilde{\Omega}(G)$. A quasicharacter in $\tilde{\Omega}(G)$ can be identified with a quasicharacter of $G_{n,A}$ where $G_n$ is the algebraic subgroup of $G$ such that every algebraic character takes values in $\text{Im } (GL_1 \rightarrow GL_1 ; x \mapsto x^n)$. We shall make this identification henceforth.

Let $\chi \in \tilde{\Omega}(G)$ and form any one-dimensional representation $\chi^*$ of $H^*_k \tilde{H}_{n,A} \times \tilde{G}_{A}$ which when restricted to $\tilde{H}_{n,A} \times \{1\}$ yields $\chi^{-1}$, and when to $\{1\} \times \tilde{G}_{n,A}$ yields $\chi$. Then

$$V_0(\omega \cdot \chi) \cong V_0(\omega) \otimes \chi^*$$

It is convenient to regard $V_0$ as a holomorphic vector bundle over $\Omega_{exc}(G)$, which, being a coset of $\tilde{\Omega}(G)$, has a complex structure.

We consider next the vector bundle $F^*$ over $\Omega_{exc}(G)$ of which the fibre $F^*(\chi)$ at $\chi \in \Omega_{exc}(G)$ consists of a certain space of maps

$$f : GL_{r,A} \rightarrow V_0(\chi)$$

such that

$$f(\gamma u g) = \nu(\gamma) \cdot (\gamma^{-1} f)(g) \quad (\gamma \in \tilde{G}_{A}, u \in U_{A}^*, g \in GL_{r,A}).$$

Here $\nu$ is the quasicharacter of $G_A$ defined as the square root of the modulus of the action of $G$ on the Lie algebra of $U$. Let $\mu_1$ be the modulus quasicharacter of $GL_{r,A}$, and $\mu$, as above, that of $G_A$; then for $g \in H$ one has $\mu_1(g) = \mu(g)\nu(g)$.

We have now to describe the space $F^*(\chi)$. Each $f \in V_0(\chi)$ is a finite sum of primitive elements uniformly in $\chi$. A primitive element is of the form $\otimes f_v$
under the identification

\[ V_0(\chi) \cong \otimes V_{0,\nu}(\chi_{\nu}) \]

where \( V_{0,\nu}(\chi_{\nu}) \) is the image of the corresponding intertwining operator from \( V_{\nu}(\chi_{\nu}) \), the corresponding induce representation — see [6] §I.2.

Let us write \( f^0_v \) for the image of the element \( f^0_v \in V_{\nu}(\chi_{\nu}) \), the standard \( K^*_v \)-invariant vector. Then, locally uniformly in \( \chi \), we require that almost all the \( f^0_v \) are \( f^0_\nu \). We also demand that \( f^0_\nu \) is smooth (resp. locally constant) if \( \nu \) is Archimedian (resp. non-Archimedian).

Note that \( F^*(\chi) \) is a \( \hat{H}_n \times \hat{G}_{L_{r,A}} \)-module and that if \( \eta \in \hat{H}_{n,A} \) then

\[(\eta, 1) \cdot f = (\omega_\mu)^{-1}(\eta) \cdot f.\]

On an open subset of \( \Omega_{\text{exc}}^e(G) \) one can define the linear form \( E^*(\chi) : F^*(\chi) \rightarrow \mathbb{C} \) by

\[ \sum_{\gamma \in \Omega^*, GL_{r,k}^*} \langle \Theta(\chi), \gamma f \rangle \]

which satisfies

\[ \langle E^*(\chi), (\eta, \gamma) f \rangle = (\chi_\mu)^{-1}(\eta) \langle E^*(\chi), f \rangle \]

when \( \eta \in H^*_n \hat{H}_{n,A} \), \( \gamma \in GL_{r,k}^* \) and \( \omega \) is extended as before. In view of [4] Lemma 4.1 the series defining \( E^*(\chi) \) converges when \( \sigma(\chi_{\text{sc}}) > n \).

Let \( F^{(1)} \) be the space corresponding to \( F(\omega) \) but for \( \hat{G}_{L_{r,A}} \) in place of \( \hat{G}_A \). An element \( f \in F^{(1)}(\omega) \) yields by the quotient map an element of \( V_0(\omega) \), and so also an element of \( F^*(\omega) \) when \( \omega \in \Omega_{\text{exc}}^e(G) \). Denote this map by \( I : F^{(1)}(\omega) \rightarrow F^*(\omega) \); one has if \( E^{(1)} \) denotes the Eisenstein series operator for \( GL_{r,A} \) then for \( \omega \in \Omega_{\text{exc}}^e(G) \)

\[ \underset{\omega}{\text{Res}} E^{(1)}, f \rangle = \langle E^*(\omega), I f \rangle; \]


From this one can derive all the analytic properties of \( E \) which we shall need. We shall not describe these in detail here but we shall formulate the functional equation as it shall later be needed.
Let \( G \) be as above and \( G' = GL_{r_2} \times GL_{r_1} \). Let \( G^{(1)} = GL_r \). Let us write
\[
\begin{bmatrix}
0 & I_{r_1} \\
I_{r_2} & 0
\end{bmatrix}
\]
where the matrix is in \((r_1 + r_2) \times (r_2 + r_1)\)-block form, and \( I_q \) denotes the \((q \times q)\)-identity matrix. Thus
\[
s(G) \cdot s(G') = I.
\]
We shall also write \( s(G) \) for the image of \( s(G) \) under \( G_k \rightarrow G_k^* \subseteq \tilde{G}_A \). For a quasicharacter \( \chi \) of \( \tilde{H}_{n,A} \) define \( \chi' \) by
\[
\chi'(h) = \chi(s(G)^{-1} \cdot h \cdot s(G));
\]
Thus \( \chi' \in \Omega^\text{exc}_e(G') \) precisely when \( \chi \in \Omega^\text{exc}_e(G) \). Let \( \psi' \) be the analogue of \( \psi \) for \( G' \).

One can define an intertwining operator
\[
I_{s(G)}: F^* (\chi, G) \rightarrow F^* (\chi', G')
\]
by passage to the quotient from the map \( V(\chi) \rightarrow V(\chi') \) defined in [6] §II.1 and denoted there by \( I_{s(G)} \).

Let \( \alpha_0 \) be the root \((r_1, r_1 + 1)\) and \( \alpha_0' = (r_2, r_2 + 1) \). Then \( I_{s(G)} \) has been regularized by
\[
\prod_{j=0}^{r_1-1} L(\chi^{\alpha_0}_0 \| A)^{r_2+j+1})L(\chi^{\alpha_0}_0 \| A)^{1-1} \varepsilon(\chi^{\alpha_0}_0 \| A)^{r_2+j+1})
\]
and is given by an Euler product so that \( I_{s(G)} \) acts on almost all the standard vectors \( \psi_0^\alpha \) by sending them identically to the corresponding standard vector. Here \( L, \varepsilon \) have the same meanings as in [6] §II.1.

For a quasicharacter \( \theta: k^\times \rightarrow \mathbb{C}^\times \) trivial on \( k^\times \) we define \( \sigma(\theta) \in \mathbb{R} \) by \( |\theta| = \| \pi^{(0)} \| \). Then the only singularity of
\[
\left\{ \prod_{i \leq j \leq r_1} L(\chi^{\alpha}_0 \| A)^{1-1} \right\} \cdot E^* (\chi)
\]
in \( \{ \chi \in \Omega^\text{exc}_e(G): \sigma(\chi^{\alpha}_0) > 0 \} \) is along \( \chi^{\alpha}_z = \| A \) and is a ‘simple pole’.
One has the functional equation

\[ \prod_{1 \leq i \leq r_1} \prod_{1 \leq j \leq r_2} L(\chi_{\alpha_0}^\alpha) \|^{(i+j-1)} E_G^*(\chi, f) = \prod_{1 \leq i \leq r_1} \prod_{1 \leq j \leq r_2} L(\chi')_{\alpha_0} \|^{1-i-j} E_G^*(\chi'(\psi)^{-2+4(r_1+r_2)}, I_{s(G)} f). \]

This can be derived by the general principles of [4] from the results of [6] §II.1.

§3. The c-distribution

We recall here, in the context of \( \tilde{G}_A \), the basic properties of the c-distribution introduced in [6] Theorem II.2.2. and also discussed in [7] §4. We shall need some notation. Let for a finite subset \( S \) of \( \Sigma(k) \) the following expressions

\( \tilde{G}_S, \tilde{H}_{n,S}, \tilde{H}_S, \tilde{B}_{n,S}, \tilde{B}_S, \) etc.

denote the \( S \)-factors of

\( \tilde{G}_A, \tilde{H}_{n,A}, \tilde{H}_A, \tilde{B}_{n,A}, \tilde{B}_A, \) etc.

Likewise we can form the \( S \)-factor of a representation of \( \tilde{G}_A \), or of \( \tilde{H}_A \times \tilde{G}_A \) etc., likewise of a quasicharacter of \( \tilde{H}_{n,A} \). Then will also be denoted by a subscript \(- S \).

Let \( G \) be as above; let \( e: \tilde{N}_A \to \mathbb{C} \) be a non-degenerate character trivial on \( \tilde{N}_k = p^{-1}(N_k) \). We form

\[ \lambda_A(e): V_0(e) \to \mathbb{C} \]

by

\[ \langle \lambda_A(e), \psi \rangle = \int_{\tilde{N}_k \setminus \tilde{N}_A} \tilde{e}(n) \Theta(n \psi) \, dn. \]

This a Whittaker functional, [6] §II.2.

We shall now define another family of linear functionals.

Let \( \tilde{M}_A \) be the normalizer of \( \tilde{H}_A \) in \( \tilde{G}_A \) and let \( \tilde{M}_{0,A} \) be the subset of elements of maximal length. Then for \( \eta \in \tilde{M}_{0,S} \) we define the linear functional \( \lambda_{n,S} \) on \( V(\omega)_S \) by

\[ \langle \lambda_{n,S}, \psi \rangle = \int_{\tilde{N}_k} \tilde{e}_S(n) \psi(\eta^{-1} n) \, dn. \]
When certain linear conditions on $c_s : \tilde{\mathcal{M}}_{0,S} \to \mathbb{C}$ satisfying $c(h\eta) = (\omega \mu)(h) \cdot c(\eta)$ (for $h \in \tilde{H}_{n,S}, \eta \in \tilde{\mathcal{M}}_{0,S}$) are satisfied then $\Sigma_{\eta \in \tilde{H}_{n,S}, \tilde{\mathcal{M}}_{0,S}} c_s(\eta) \cdot \lambda_{\eta,S}$ factors through $V_0(\omega)_S$. Let us denote the space of such functions by $U_S(\omega, e)$; one has a natural restriction map

$$\text{res} : U_{S'}(\omega, e) \to U_S(\omega, e) \quad (S' \supset S).$$

The space $U_S(\omega, e)$ is finite-dimensional and one can make certain statements as to its dimension – cf. [6] §§1.4, I.6, II.2. Let

$$U(\omega, e) = \lim U_S(\omega, e)$$

which is a vector space endowed with morphism $U(\omega, e) \to U_S(\omega, e)$ compatible with the restriction map, and universal with respect to this property. An element $c \in U(\omega, e)$ can be regarded as a function $c : \tilde{\mathcal{M}}_{0,A} \to \mathbb{C}$ satisfying

$$c(h\eta) = (\omega \mu)(h) \cdot c(\eta) \quad (h \in \tilde{H}_{n,A}, \eta \in \tilde{\mathcal{M}}_{0,A})$$

and the further linear conditions indicated above. Let

$$T(S) = \prod_{\nu \in S - \Sigma(k)} \prod_{i=1}^{r} \prod_{j=1}^{r-1} (1 + q_{\nu}^{-1} + \ldots + q_{\nu}^{-j}).$$

Then one has that there exists $c \in U(\omega, e)$ such that $\lambda_A(e)$ can be represented as follows. For $\nu \in V_0(\omega)$ we choose $S' \supset \{\nu \in \Sigma(k) | |n|_{\nu} \neq 1\}$ such that $\omega$ is unramified on $\Sigma(k) - S$, and such that in the representation

$$V_0(\omega) \cong V_0(\omega)_S \otimes_{\mu_0(k)} \otimes_{\nu \in \Sigma(k) - S} V_{0,\nu}(\omega)$$

the vector $v$ can be represented as $v_S \otimes v_0^S$ where $v_0^S$ is the image under the intertwining operator of the standard $\Pi_{\nu \in \Sigma(k) - S} K_{\nu}^*$-invariant vector of $\otimes_{\nu \in \Sigma(k) - S} V_0(\omega)$. Then from [6] Theorem II.2.2, one of the main results of that paper, one has

$$\langle \lambda_A(e), v \rangle = T(S)^{-1} \sum_{\eta \in \tilde{H}_{n,S}, \tilde{\mathcal{M}}_{0,S}} c(\eta) \langle \lambda_{\eta,S}, v_S \rangle$$

where we have written $e$ also for the restriction of $e$ to $\tilde{H}_S$. Note that in [6] Theorem II.2.4 this result was given incorrectly and should be altered as in the Corrigendum.
Let $\tilde{G}_{i,A} = \text{GL}_r(k_A)$ for $1 \leq i \leq s$. Let $\tilde{H}_A = \tilde{H}_A \cap \tilde{G}_{i,A}$. Let $D_i: \tilde{G}_{i,A} \xrightarrow{\rho} \text{GL}_r(k_A) \xrightarrow{\text{det}} k_A^\times$. Then one has:

**Theorem 3.1.** With the notations above $c$ is supported on $\tilde{H}_{n,A} \cdot (\prod_{i=1}^{s} D_i^{-1}(k^\times))$. Let $c'$ be the corresponding function for $\text{GL}_r(1 \leq i \leq s)$ with restriction of $e$ to $G_{i,A}$. Then if $\eta \in H_A$ is of the form $\eta = \eta_1 \ldots \eta_s$ where $\eta_i \in D_i^{-1}(k^\times) \cdot (\tilde{H}_{n,A} \cap \tilde{H}_A)$ we have

$$c(\eta) = c'(\eta_1) \ldots c'(\eta_s).$$

Note that in this theorem the $\eta_i$ commute with one other. A primitive version of this theorem is to be found in [7].

**Proof.** Let $S$ be a finite subset of $\Sigma(k)$ containing $\{v | |n|_v \neq 1\}$. Let $N^S$ be the complementary factor to $N^\times_S$ in $N^\times_A$. Let $e^3 = e|N^S$. Suppose $f \in F(\omega)$ can be written as

$$f_S \otimes F_0^S: \tilde{G}_S \times \mu(k) \left( \prod_{v \in \Sigma(k) - S} \tilde{G}_v \right) \to \mathbb{C}$$

and

$$f_0^S = \otimes_{v \in \Sigma(k) - S} f_0^v.$$

Then as in [6] p. 120 one has

$$\int_{N^\times_S} \tilde{e}(n) \langle E(\omega), nf \rangle \, dn = \sum_{\eta \in H_{n,A} \cdot M_{\omega,\rho}} \int_{N^\times_S} \tilde{e}_S(n)f_S(\eta n) \, dn \cdot \int_{N^\times_S} \tilde{e}^3(n)f_0^S(\eta n) \, dn.$$

The final factor here is not relevant to our considerations and we denote it by $J_S(\omega, e, \eta)$; it can be evaluated fairly explicitly.

The factor

$$\int_{N^\times_S} \tilde{e}_S(n)f_S(\eta n) \, dn$$

depends only on the restriction of $f_S$ to a certain subgroup of $\tilde{G}_S$ which we shall now describe, at least for sufficiently large $S$.

Let $r' = \prod_{v \in \Sigma(k) - S} r_v$ where $r_v$ is the ring of integers of $k_v$. Let $U_S$ be the topological semidirect product of the $k_v^\times r_v^\times$ for $v \in \Sigma(k) - S$. This is a subgroup of $k_A^\times$. 
The \( \eta \) which occur in the sum above and which give a non-zero factor \( J_S(\omega, e, \eta) \) are restricted to have

\[
D_i(\eta) \in U_S
\]

by definition of \( f_v^9 \). Thus as \( D_i(\eta) \in k^\times \) we also have to have

\[
D_i(\eta) \in U_S \cap k^\times.
\]

We assume now that \( S \) is so large that \( r^s \) is a principal ideal domain. Then

\[
U_S \cap k^\times = \{ x \in k^\times | \text{ord}_v(x) \equiv 0(\text{mod.} n), v \in \Sigma(k) - S \}
\]

so that the fractional ideal generated by \( x \) in \( r^s \) is an \( n \)th power. Consequently one has

\[
U_S \cap k^\times = k^{n \times} \cdot (k \cap r^S)^\times.
\]

Hence the sum depends only on the restriction of \( f_S \) to the subgroup

\[
\{ g \in \tilde{G}_S | D_i(g) \in k^{n \times}_S \cdot (k \cap r^S)^\times, \quad 1 \leq i \leq s \}.
\]

and the same holds for the residues of \( E(\omega) \) since this is an open subgroup. Hence

\[
\sum_{\eta \in \tilde{H}_{n,S} / \tilde{H}_S} c(\eta) \cdot \langle \lambda_{\eta,S}, f_S \rangle
\]

depends only on the restriction of \( f_S \) (at an exceptional \( \omega \)). This means that we have to have in the sum \( D_i(\eta) \in k^{n \times}_S \cdot (k \cap r^S)^\times \) whenever \( c(\eta) \neq 0 \). If we now let \( S \) increase it follows that \( c \) is supported on

\[
\{ \eta \in \tilde{M}_{0,A} | D_i(\eta) \in k^{n \times}_A \cdot k^\times \}.
\]

The same argument can be applied to

\[
\tilde{G}_{1,A} \times_{\mu_n(k)} \tilde{G}_{2,A} \times_{\mu_n(k)} \cdots \times_{\mu_n(k)} \tilde{G}_{S,A}
\]

and we obtain an analogous formula. Since the subgroups of each of these as above are isomorphic the computation of \( c \) cannot distinguish between this group and \( \tilde{G}_A \). Thus the resulting \( c \)'s are also equal, and the \( c \) for
$G_{1,A} \times \mu_n(k) \cdots \times \mu_n(k) \tilde{G}_{S,A}$ is simply $c^1(\eta_1) \cdots c^r(\eta_r)$. This completes the proof of the theorem.

It will be useful to write $c(r_1, r_2, \ldots, r_s; e, \chi, \eta)$ where we make the dependence on the additive character $e$ and the exceptional quasicharacter explicit. One can easily verify that for $\theta \in H^*_k$

$$c(r_1, \ldots, r_s; \theta e, \chi, \theta \eta) = c(r_1, \ldots, r_s; e, \chi, \eta).$$

We shall find it useful to define $c(1)$ to be the function defined on $H^*_k \tilde{H}_{n,A}$ such that

$$c(1; e, \chi, h) = 1 \quad h \in H^*_k$$

and $H \cong GL_1$. This is then consistent with all the results given above.

Moreover if $\chi_1$ and $\chi_2$ are exceptional for $G$ then $\chi_1 = \chi_2 \psi$ where $\psi$ is $W(G)$-invariant, where $W(G)$ is the Weyl group of $G$ (relative to $H$). We extend this $H^*_k \tilde{H}_{n,A}$ by demanding that it be trivial on $H^*_k \cdot \tilde{H}_{n,A}$. As it is trivial on $i(\mu_n(k))$ it can be extended to a $W(G)$-invariant quasicharacter on $H^*_A$ (not uniquely). One can then verify that

$$c(r_1, \ldots, r_s; e, \chi_1, \eta) = c(r_1, \ldots, r_s; e, \chi_2, \eta)\psi(\eta)^{-1}.$$

Note that this also yields the assertion of Theorem 3.1 as to the support of $c$.

§4. The Fourier coefficients

In this section we shall give an expression for

$$\int_{N_t^* \setminus N_t^*} \left\langle E^*(\chi), nf \right\rangle \tilde{e}(n) \, dn$$

where $e$ is a non-degenerate character of $N_t^*$ trivial on $N_t^*$. Let $S \subset \Sigma(k)$ be finite; for any adelic object $X_A, \tilde{X}_A$ (here an algebraically defined subset of $GL_r, A$ or a metaplectic cover of such a set) we write $X_S$ and $\tilde{X}_S$ for the component corresponding to $S$, and $X^S$ or $\tilde{X}^S$ for the complementary factor. For a function $f$ on $X_A$ which can be written as $f(x_S \times x^S) = g_1(x_S)g_2(x^S)$ where $x_S \in X_S$ or $\tilde{X}_S, x^S \in X^S$ or $\tilde{X}^S$ we shall write $f_S = g_1, f^S = g_2$. In general $f_S$ and $f^S$ are not uniquely determined, but the notation
will in general be used only when $S$ is so large that $f^S$ is a ‘standard function’. In particular we can define $e^S$, $e^S$, $\chi^S$, $\chi^S$ and all the components are quasicharacters.

We shall need some definitions and results concerning Whittaker functionals of representations of $(G^{(1)})_S$. We define, in a notation more precise than the previous $\lambda_{w,S}$

$$L^S(e^S, w, \omega^S): V(\omega^S) \to \mathbb{C}$$

as

$$\langle L^S(e^S, w, \omega^S), f \rangle = \int_{S^S} f(w^{-1}n)\tilde{\epsilon}_S(n) \, dn$$

and

$$w \in \tilde{M}_{0,S}.$$ 

This can be expressed as a product over the places in $S$. Likewise we can define for $w \in \tilde{M}_S$ the intertwining operator

$$I_{w,S}(\omega^S): V(\omega^S) \to V(\omega^S)$$

as in [6] §1.2. Define the scalar $\theta_{S,w_1,w_2}(\omega^S)$ by

$$I_{w_1,S}(\omega^S)I_{w_2,S}(\omega^S) = \theta_{S,w_1,w_2}(\omega^S)I_{w_1w_2,S}(\omega^S)$$

and $\tau_S(e^S, \omega^S, w; w_1, w_2)$ for $w \in \tilde{M}_S$ and $w_1, w_2 \in \tilde{M}_{0,S}$ by

$$\langle L_S(e^S, w^S\omega^S), I_{w,S}(\omega^S)f \rangle = \sum_{w \in \tilde{M}_{0,S}/\tilde{H}_S} \tau_S(e^S, \omega^S, w; w_1, w_2) \langle L_S(e^S, w_1^S\omega^S, w_2^S), f \rangle$$

valid for all $f \in V(\omega^S)$ as in [6] p. 75 or in [7]. One has the following properties of $\theta_S$ and $\tau_S$:

**Proposition 4.1.** One has

i) $\theta_{S,w_1,w_2,w_3}(\omega^S)\theta_{S,w_2,w_3}(\omega^S) = \theta_{S,w_1,w_2}(\omega^S)\theta_{S,w_1,w_2,w_3}(\omega^S)$,

ii) if for each $v \in S$ one has $l_v(w_1) + l_v(w_2) = l_v(w_1w_2)$ where $l_v(w)$ denotes the length of the $v$th component of $w$ in $\tilde{M}_v/\tilde{H}_v \subset \tilde{M}_S/\tilde{H}_S$ then $\theta_{S,w_1,w_2}(\omega^S) = 1$
iii) if \( h_1, h_2 \in \tilde{H}_{n,S} \) then

\[
\tau_S(e_S, \omega_S, w; w_1 h_1, w_2 h_2)
\]

\[
= (\omega_S \mu_S)(h_1)^{-1}(\omega_S \mu_S)(h_2)\tau_S(e_S, \omega_S, w; w_1, w_2),
\]

iv) for \( \eta \in \tilde{H}_S \) one has

\[
\tau_S(\eta e_S, \omega_S, w; w_1, w_2) = \tau_S(e_S, \omega_S, w; \eta^{-1} w_1, \eta^{-1} w_2),
\]

\[
\tau_S(e_S, \omega_S, \eta w; w_1, w_2) = \mu_S(\eta) \mu_S(\eta^w)^{-1} \tau_S(e_S, \omega_S, w; w_1 \eta, w_2)
\]

and

\[
\tau_S(e_S, \omega_S, w; w_1 \eta, w_2 \eta^w) = \mu_S(\eta^w) \mu_S(\eta)^{-1} \tau_S(e_S, \omega_S, w; w_1, w_2),
\]

v) \( \tau_S(e_S, \omega_S, w' w^w; w_1, w_2) \theta_{S,w',w}(\omega_S) \)

\[
= \sum_{w \in M_{0,\tilde{H}_S}} \tau_S(e_S, \omega_S, w'; w_1, w) \tau_S(e_S, \omega_S, w''; w, w_2)
\]

vi) \( \tau_S(e_S, \omega_S, I; w_1, w_2) = 1 \) if \( w_1 = w_2 \)

\[
= 0 \quad \text{if} \quad w_1^{-1} w_2 \notin \tilde{H}_{n,S}.
\]

The properties listed here follow directly from the definitions and can be left to the reader. It is worth noting that \( \theta_S \) can be computed explicitly using [6] Theorem 1.2.6 but we shall not need that here. For a technique for computing \( \tau_S(e_S, \omega_S, w; w_1, w_2) \) when \( w \) is a simple reflection see [7] §3.4 and [6] Lemma 1.3.3. Note that all the necessary information about the complex places is given in [6] §1.6; we shall not need to discuss real places as we shall concentrate on the case \( n > 2 \).

We now define the subset \( W^+ \) of \( W(G^{(1)}) \) by demanding that \( w(\alpha) > 0 \) for every positive root in \( G \cap N \). One has then \( W^+ \) is a set of representatives for \( W(G^{(1)})/W(G) \). Next let \( \tilde{M}^+ \) be the lift of \( W^+ \) back to \( M \). Let \( \tilde{M}^+ = p^{-1}(M^+) \).

Let \( w_0 \) be the element

\[
\kappa \begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & 0 \\
1 & 0 & \ldots & 0
\end{pmatrix} \in \tilde{M}_0(G^{(1)}).
\]
representing the longest element of $W(G^{(1)})$. Let $s(G)$ be as in §2. Then the maps

$$\tilde{M}^0(G) \to \tilde{M}^0(G^{(1)}); \quad w \mapsto s(G)^{-1}w \quad \text{and} \quad w \mapsto ws(G)$$

are bijections. These sets, in which we have specified the reference group, can be understood globally, locally or semilocally. In the latter two cases $s(G)$ is the standard lift, [6] §0.1.

We now need a purely local theorem dealing with the generic places; it is a generalization of the main theorem of [3]. Suppose $F$ is a local field in which $|n|_F = 1$ and that Card $\mu_n(F) = n$. We call a quasicharacter $\omega$ of $\tilde{H}_{n,F}$ unramified if $\omega|\tilde{H}_{n,F}K^* = 1$. Let $e$ be a non-degenerate character of $N^*$. We call $e$ unramified if $e|N^* \cap K^* = 1$ but $\eta e|N^* \cap K^* \neq 1$ for any $\eta \in \tilde{H}_F$ for which there exists a positive root $\alpha$ with $|\eta^\alpha|_F < 1$. Concepts defined over $k_s$ above can be transferred to the case of $F$ without any trouble.

**Theorem 4.2.** Suppose that $\omega$ and $e$ are unramified. Then one has for $f \in V(\omega)K^*$

$$\langle L(e, w, \omega), \eta f \rangle = \sum_{w' \in \tilde{M}^0(H_n)} \tau(e, \omega, w^{-1}, w, w) I_{\eta^{-1}w^*} f (I)$$

if $|\eta^\alpha|_F \leq 1$ for all positive $\alpha$, 

$$= 0 \quad \text{otherwise.}$$

The proof of Theorem 4.2 will be based on the following proposition:

**Proposition 4.3.** Suppose that $\omega$ is unramified and dominant. Then for $f \in V(\omega)K^*$ one has that $\langle L(e, w, \omega), f \rangle$ can be written as

$$\sum_{w' \in W} \langle L^*_w(e, w, \omega), f \rangle$$

if $e|N^* \cap K^* = 1$, where, for $h \in H_n$

$$L^*_w(he, w, \omega) = (\omega/h^\omega)(h^\omega) L^*_w(e, w, \omega).$$

**Proof.** The proof follows [3] and is based on an idea of Casselman [2]. Let $B_i \subset K$ be the Iwahori subgroup

$$B_i = \{k \in K; k \equiv I \mod P_F \} \cdot (B_n \cap K).$$

Then $B_n \backslash G/B_i$ can be identified set-theoretically with $\tilde{H}_n \backslash \tilde{M}$. Let $B_i^*$ be $B_i$ considered as a subgroup of $K^*$. 
Let $V(\omega)$ be as above and let $V(w) \to V(\omega)_N$ be the map induced by the Jacquet functor. Let $V(\omega)^{B_I}$ be the space of $B_I^*$-invariants in $V(\omega)$ and consider the composite map

$$j: V(\omega)^{B_I} \hookrightarrow V(\omega) \rightarrow V(\omega)_N.$$ 

Casselman proves in [2] §2 that this map is an isomorphism of vector spaces when $n = 1$. Both proofs in [2] can be adapted to the case of covering groups using [1] Theorem 5.2. Here we shall give a fairly elementary proof of this isomorphism.

First of all we show that $V(\omega)^{B_I}$ and $V(\omega)_N$ have the same dimension. This is so as by [6] Proposition 1.2.1. one has

$$\dim (V(\omega)_N) = n^{2r} \cdot r!,$$

and, since $\tilde{G}$ is the disjoint union of the open cosets $\tilde{B}_n \cdot w \cdot B_I^*$ where $w \in \tilde{H}_n \setminus \tilde{M}$ one also has

$$\dim (V(\omega)^{B_I}) = \text{Card} (\tilde{H}_n \setminus \tilde{M}) = n^{2r} \cdot r!.$$ 

This proves the identity of the dimensions. Thus it is only necessary to show that $j$ is injective. To do this we need some geometrical facts. Let $W$ be the Weyl group of $GL_r$ and write $w_1 \leq w_2$ ($w_1, w_2 \in W$) if $l(w_1) + l(w_2 w_1^{-1}) = l(w_2)$, and $w_1 < w_2$ if $w_1 \leq w_2, w_1 \neq w_2$. We lift these relations to $\tilde{M}$. Then we have firstly that

i) $w_1 N^* \cap (\tilde{B}_n \cdot w \cdot B_I^*) = \emptyset$ only if $w_1 \geq w$; this follows directly from [2] Proposition 1.3(a). Moreover the proof of [2] Proposition 1.3 (b) can easily be adapted to show that

ii) if $\eta \in \tilde{H}$ and $(\eta w N^*) \cap (\tilde{B}_n \cdot w \cdot B_I^*) = \emptyset$, then $\eta \in \tilde{H}_n$,

and

iii) $(\tilde{B}_n w N^*) \cap (\tilde{B}_n w B_I^*) = \tilde{B}_n w (N^* \cap K^*)$.

Let now $\chi_w$ be that function in $V(\omega)^{B_I}$ which is supported on $\tilde{B}_n \cdot w \cdot B_I^*$ with $\chi_w(w) = 1$. Then the $\chi_w$ form a base of $V(\omega)^{B_I}$ as $w$ runs through a set of representatives of $\tilde{H}_n \setminus \tilde{M}$. Suppose now that $\sum c(w) \chi_w$ were a non-trivial linear combination of the $\chi_w$ where the $w$ lie in a fixed set of representative of $\tilde{H}_n /\tilde{M}$, such that it maps to zero in $V(\omega)_N$ under $j$. By Jacquet’s Lemma there exists an open compact subgroup $U$ of $N^*$ so that

$$\int_U \sum c(w) \chi_w(gu) \, du = 0.$$
Without loss of generality we may assume that $U \supset N^{*} \cap K^{*}$. Let $w_1$ be minimal so that $c(w_1) \neq 0$. Set $g = w_1$ and by i), ii) we have

$$c(w_1) \int_{U} \chi_{w_1}(w_1 u) \, du = 0.$$ 

By iii) the integral is non-zero and hence $c(w_1) = 0$. This is a contradiction; hence no such linear combination exists, and $f$ is injective, as claimed.

We shall now consider the representations $V(\omega)$ in Bernstein's generic sense described in [6] §1.2. There exists, as $j$ is an isomorphism, a basis of $V(\omega)^{\text{BT}}$ dual to the linear forms $f \mapsto (I_w f)(I)$ induced by the intertwining operators. Denote the elements of this basis by $f_w$. If $f \in V(\omega)^{K^{*}}$ then $f \in V(\omega)^{\text{BT}}$ and hence there is a representation of $f$ as

$$\sum c(w_1, f) \cdot f_{w_1}.$$ 

It now follows that

$$\langle L(e, w, \omega) f \rangle = \sum_{w_1} c(w_1, f) \cdot \langle L(e, w, \omega), f_{w_1} \rangle.$$ 

We now observe that

$$g \mapsto \int_{N^{*} \cap K^{*}} f_w(ga) \, dn$$

for a satisfying $|a^{*}|_{F} \leq 1$ for every positive root $\alpha$ is itself in $V(\omega)^{\text{BT}}$ and can therefore be represented as $\sum t(w_1) f_{w_1}$. Apply the functional $f \mapsto (I_{w_2} f)(I)$ the element of $V(\omega)$ defined by (\star); it is clear that since this functional factors through the Jacquet functor $V(\omega) \to V(\omega)^{N^{*}}$ that the result is

$$(I_{w_2} f_w)(a) = \mu(a)^{2} (I_{w^{-1} w_2} f_{w})(I).$$

On the other hand, by the definition of the $t(w_1)$ this is also equal to $t(w_2)$. Thus we have shown that

$$\int_{N^{*} \cap K^{*}} f_w(ga) \, dn = \mu(a)^{2} \cdot f_{w}(a).$$

We now replace $g$ by $w'^{-1} \cdot n'$, multiply by $\tilde{e}(n')$ and integrate over $n'$ in $N^{*}$. Suppose that $e|N^{*} \cap K^{*} = 1$ and that a is as above; then we obtain

$$\langle L(e, w', \omega), af_w \rangle = \mu(a)^{2} \langle L(e, w', \omega), f_{aw} \rangle.$$
Thus if \( \eta \) satisfies the condition that \( |\eta^z|_F \geq 1 \) for all positive roots \( z \) we have that

\[
\langle L(\eta e, w', \omega), f \rangle = \sum_w I_w f(I) \langle L(\eta e, w', \omega), f_w \rangle
\]

\[
= \sum_w I_w f(I) \langle L(e, \eta^{-1} w', \omega), \eta^{-1} f_w \rangle \mu(\eta)^2
\]

\[
= \sum_w I_w f(I) \langle L(e, \eta^{-1} w', \omega), f_{\eta w} \rangle.
\]

The assertion of the proposition follows if we take

\[
\langle L^*_w(\eta e, w', \omega), f \rangle
\]

to be

\[
\sum \langle L(e, \eta^{-1} w', \omega), f_w \rangle I_{\eta w} f(I)
\]

where the summation is taken over those \( w \) in \( \tilde{H}_n \setminus \tilde{M} \) which project to \( w^* \) in \( H \setminus M \).

We shall next identify the \( \langle L^*_w(e, w, \omega), f \rangle \) of Proposition 4.3. Suppose first that \( \omega \) is dominant. Then we see that for \( w' \neq 1 \)

\[
L^*_w(he, w, \omega) \to 0
\]

if \( h \) runs through a sequence such that for each positive \( z \) one has \( |(h^w)^z| \to 0 \). If this is so \( he \) converges boundedly and locally uniformly to 1 and hence in the limit we have

\[
I_w(\omega)(f)(I) = \langle L^*_w(e, w, \omega), f \rangle.
\]

This identifies \( L^*_w \). By regularization this will remain valid for all \( \omega \).

Next we have for \( w_1 \in \tilde{M} \) that

\[
\langle L(e, w, ^{-1} w_1), I_{w_1} f \rangle = \sum_{w_2 \in \tilde{M}_0 \setminus \tilde{R}_n} \tau(e, \omega, w_1; w, w_2) \cdot \langle L(e, w_2, \omega), f \rangle.
\]

This yields

\[
\langle L^*_w(e, w, ^{-1} w_1), I_{w_1} f \rangle = \sum_{w_2 \in \tilde{M}_0 \setminus \tilde{R}_n} \tau(e, \omega, w_1; w, w_2) \langle L^*_{w_1 w}(e, w_2, \omega), f \rangle.
\]
This we apply with \( w_1 = w'^{-1} \). We obtain

\[
\langle L_w^*(e, w, w'^{-1} \omega), I_{w^{-1}f} \rangle = \sum_{w_2 \in M/H_n} \tau(e, \omega, w'^{-1}, w, w_2) I_{w_2}(f)(I).
\]

We now replace \( f \) by \( I_w f \) and \( \omega \) by \( \omega' \). This yields

\[
\langle L_w^*(e, w, \omega), f \rangle = \sum_{w_2 \in M/H_n} \tau(e, \omega, w'^{-1}; w, w_2) I_{w_2w'}(f)(I)
\]

\[
\times \theta_{w_2w', \omega} \cdot \theta_{w'^{-1}, w}(\omega)^{-1}.
\]

Observe that

\[
\theta_{w_2, \omega'} \cdot \theta_{w_2w', \omega'^{-1}}(\omega') = \theta_{w', \omega'^{-1}}(\omega),
\]

\[
= \theta_{w'^{-1}, w}(\omega).
\]

This means that if we sum over \( w' \) in \( W \) and set \( w_2 = w \cdot \eta \) we obtain, as \( w \) is of maximal length, from Proposition 4.1(ii)

\[
\langle L(e, w, \omega), f \rangle = \sum_{w} \sum_{\eta \in H/H_n} \tau(e, \omega, w^{-1}, w, w_2) I_{w\eta w'}(f)(I)
\]

\[
= \sum_{w'} \sum_{\eta} \tau(e, \omega, w'^{-1}\eta^{-1}; w, w) I_{w\eta w'}(f)(I)
\]

This we regard as a sum over \( \tilde{M}/\tilde{H}_n \) and we obtain now

\[
\langle L(e, \omega, w), f \rangle = \sum_{w' \in \tilde{M}/\tilde{H}_n} \tau(e, \omega, w'^{-1}; w, w) I_{w'}(f)(I)
\]

The assertion of the theorem now follows from the elementary relation

\[
\langle L(e, \eta, w), \eta f \rangle = \mu(\eta)^2 \langle L(\eta^{-1} e, \omega \eta^{-1} w), f \rangle.
\]

and Proposition 4.1(iv).

REMARK. As in [3] the proof here uses no features of the series \( GL_r \), which are special to this series. Therefore the results will remain valid also for the metaplectic covers of other Chevalley groups. It is also worth noting that we have also not used the non-degeneracy of \( e \) so that this result can also be applied in cases where \( e \) is degenerate. This is useful in other applications.
We are now in a position to carry out the computation which is the main purpose of this section.

**Theorem 4.4.** Suppose \( \chi \in \Omega^\text{exc}_k(G) \) is such that

\[
\sigma(\chi_{\text{ad}}) > n.
\]

Suppose that \( S \subset \Sigma(k) \) is such that \( \omega \) and \( e \) are unramified outside \( S \) and that \( f \in F^*(\chi) \) can be represented by \( f_S \otimes f^S \) where \( f^S = \otimes_{v \in \Sigma(k) - S} f_v^0 \). Let \( U \) be the unipotent subgroup of the standard parabolic subgroup of \( G^{(1)} \) with Levi component \( G \). Let \( \Phi_+(U) \) be the set of positive roots associated with \( U \). Then

\[
\int_{N^*_A \backslash N^*_A} \bar{e}(n) \left< E^*(\chi), nf \right> \, dn
\]

is equal to

\[
\lim_{S_1, S_2 \to S} \sum_{r \in H_0, v} \epsilon(c_1, r_2; e', \chi, \eta) \cdot \int_{U^S} \bar{e}(u) \left< \lambda \eta_S, s(G)^{-1} u f_S \right> T(S)^{-1} \, du
\]

\[
\cdot \prod_{v \in S_1 - S} \sum_{w \in (M^*_v \cap K^*_v)} \tau(e_v, w, \chi_v; (s(G) \eta)_v, w_0) \prod_{w \in \Phi_+(U)} L((\chi^w_\alpha)_v) L(\| \chi^w_\alpha \|^{-1})
\]

**Proof.** We have that

\[
\left< E^*(\chi), f \right> = \sum_{\gamma \in \Pi^*_A \cap G^{(1)}} \left< \Theta(\chi), \gamma f \right>.
\]

Since \( e \) is non-degenerate a standard computation using the Bruhat decomposition shows that

\[
\int_{N^*_A \backslash N^*_A} \bar{e}(n) \left< E^*(\chi), nf \right> \, dn
\]

\[
= \int_{U_\Lambda} \bar{e}(u) \int_{N^*_A \backslash N^*_A} \bar{e}'(n) \left< \Theta(\chi), ns(G)^{-1} u f \right> \, dn \, du
\]

where \( N' = N \cap G \) and if \( \Pi' \) is the standard parabolic subgroup associated with \( G' \) then \( U' \) is its unipotent radical. Here \( e'(n) = e(n^{(G)}) \) for \( n \in N^*_A \). The
Integral on the right has remained absolutely convergent, and so we can rewrite it as

$$\lim_{S_1 \to S} \int_{U_{S_1}} \bar{e}(u) \int_{N_{s,t}} e'(n) \langle \Theta(\chi), ns(G)^{-1}u \rangle \ dn \ du.$$ 

If now we restrict $S_1$ by $S_1 \supset S$ then we have by the results recalled in §3 that this is equal to

$$\lim_{S_1 \to S} \int_{U_{S_1}} \bar{e}(u) \cdot \sum_{\eta \in \tilde{H}_{n,S_1} \backslash \tilde{H}_{S_1}} c(r_1, r_2; e', \chi, \eta) \langle \lambda_{n,S_1}, s(G)^{-1}u \rangle_{S_1} \ du \cdot T(S)^{-1}.$$ 

Since $\tilde{H}_{n,S_1} \backslash \tilde{H}_{S_1}$ is finite the integral and sum are absolutely convergent ([6] §I.3).

We separate the places in $S$ from the others and obtain

$$\lim_{S_1 \to S} \sum_{\eta \in \tilde{H}_{n,S_1} \backslash \tilde{H}_{S_1}} c(r_1, r_2; e', \chi, \eta) \cdot \int_{U_S} \bar{e}(u) \cdot \langle \lambda_{n,S}, s(G)^{-1}u \rangle_{u} \ du \cdot T(S)^{-1}$$

$$\times \prod_{v \in S_1} \left\{ \int_{U_v} \langle \lambda_{n,v}, s(G)^{-1}u \rangle_{v} \bar{e}_v(u) \ du \cdot T_v^{-1} \right\}$$

where $\lambda_{n,v}$ is formed with respect to $e'$ and

$$T_v = \left\{ \prod_{1 \leq j \leq r_1 - 1} \left( 1 + q_v^{-1} + \cdots + q_v^{-j} \right) \right\}$$

$$\times \left\{ \prod_{1 \leq k \leq r_2 - 1} \left( 1 + q_v^{-1} + \cdots + q_v^{-k} \right) \right\}.$$ 

By Fubini’s theorem the $v$th factor in the latter integral is precisely what is denoted by

$$\langle L(e_v, (s(G)\eta)_v, \chi_v) \rangle f_v^{0} \cdot T_v^{-1}.$$ 

We can now apply Theorem 4.2 since $e_v$ is unramified at $v$. We obtain then for this factor:

$$\sum_{w' \in M_v / \tilde{H}_{n,v}} \tau(e_v, w' \chi_v, w'^{-1}, (s(G)\eta)_v, (s(G)\eta)_v) T_v^{-1} (I_{(s(G)\eta)_v w' f^0}) (I)$$
By [6] Theorem 1.2.4 we can restrict the summation in $w'$ to $w'$ of the form
$(s(G)\eta)_{\chi_w}^{-1}w_0 \cdot w''$ where $w''$ runs through $(\tilde{M}_\nu \cap K^s_\nu)/(\tilde{H}_{n,v} \cap K^s_\nu)$. We note further by Proposition 4.1.iv) that

$$\tau(e_v, w' \chi_v, w'^{-1}, (s(G)\eta)_{\chi_v}, (s(G)\eta)_{\chi_v}) = \tau(e_v, w' \chi_v, w'^{-1}, (s(G)\eta)_{\chi_v}, w_0).$$

From [6] Theorem 1.2.4 we now obtain

$$\sum_{w'} \tau(e_v, w' \chi_v, w'^{-1}, (s(G)\eta)_{\chi_v}, w_0) \cdot \prod_{\chi_v \neq 0} \frac{L((\chi_v^\alpha)_{\chi_v})}{L(\mid_{\nu}(\chi_v^\alpha)_{\chi_v})} \cdot T_v^{-1}$$

for the $v$th factor above.

In view of the linear relations satisfied by $c$ we see that any $w''$ for which there exists an $s \in W(G) \subset W(G^{(1)})$, a simple reflection, with

$$l(w''s) < l(w'')$$

can be omitted from the summation. But this means precisely that $w''$ should represent an element of $W^+$. Next, note that if $\alpha$ is a root of $G$ then for exceptional $\chi$ the $L$-factors are explicitly computable. These factors together yield $T_v$. Thus our expression reduces to

$$\sum_{w'' \in (\tilde{M}_\nu^+ \cap K^s_\nu)/(\tilde{H}_{n,v} \cap K^s_\nu)} \tau(e_v, w'' \chi_v, w''^{-1}, (s(G)\eta)_{\chi_v}, w_0) \cdot \prod_{\alpha \in \Phi_+(U)} \frac{L((\chi_v^\alpha)_{\chi_v})}{L(\mid_{\nu}(\chi_v^\alpha)_{\chi_v})}$$

§5. The main theorem

We can now assemble all that we have done to prove the main theorem. To formulate it we introduce a semilocal space of functions, $\mathcal{K}_S(\chi_S)$ where $S$ is a finite subset of $\Sigma(k)$ as in §4 and $\chi$ exceptional. This space consists of functions $\varphi: \tilde{M}_{0,S} \to \mathbb{C}$ satisfying

$$\varphi(h\eta) = (\chi\mu)^{-1}(h)\varphi(\eta) \quad (h \in \tilde{H}_{n,S}, \eta \in \tilde{M}_{0,S}).$$
It is finite dimensional. We can regard this as a fibre of a holomorphic vector bundle over the complex manifold of exceptional quasicharacters $\tilde{H}_{n,S} \to \mathbb{C}^\times$ which we denote by $\Omega^\text{exc}_S(G, S)$. This is one-dimensional but not connected (if $n > 1$).

In particular we have an operator

$$\tau_S(e_S, \chi_S): \mathscr{H}_S(\chi_S) \to \mathscr{H}_S(\chi'_S)$$

defined by

$$(\tau_S(e_S, \chi_S)\varphi)(\eta) = \sum_{\eta' \in \tilde{M}_{0,S}/\tilde{H}_{n,S}} \tau_S(e_S, \chi_S; s(G); \eta, \eta')\varphi(\eta')$$

$$\times \prod_{v \in S} \prod_{\sigma \in \Phi_+^{(U)}} \varepsilon_v(\chi'_S, e_v)L_v(\|\chi'_S\|/\Lambda_{\chi'_S}^{\sigma})$$

where $\tau_S$ is the semilocal analogue of the $\tau$ defined in §4 and $\varepsilon$ is as in [6] §II.1, $e$ being a character of $k_\Lambda$ trivial on $k$ unramified outside $S$.

We define next a local modification $\tilde{\tau}$ as follows:

$$\tilde{\tau}(e^w, \chi, w^{-1}; \eta, \eta') = \tau(e^w, \chi, w^{-1}; \eta, \eta')$$

$$\times \prod_{x \in \Phi_+^{(U)}} L(\chi_S^n) \cdot \prod_{w \in M^+} L(\|\chi_S^n\|/\Lambda_{\chi_S^n})$$

where $\chi$ is in $\Omega^\text{exc}_S(G)$ and $w \in \tilde{M}^+$.

For $S \subseteq \Sigma(k)$ we define $L_S(\omega)$ for a quasicharacter $\omega$ of $k_\Lambda^*$ trivial on $k^*$ to be the product taken over all $v \in S$ of the local L-functions.

We can now define for $\varphi \in \mathscr{H}_S(\chi_S)$, $\sigma(\chi_{0^0}) > n$

$$\psi_S(\varphi, e, \chi) = \lim_{S_1 \supset \eta, S_1 \supset S} \left\{ \sum_{g \in \tilde{M}_{0,S_1}/\tilde{H}_{n,S_1}} c(r_1, r_2; e', \chi, \eta) \cdot (\eta_S) \right\}$$

$$\times \prod_{v \in S_1 - S} \left[ \sum_{w \in (\tilde{M}_v^+ \cap k_v^*)/(\tilde{H}_{n_S} \cap k_v^*)} \tilde{\tau}(e_v, w, \chi_v, \eta_v, \Lambda_{\chi_v}) \right].$$

The corresponding construction with respect to $G(r_2, r_1)$ will be denoted by $\psi'$.

**Theorem 5.1.** The functional $\psi_S(e)$ has a meromorphic continuation to $\Omega^\text{exc}_S(G, S)$ as an integral function of finite order. It satisfies the functional equation

$$\psi'(\tau_S(\varphi, e', \chi'v^{-2+4(n_1+r_2)}) = \psi_S(\varphi, e, \chi).$$
The only singularity of $\psi_S(\varphi, e, \chi)$ in

$$\{ \chi: \sigma(\chi^n) > 0 \}$$

is on \( \{ \chi: \chi^n_{x_0} \parallel \parallel_A \} \). Let

$$T^*(S, r_1, r_2) = \prod_{\upsilon \in S-S_\chi(k)} \prod_{1 \leq i \leq r_1} (1 - q^{-i-j})$$

$$\cdot \prod_{1 \leq k \leq r_1} (1 - q^{-k}) \prod_{1 \leq l \leq r_2} (1 - q^{-l}) \prod_{1 \leq m \leq r_1 + r_2} (1 - q^{-m}).$$

Then if $\chi_0$ is such that $(\chi_0)^{x_0}_{y_0} = \parallel \parallel_A$ one has

$$\lim_{\chi \to \chi_0} L(\chi^n_{x_0})^{-1} \cdot \psi_S(\varphi, e, \chi)$$

$$= \sum_{\eta \in M_0/S/\hat{B}_{n,S}} c(r_1 + r_2; e, \chi_0, \eta)\varphi(\chi_0, \eta) T^*(S, r_1, r_2) \prod_{1 \leq i \leq r_1} \zeta(i + j).$$

**REMARKS.**

1. The analytic continuation, functional equation and, at the moment, hypothetical bounds for $c(r_1, r_2)$ would yield via the Phragmén-Lindelöf theorem (in the number-field case) or the maximum principle (in the function-field case) bounds for $c(r_1 + r_2)$. These bounds would have amongst other consequences for large enough $\sigma(\chi^n_{x_0})$ that one could define $\psi_S(\varphi, e, \chi)$ by following expression which would be more satisfactory than the one we have had to use:

$$\left\{ \sum_{\eta \in \hat{B}_{n,A} / \hat{B}_{n,A}} c(r_1, r_2; e', \chi, \eta)\varphi(\eta_S) \right\}$$

$$\cdot \prod_{\upsilon \in S-S_\chi} \left[ \sum_{w \in M_0^+ \cap \hat{K}_n^+ \cap \hat{K}_n^+} \hat{\tau}(e, w, \chi, w^{-1}, (s(G)\eta)_e w_0) \right].$$

This estimate would have other useful consequences. It will be give in the later papers of this series.

2. In the case of a function-field an integral function of finite order is to be understood as a ‘rational function’.
3. One could investigate the further singularities of $\psi_s(e)$ by related methods. We shall not undertake this investigation here, as it plays no role in the further investigation of the $c$.

**Proof.** The meromorphic continuation follows more or less directly from Theorem 4.1 and [6] Lemma I.3.1 which shows that pointwise in $\chi$ any $\varphi \in \mathcal{H}_s(\chi)$ can be represented by a suitable expression of the form:

$$\int_{U_S} \tilde{e}_S(u) \langle \lambda_{nS}, s(G)^{-1} \cdot u \cdot f_S \rangle \, du \, T(S)^{-1}.$$ 

That the function is integral of finite order follows from the estimates of the truncated Eisenstein series given by the Maaß-Selberg relations – cf. [5] §6. The functional equation follows from the functional equation for the Eisenstein series ([5] §§6,7) and the definition of $\tau_S$.

Finally we can derive the final formula by substitution in the definition of $c(r_1 + r_2)$ combined with the identification of $E^*$ with a residue of the full Eisenstein series.

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**References**