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On the Siegel modular function field of degree three

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Introduction

Let $H_n$ be the Siegel space of degree $n$, and let $\Gamma_n$ be the modular group. A (Siegel) modular function $f$ is defined to be a meromorphic function on $H_n$ which is invariant under $\Gamma_n$, where for $n = 1$, we need an additional condition that $f$ is meromorphic also at the cusp. Let $K_n$ denote the Siegel modular function field over $\mathbb{Q}$, namely the field generated over $\mathbb{Q}$ by modular functions with the rational Fourier coefficients. Then the modular function field is given by $K_n \cong \mathbb{Q}C$. When $n = 1$, namely the elliptic modular case, it is well-known that $K_1$ is generated by the absolute invariant, which has a nice arithmetic property, e.g. an elliptic curve $E$ has a model over the field generated over $\mathbb{Q}$ by its special value attached to $E$. In the higher dimensional case, several ways to get $K_n$ are known: for example, Siegel [16], [18] showed that $K_n$ is generated by $E_{kl}/E_k^l$ (even $k > n + 1$, $l = 1, 2, \ldots$) where $E_k$ denotes the Eisenstein series of weight $k$. Besides this, if we denote by $K(\Gamma_n(l))$ the modular function field for the principal congruence subgroup $\Gamma_n(l)$ of level $l$, then it is shown (Siegel [17]) that $K(\Gamma_n(l))$, $l \geq 3$, is generated by ratios of theta constants. Then $K_n$ is given as the invariant subfield $K(\Gamma_n(l))^{\Gamma_n/\Gamma_n(l)}$. However, these methods seem not very effective to get a finite number of generators explicitly. In the case of $K_2$, Igusa determined three generators in his paper [3], [4], where they are written by Eisenstein series, or also by theta constants. In particular, $K_2$ is shown to be purely transcendental. In a previous paper [19], we gave 34 generators of the graded ring of Siegel modular forms of degree three. By this, we are able to find generators of $K_3$ systematically. However, a systematic calculation gives too many (actually thrity three) generators. The purpose of the presence paper is to give seven generators of $K_3$ explicitly, which are ratios of modular forms of weight at most 30.

The quotient space $H_3/\Gamma_3$ is naturally equipped with the structure of the moduli variety over $\mathbb{Q}$, of three-dimensional principally polarized Abelian
It is still an open problem if the number of generators of $K_3$ can
be reduced one more, to six, which amounts to the rationality problem of
$H_3/\Gamma_3$ since $K_3$ is the rational function field of the variety $H_3/\Gamma_3$. The moduli
variety of curves of genus three is regarded as an open subvariety of $H_3/\Gamma_3$
by means of the Torelli map. Using the moduli theory of curves, Riemann
[11], Weber [20], Frobenius [2] studied $K(\Gamma_3(2))$. They showed the rationality
of the variety $H_3/\Gamma_3(2)$, and moreover gave six generators of $K(\Gamma_3(2))$
explicitly written in terms of derivatives of odd theta functions at the origin.
Prof. R. Sasaki has given a nice mimeograph [12] surveying this topic. So
$H_3/\Gamma_3$ is a unirational variety with a Galois covering of a rational variety of
degree $[\Gamma_3:\Gamma_3(2)] = 1451520$, in other words, $K_3$ has a Galois extension of
degree 1451520 which is purely transcendental. Also by the moduli theory
of curves, $H_3/\Gamma_3$ is proved to be even stably rational (Kollár and Schreyer
[6], see also Bogomolov and Katsylo [1]).

In some cases, generators of $K_n$ work as the absolute invariant of the
elliptic modular case. More precisely by Shimura [13], [14] it is shown that
if a principally polarized Abelian variety $A$ is with sufficiently many complex
multiplication, under a certain condition, or generic of odd dimension (our case),
then $A$ has a model over the field generated over $\mathbb{Q}$ by their special
values attached to $A$ (see also [15], Theorem 9.5, Corollary 9.6). The author
hopes that the result of the present paper will be of use for study of the
rationality problem of $H_3/\Gamma_3$, or for that of arithmetic properties of three-
dimensional Abelian varieties.

1. Notation and preliminary

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ denote as usual the ring of integers, the rational number field,
the complex number field respectively. Let $A = \bigoplus A_k, B = \bigoplus B_k$ be graded
$\mathbb{C}$-algebras. Then the tensor product $A \otimes B$ denotes a graded $\mathbb{C}$-algebra
$\bigoplus_k A_k \otimes B_k$. For an integral graded algebra $A$, $F_0(A)$ denotes the field
formed by elements of degree 0 in the field of fractions of $A$. We denote by
$M_{k,l}(\ast)$, the set of $k \times l$ matrices with entries in $\ast$, and by $M_k(\ast)$, the set of
square matrices of size $k$.

Let $H_n$ denote the Siegel space of degree $n$ \{\(Z \in M_n(\mathbb{C})|Z = Z,\)
\(\text{Im } Z > 0\}\}, and let $\Gamma_n$ denote the modular group $Sp_{2n}(Z)$. $\Gamma_n$ acts on $H_n$ by
the usual modular substitution

$$Z \rightarrow MZ = (AZ + B)(CZ + D)^{-1}, \quad M = (A^B) \in \Gamma_n.$$  

$\Gamma_n(l)$ denotes the principal congruence subgroup of level $l$ \{\(M \in \Gamma_n|M \equiv 1_{2n}\)
\(\mod l\}\}, $1_{2n}$ being the identity matrix of size $2n$. For a congruence subgroup
of $\Gamma_n$, a holomorphic function $f$ on $H_n$ is called a (Siegel) modular form for $\Gamma$ of weight $k$ if $f$ satisfies

$$f(MZ) = |CZ| + D^{|ijf(Z)|} \text{ for } M \in \Gamma$$

and if $f$ is holomorphic also at cusps which is automatic when $n > 1$. In the present paper, weight $k$ of a modular form is always supposed to be even. $A(\Gamma)_k$ denotes the vector space of modular forms of weight $k$, and $A(\Gamma) = \bigoplus A(\Gamma)_k$, the graded ring of modular forms. For $f \in A(\Gamma)_k$, and for $M \in \Gamma_n$, we define $(Mf)(Z)$ to be $|CZ| + D^{-|f|}(MZ)$.

Let $m = \binom{m'}{m''} \in M_{2,n}(\mathbb{Z})$. We define a theta function with a theta characteristic $m$ by setting

$$\theta[m](Z, x) = \sum_{g \in \mathbb{Z}^n} e^{\left(\frac{1}{2}(g + \frac{1}{2}m')Z'(g + \frac{1}{2}m') + (g + \frac{1}{2}m')^2(x + \frac{1}{2}m'')\right)}$$

where $x = (x_1, \ldots, x_n)$ is a variable on $\mathbb{C}^n$, and $e(\cdot) = \exp(2\pi i \cdot)$. $m$ is called even or odd according as $e(\frac{1}{2}m'm'')$ equals 1 or $-1$. We put $\theta[m](Z) = \theta[m](Z, 0)$, which is called a theta constant and which is not identically zero if and only if $m$ is even. $\theta[m](Z)$ has the integral Fourier coefficients. If $m$ is odd, then $(1/2\pi)\partial/(\partial x)\theta[m](Z, 0)$ does not vanish identically and has the integral Fourier coefficients.

Let $\xi_0, \ldots, \xi_{r-1}$ be variables, and let $h$ be a homogeneous polynomial in $\xi_0, \ldots, \xi_{r-1}$, of degree $k$ in $\xi_0$, and of degree $s$ in each of $\xi_1, \ldots, \xi_{r-1}$ such that the identity

$$h \left( \frac{a_{i,j} + b}{c_{i,j} + d}, \ldots \right) = (c\xi_0 + d)^{-k} \prod_{i=1}^{r-1} (c\xi_i + d)^{-s}h(\ldots, \xi, \ldots)$$

is satisfied for $(a_{ij}) \in SL_2(\mathbb{C})$. Let $S(\mathfrak{r})$ denote the $\mathbb{C}$-algebra of such $h$ with $k = s$. $S(\mathfrak{r})$ becomes a graded $\mathbb{C}$-algebra in terms of $s$. $S(2, r)$ is defined to be a subring of $S(\mathfrak{r})$ composed of $h$ which is symmetric in $\xi_0, \ldots, \xi_{r-1}$, namely $S(2, r)$ is the invariant subring $S(\mathfrak{r})^\mathfrak{S}_r$ where the symmetric group $\mathfrak{S}_r$ acts naturally on $\xi_0, \ldots, \xi_{r-1}$ as permutations. $S(2, r)$ is nothing else but the graded ring of invariants of a binary $r$-form (cf. Tsuyumine [19], Sect. 1), and its homogeneous element is called a (projective) invariant.

An element $h$ satisfying (1) is called a $(k, s)$-covariant if $h$ is symmetric in $\xi_1, \ldots, \xi_{r-1}$. The ring of $(s, s)$-covariants ($s \geq 0$) is equal to $S(\mathfrak{r})^{\mathfrak{S}_{r-1}}$ where $\mathfrak{S}_{r-1}$ acts on $\xi_1, \ldots, \xi_{r-1}$ as permutations. We have inclusions of rings; $S(2, r) \subset S(\mathfrak{r})^{\mathfrak{S}_{r-1}} \subset S(\mathfrak{r})$. 


2. Modular forms of degree three

Let us recall some structures of the graded ring $A(\Gamma_3)$ of modular forms of degree three. The details are found in Tsuyumine [19]. For simplicity we write $A$ for $A(\Gamma_3)$ in what follows.

We decompose $Z \in H_3$ into

$$Z = \left( \begin{array}{c} Z_1 \\ \tau \\ Z_3 \end{array} \right), \quad Z_1 = \left( \begin{array}{cc} z_1 & z_{12} \\ z_{12} & z_2 \end{array} \right) \in H_2, \quad z_3 \in H_1, \quad \tau = \left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right) \in \mathbb{C}^2.$$  

$R$ denotes the subset of $H_3$ given by $\tau = 0$. A point of $H_3$ equivalent to some point in $R$ is called reducible, and the set of images of such points by the canonical projection of $H_3$ to $H_3/\Gamma_3$ is its algebraic subset, and called the reducible locus. Let $V \subset H_3$ denote the irreducible component of zeros of a theta constant $\theta[111]$ which contains $R$. For a modular form $f \in A$, we define $v(f)$ to be the vanishing order of $f|_V$ at $R$ ($v(f) = \infty$ if $f|_V \equiv 0$). $v(f)$ is called the order of $f$. If $f|_V \neq 0$, then $v(f)$ is a non-negative even integer since $f$ is of even weight, namely $f$ is invariant by changing $i$ for $-i$. For even $v \geq 0$, we define $A(v)$ to be a graded ideal generated by modular forms $f$ with $v(f) = v$. We have a sequence of inclusions $A = A(0) \supset A(2) \supset A(4) \supset \cdots$. Let

$$\chi_{18}(Z) = \prod_{m \text{ even}} \theta[m](Z).$$

Then $\chi_{18}$ is a modular form of weight 18, and it is a prime element of the ring $A$ (Igusa [5]). If $f \in A$ vanishes identically on $V$, then $f$ is divisible by $\chi_{18}$, i.e., $f/\chi_{18}$ is an element of $A$. $\chi_{18}$ is involved in every $A(v)$. Let us put

$$\bar{A}(v) = A(v)/A(v + 2).$$

$\bar{A}(0)$ is a graded $\mathbb{C}$-algebra and $\bar{A}(v)$'s can be regarded as $\bar{A}(0)$-modules. We have an isomorphism

$$A/(\chi_{18}) \simeq \bar{A}(0) \oplus \bar{A}(2) \oplus \cdots$$  \hspace{1cm} (2)

of vector spaces, or more strongly, of (infinite) graded modules over some ring of Krull dimension five. If $f$ is a modular form of weight $k$ with $v(f) > \frac{3}{2}k$, then $f$ vanishes identically on $V$ ([19], Cor. 2 to Prop. 7) and hence $f$ is divisible by $\chi_{18}$. So the vector space $(A/(\chi_{18}))_k$ corresponding to modular forms of weight $k$ is isomorphic to the direct sum $\bar{A}(0)_k \oplus \bar{A}(2)_k \oplus \cdots \oplus A([\frac{3}{2}k])'_k$, $[\frac{3}{2}k]'$ denoting the maximal even integer not exceeding $\frac{3}{2}k$. To know the structure of $\bar{A}(v)$, we exhibit them as
subspaces of $A(\tau_2) \otimes A(\Gamma_1)$ in the following way where $\Gamma_2$ is the maximal congruence subgroup of $\Gamma_2$ which stabilizes an odd theta characteristic $(\frac{1}{10})$ mod 2.

Suppose that $g$ is a meromorphic modular form, but holomorphic on $V - \Gamma_3 R$, $\Gamma_3 R$ being the union $\cup M \cdot R$, $M \in \Gamma_3$, and that $g|_{v - \Gamma_3 R}$ is locally bounded at $R$, hence at $\Gamma_3 R \cap V$. For such $g$, and for $(Z_1, z_3) \in H_2 \times H_1$ we define

$$(\Psi g)(Z_1, z_3) = \lim_{Z \to Z_0} g(Z), \quad Z_0 = \begin{pmatrix} Z_1 & 0 \\ 0 & z_3 \end{pmatrix} \in R.$$ 

By Riemann's removable singularity theorem $g|_{v - \Gamma_3 R}$ extends to a holomorphic function on $V$, and hence $\Psi g$ is well-defined. $\Psi g$ is an element of the tensor product $A(\tau_2) \otimes A(\Gamma_1)$ ([19], Sect. 14). Let $\chi_{28}$ be a modular form of weight 28 defined in Section 5 of the present paper (or [19], Sect. 22). It is a modular form of lowest weight having the property that $\chi_{28}|_{v}$ vanishes only at $\Gamma_3 R \cap V$. Its order $v(\chi_{28})$ is eight. Now let us fix three modular forms $\beta', \gamma, \delta$ with $\beta' \in A(2) - A(4)$, $\gamma \in A(4) - A(6)$, $\delta \in A(6) - A(8)$. Then if $f \in A$ is of order $v \equiv 0 \mod 8$ (resp. 2, 4, 6 mod 8), then

$$f|_{\chi_{28}}$$

is obviously holomorphic on $V - \Gamma_3 R$ and moreover its restriction to $V - \Gamma_3 R$ is locally bounded at $R$ ([19], Sect. 13). So its image by $\Psi$ is well-defined. We denote by $\Psi(v)$, the map $f \mapsto \Psi(f|_{\chi_{28}^{(8)}})$ (resp. $\Psi(f|_{\chi_{28}^{(v+6)/8}})$, $\Psi(f|_{\chi_{28}^{(v+4)/8}})$, $\Psi(f|_{\chi_{28}^{(v+2)/8}})$), where we shall write simply $\Psi$ instead of $\Psi(0)$. (In [19], we have taken as $\beta', \gamma, \delta$, some particular modular forms.) $\Psi(v)$ is a map of $A(v)$ to $A(\Gamma_2) \otimes A(\Gamma_1)$, and by definition the kernel of $\Psi(v)$ is just $A(v + 2)$. So $\Psi(0)$ is also considered to be an embedding of $\tilde{A}(v)$ to $A(\Gamma_2) \otimes A(\Gamma_1)$. By definition $(\Psi(v))(Z_1, z_3) = f(Z_1/z_3)$, hence $\Psi(\tilde{A}(0))$ is contained in $A(\Gamma_2) \otimes A(\Gamma_1)$. If we identify $\tilde{A}(0)$ with $\Psi(\tilde{A}(0))$, then the map $\Psi(v)$ of $\tilde{A}(v)$ to $A(\Gamma_2) \otimes A(\Gamma_1)$ can be regarded as an $A(0)$-module morphism since $\Psi(v)(fg) = \Psi(f) \cdot \Psi(v)g$ for $f \in A$, $g \in A(v)$. $\tilde{A}(0) \subseteq A(\Gamma_2) \otimes A(\Gamma_1)$ is equal to $\{ \Sigma j \otimes j \in A(\Gamma_2) \otimes A(\Gamma_1) | \Sigma j \otimes j(z_3) \text{ is symmetric in } z_1, z_2, z_3 \}$ ([19], Sect. 16), over which $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite as a module, hence $A(\Gamma_2) \otimes A(\Gamma_1)$ is. Since $\chi_{28}A(v) \subseteq A(v + 8)$, we have sequences of inclusions of $A(0)$-submodules of $A(\Gamma_2) \otimes A(\Gamma_1)$ by definition of $\Psi(v)$;

$$\Psi(\tilde{A}(0)) \subset \Psi(8)\tilde{A}(8) \subset \cdots$$

$$\Psi(2)\tilde{A}(2) \subset \Psi(10)\tilde{A}(10) \subset \cdots$$
Since $A(\Gamma_1) \otimes A(\Gamma_1)$ is a Noetherian $A(0)$-module, there is a positive even integer $v_0$ such that if $v \geq v_0$, then $\Psi(v)A(v) = \Psi(v - 8)A(v - 8)$, in other words

$$A(v) = \chi_{28}A(v - 8) \quad \text{for} \quad v \geq v_0. \quad (3)$$

Then it is not difficult to see that any modular form $f \in A(v)$, $v \geq v_0$, is written as $f = g\chi_{18} + h\chi_{18}$ for some $g, h \in A$, combining (3) with the fact that $f$ is divisible by $\chi_{18}$ if $v(f) > \frac{v}{2}$ weight $(f)$. $v_0$ is actually taken to be 14, and hence the isomorphism (2) becomes

$$A/(\chi_{18}) \simeq A(0) \oplus A(2) \oplus A(4) \oplus \left( \bigoplus_{\mu=0}^{\infty} (A(6) \oplus A(8) \oplus A(10) \oplus A(12))\chi_{28}^{\mu} \right).$$

All the structures of $A(v)$, $v \leq 12$, have been determined in [19], and from this the structure of $A/(\chi_{18})$ is given, and that of $A$ is too.

Finally in this section we give a comment on an alternate definition of $\Phi(2)$. Restricting to $V$, the Taylor expansion of $\theta[110](Z)$ at $Z_0 = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}$ in terms of $\tau$, we get

$$0 = \sum_{i=1}^{2} \left( \frac{\partial}{\partial \tau_i} \theta[110](Z_1, 0)(\theta[0][\theta[0][\theta[0)])(z_3) \right) \tau_i$$

$$+ \text{(higher degree terms of } \tau).$$

At least one of $\partial/(\partial \tau_i)\theta[110](Z_1, 0)$ is not zero since the theta divisor of degree two is nonsingular, and $\theta[0][\theta[0][\theta[0]$ vanishes nowhere on $H_1$. Hence one of the $\tau_i$ is written as an analytic function of another on some neighborhood at $Z_0$. Let $f \in A(2)$. Substituting it in the expansion of $(f \delta/\chi_{28})_v$ in terms of $\tau$, and taking the limit as $\tau_i \to 0$, we get

$$(\Phi(2)f)(Z_1, z_3) = (F_2F_6/F_8)(Z_1, z_3)$$
where

\[
F_2(Z_1, z_3) = \frac{1}{2!(2\pi i)^4(\sqrt{-1})^2} \sum_{l=0}^{\frac{n}{2}} (-1)^l \binom{2}{l} \frac{\partial}{\partial \tau^l_1} \frac{\partial}{\partial \tau^l_2} f(Z_1)
\]

\[
\times \left( \frac{\partial}{\partial x_1} \theta[\frac{1}{i}] \right)^{2-l} \left( \frac{\partial}{\partial x_2} \theta[\frac{1}{i}] \right)^l,
\]

\[
F_6(Z_1, z_3) = \frac{1}{6!(2\pi i)^6(\sqrt{-1})^6} \sum_{l=0}^{6} (-1)^l \binom{6}{l} \frac{\partial}{\partial \tau^l_1} \frac{\partial}{\partial \tau^l_2} \delta(Z_1)
\]

\[
\times \left( \frac{\partial}{\partial x_1} \theta[\frac{1}{i}] \right)^{6-l} \left( \frac{\partial}{\partial x_2} \theta[\frac{1}{i}] \right)^l,
\]

\[
F_8(Z_1, z_3) = \frac{1}{8!(2\pi i)^8(\sqrt{-1})^8} \sum_{l=0}^{8} (-1)^l \binom{8}{l} \frac{\partial}{\partial \tau^l_1} \frac{\partial}{\partial \tau^l_2} \chi_{28}(Z_1)
\]

\[
\times \left( \frac{\partial}{\partial x_1} \theta[\frac{1}{i}] \right)^{8-l} \left( \frac{\partial}{\partial x_2} \theta[\frac{1}{i}] \right)^l
\]

(\(\binom{\cdot}{\cdot}\)) denoting a binomial coefficient. \(\Psi(2)f\) is holomorphic and has a Fourier expansion on \(H_2 \times H_1\), and each of \(F_2, F_6, F_8\) has too. By definition \(30\chi_{28}\) has integral Fourier coefficients. Now let us suppose that \(\delta\) has rational Fourier coefficients (with a bounded denominator). Then both of \(F_6, F_8\) have rational Fourier coefficients (with a bounded denominator). Hence there is a rational number \(N\) such that \(N\Psi(2)f\) has integral Fourier coefficients if and only if \(F_2\) does. In particular, for such \(N\), \(2N\Psi(2)f\) has the integral Fourier coefficients if \(f\) does.

Let us calculate a first term of \(F_2\) explicitly in terms of the Fourier coefficient of \(f \in A(2)\) for the identity matrix, i.e. for \(e(\text{tr}(Z))\). There are 23 positive symmetric semi-integral ternary matrices with merely one as their diagonal components, each of which is equivalent under the action \(S \rightarrow {}^tUSU, U \in GL_3(\mathbb{Z})\), to one of the following three matrices; the identity matrix; the matrix with 0 as its (1, 2), (1, 3)-components and with \(1/2\) as its (2, 3)-component; the matrix with 0 as its (1, 2)-component and with \(1/2\) as its (1, 3), (2, 3)-components. Let \(a_0, a_1, a_2\) be the Fourier coefficients of \(f\) corresponding to the first, second, third matrix respectively. From \(\Psi\beta = 0\), two relations among \(a_0, a_1, a_2\) are derived; \(a_0 + 4a_1 + 4a_2 = a_1 + 6a_2 = 0\), hence \(a_0:a_1:a_2 = 20:-6:1\) if \(a_0 \neq 0\). Then a direct calculation shows

\[
F_2(Z_1, z_3) = -\frac{2}{3} a_0 e(\text{tr}(\pm \frac{1}{4} \pm \frac{3}{4})Z_1))e(z_3) + \cdots
\]
3. A subring of $A(\Gamma_3)$

$\Gamma_2/\Gamma_2(2)$ is isomorphic to the symmetric group $S_6$ of degree six, and it acts on the set of six odd theta characteristics (mod 2) of degree two as permutations. $\Gamma_2$ has been defined to be a stabilizer subgroup of $\Gamma_2$ at an odd theta characteristic $\binom{11}{10}$, and hence $\Gamma_2/\Gamma_2(2)$ is isomorphic to $S_5$.

There is an injective homomorphism $\varphi_2$ of $A(\Gamma_2(2))$ to $S(6) \subset \mathbb{C}[\zeta_0, \ldots, \zeta_5]$ which is equivalent under $S_6$ (Igusa [5], Tsuyumine [19], Sect. 9, 11), where $\varphi_2$ induces an isomorphism between the field of fractions of $A(\Gamma_2(2))$ and that of $S(6)^{(2)}$, $S(6)^{(2)}$ denoting the subring of $S(6)$ consisting of homogeneous elements of even degree. We may assume that $S_5 \simeq \Gamma_2/\Gamma_2(2)$ acts on $\{\xi_1, \ldots, \xi_5\}$ as permutations. Hence we have a commutative diagram:

$$A(\Gamma_2(2)) \xrightarrow{\varphi_2} S(6) \supseteq A(\Gamma_2) \xrightarrow{\varphi_2} S(6)^{S_5} \supseteq A(\Gamma_2) \xrightarrow{\varphi_2} S(2, 6) = S(6)^{S_6}.$$ 

In particular, there is no proper intermediate field between $F_0(A(\Gamma_2^2))$ and $F_0(A(\Gamma_2))$, and hence $F_0(A(\Gamma_2^2)) = F_0(A(\Gamma_2))$ for any $\psi \in A(\Gamma_2^2) - A(\Gamma_2)$.

**Lemma 1.** Let $\beta$ be a modular form for $\Gamma_3$ of order $v$ with $v \equiv 2$ or 6 mod 8. Let us fix $z_3 \in H_1$ so that $\psi(z_1) := (\Psi(4v)\beta^4)(z_1, z_3)$ is not identically zero. Then $\psi \notin A(\Gamma_2)$. In particular, $F_0((A(\Gamma_2) \otimes A(\Gamma_1))[\Psi(4v)\beta^4]) = F_0(A(\Gamma_2) \otimes A(\Gamma_1))$.

**Proof.** We treat only the case $\nu \equiv 2$ mod 8, since a similar argument is applicable to the case $\nu \equiv 6$ mod 8. By the argument [19], Sect. 14, the proof of Lemma 12, $\varphi_2 \phi$ is the form $H^4\mathcal{D}_0$ where $H$ is an $(s + 2, s)$-covariant and $\mathcal{D}_0$ denotes the $(0, 8)$-covariant $\Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$. It is enough to show that $H^4\mathcal{D}_0 \not\in S(2, 6)$. Suppose otherwise. Dividing $H^4$ by a power of the discriminant $\Pi_{0 \leq i < j \leq 5}(\xi_i - \xi_j)^2 \in S(2, 6)$ if necessary, we may assume that $H$ is not divisible by $\Pi_{0 \leq i < j \leq 5}(\xi_i - \xi_j)^2$. Since $H^4\mathcal{D}_0$ obviously has factors $(\xi_i - \xi_j)^2$ ($1 \leq i < j \leq 5$) and since $H^4\mathcal{D}_0$ is symmetric in $\xi_0, \ldots, \xi_5$ by our assumption, it has a factor $\Pi_{i=1}^5(\xi_0 - \xi_i)^2$. Then $H$ is divisible by $\Pi_{i=1}^5(\xi_0 - \xi_i)$, and hence $H^4\mathcal{D}_0$, by $\Pi_{i=1}^5(\xi_0 - \xi_i)^4 \times \Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$. Again by symmetry $H^4\mathcal{D}_0/\Pi_{i=0}^5(\xi_0 - \xi_i)^4 \times \Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$ is still divisible by...
Let $\Lambda$ be a graded subring of $A$ such that $A(2) \otimes A(1)$ is finite integral over $\overline{\Lambda} := \Psi \Lambda$, and that $\chi_{28}, \chi_{18} \in \Lambda$.

**Lemma 2.** $A$ is finite integral over $\Lambda$.

**Proof.** $\Psi(v)A(v)$ is a finite $\overline{\Lambda}$-module for every even $v \geq 0$. Let $\{f_{i,v}\}_i$ be a finite number of modular forms in $A(v)$ such that $\{\Psi(v)f_{i,v}\}_i$ generates $\Psi(v)A(v)$ over $\overline{\Lambda}$. We show that $A$ is generated as a $\Lambda$-module, by $f_{i,v}$’s with $v \leq v_0$, $v_0$ being as in (3).

We prove that any modular form $f$ of weight $k$ is written as a linear combination of $f_{i,v}$’s ($v \leq v_0$) over $\Lambda$, by induction on $k$. $\Psi f \in \Psi A(0)$ is written as $\Psi f = \Sigma \Psi(P_i f_{i,0})$ with $P_i \in \Lambda$. By taking $f - \Sigma P_i f_{i,0}$ instead of $f$, we may assume $\Psi f = 0$, namely $\nu(f) \geq 2$. Then $\Psi(2)f$ is written as $\Sigma \Psi(2)(P'_i f_{i,2})$ with $P'_i \in \Lambda$. By a similar argument as above, we may assume $\Psi(2)f = 0$, and by a recursive argument, we may assume $\nu(f) > \frac{k}{2}$, where we make use of such elements as $\chi_{28} f_{i,v}$ ($m > 0$) instead of $f_{i,v}$ if the order $\nu(f)$ exceeds $v_0$. Then $f|_v$ vanishes identically and $f$ is written as $f = g \chi_{18}$ for some $g \in A$. By the induction hypothesis $g$ is a linear combination of $f_{i,v}$’s ($v \leq v_0$) over $\Lambda$, and hence $f$ is.

**Q.E.D.**

**Corollary.** $A(v)$ is a finite $\Lambda$-module for any even $v \geq 0$.

**Proposition 1.** Let $\Lambda$ be a graded subring of $A$ containing $\chi_{28}, \chi_{18}$ such that $A(2) \otimes A(1)$ is finite integral over $\overline{\Lambda} := \Psi \Lambda$, and that $g.c.d \{k | \chi_k \neq 0\} = 2$ for $\overline{\Lambda} = \bigoplus \chi_k$. If $\beta$ is a modular form of order two such that $F_0(\Psi(8)\beta^2) = F_0(A(2) \otimes A(1))$, then the modular function field of degree three is given by $F_0(\Lambda[\beta])$.

**Proof.** At first we show that there are a positive integer $v_1$ and a modular form $P \in \Lambda$ of order 0 such that

$$\Psi(v + v')(\beta^{v/2}PA(v)) \subset \Psi(v + v')(\beta^{v/2}(\Lambda[\beta] \cap A(v)))$$

for any even $v \geq v_1$ where $v' \in \{0, 2, 4, 6\}$ is determined by $v + v' \equiv 0 \mod 8$. By our assumption, we can take $P \in \Lambda, \neq 0$ such that $\overline{P}(A(2) \otimes A(1))$ is contained in a $\overline{\Lambda}$-module generated by $\Psi(8)\beta^4, (\Psi(8)\beta^4)^2, \ldots, (\Psi(8)\beta^4)^m$ with $m = [F_0(A(2) \otimes A(1)) : F_0(\Lambda)]$. Since $\Lambda[\beta] \cap A(v)$ has as a subset

$$\sum_{2n_1 + 5n_2 \geq v} \beta^{n_1} \chi_{28} \Lambda,$$
\( \Psi(v + v')(\beta^{v/2}(\Lambda[\beta] \cap A(v)) \) contains the \( \tilde{\Lambda} \)-module generated by \( \Psi(8)\beta^t, \ldots, (\Psi(8)\beta^t)^m \) if \( v \) is large enough. If \( P \in \Lambda \) is such that \( \tilde{P} = \Psi P \), then \( \Psi(v + v')(\beta^{v/2} PA(v)) = \tilde{P}\Psi(v + v')(\beta^{v/2} A(v)) \subset \tilde{P}(A(\Gamma_2) \otimes A(\Gamma_1)) \).

Thus we have proved (5).

\( A(2) \) is the prime ideal of \( A \) defining the reducible locus of \( H_3/\Gamma_3 \), and hence \( A(2) \cap \Lambda[\beta] \) is prime in \( \Lambda[\beta] \). Let us take the ring \( \Lambda_0 := \Lambda[\beta, \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots)] \). The ideal of \( \Lambda_0 \) generated by \( A(2) \cap \Lambda[\beta], \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots) \) is prime since \( \tilde{\Lambda} = \Lambda_0/(A(2) \cap \Lambda[\beta], \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots)) \) is an integral domain. Let \( \Lambda' \) be the localization of \( \Lambda_0 \) at the prime ideal. Let \( v_2 \) be an even integer equal to or greater than each of \( v_0 \) and \( v_1 \), \( v_0 \) being as in (3). Since \( \Lambda \subset \Lambda'_0 \), by Corollary to Lemma 2 there are a finite number of holomorphic modular forms \( f_1, \ldots, f_t \in A(v_2) \) such that \( A(v_2) \subset \Lambda'_0 f_1 + \cdots + \Lambda'_0 f_t \). We may assume that \( \{f_1, \ldots, f_t\} \) is a minimal system with this property. Then we show \( t = 1 \). Suppose \( t \geq 2 \). Since \( v := v(f_i) \) is larger than \( v_1 \), we have \( \Psi(v + v')\beta^{v/2} P f_i = \Psi(v + v')\beta^{v/2} q \) for some \( q \in A(v) \cap \Lambda[\beta] \). Since \( \Psi(v + v')\beta^{v/2} (P f_i - q) = 0 \), the order of \( P f_i - q \) is at least \( v + 2 \). By repeating the similar argument four times, it is shown that there is \( Q \in \Lambda[\beta] \) satisfying the inequality \( v(P^4 f_i - Q) \geq v + 8 \). Since \( v \geq v_0 \), by (3) there are \( g, h \) such that \( P^4 f_i - Q = g\chi_{28}^k + h\chi_{18}^k \). \( v(g) \) is obviously greater than or equal to \( v \), and in particular \( g \in A(v_2) \) because \( v \geq v_2 \). \( h\chi_{28}^k \) is also involved in \( A(v_2) \) if \( k \) is sufficiently large. \( g, h\chi_{28}^k \in A(v_2) \subset A'_0 f_1 + \cdots + A'_0 f_t \) is written as \( g = \Sigma_{i=1}^t a_i f_i, h\chi_{28}^k = \Sigma_{i=1}^t b_i f_i \) with \( a_i, b_i \in \Lambda'_0 \). Hence we have

\[
(P^4 - a_i\chi_{28}^k - b_i\chi_{18}/\chi_{28}^k) f_i = Q + \sum_{i=1}^{t-1} a_i\chi_{28} f_i + \sum_{i=1}^{t-1} b_i(\chi_{18}/\chi_{28}^k) f_i.
\]

Since \( P \) is of order 0, \( P^4 - a_i\chi_{28}^k - b_i\chi_{18}/\chi_{28}^k \) is a unit of the ring \( \Lambda'_0 \). So \( f_i \) is written as a linear combination of other \( f_j \). This contradicts to the minimality of a system of \( \{f_1, \ldots, f_t\} \). Thus \( t = 1 \).

Now we have \( A(v_2) \subset A'_0 f_1 \). \( A(v_2) \) and \( A'_0 \) have a common non-trivial element (e.g., \( \chi_{18} \)). This implies that \( f_1 \) is contained in the field of fractions of \( \Lambda'_0 \), and that \( A(v_2) \) is a subset of the field of fractions of \( \Lambda[\beta] \). Since \( \chi_{28}^k A \subset A(v_2) \) for large \( k \), the modular function field \( F_0(A) \) is equal to \( F_0(\Lambda[\beta]) \).

Q.E.D.

Combining Proposition 1 with Lemma 1, we have the following corollary.

**Corollary.** Let \( \Lambda \) be a ring as in Proposition 1 satisfying the additional condition that \( F_0(\tilde{\Lambda}) = F_0(A(\Gamma_2) \otimes A(\Gamma_1)) \). Let \( \beta \) be any modular form with \( v(\beta) = 2 \). Then the modular function field is given by \( F_0(\Lambda[\beta]) \).
4. Main theorem

$\mathcal{A}(\Gamma_1)$ is generated by two algebraically independent modular forms $j_4, j_6$ of weight 4, 6 respectively where

$$j_4 = \frac{1}{2} \sum_{m: \text{even}} \vartheta[m]^8, \quad j_6 = \sum_{M: \Gamma_1/\Gamma_1(2)} M(\theta[0010]\theta[10]^4).$$

As Igusa [3], [4] showed, $\mathcal{A}(\Gamma_2)$ is generated by four algebraically independent modular forms $\psi_4, \psi_6, \psi_{10}, \psi_{12}$ with their subscript as their weight where

$$\psi_4 = \left(\frac{1}{4} \sum_{m: \text{even}} \vartheta[m]^8, \quad \psi_6 = \frac{1}{2} \sum_{M: \Gamma_1/\Gamma_1(2)} M(\theta[0010]\theta[10]^2 \theta[0101]^2 \theta[111]^2), \right.$$

$$\psi_{10} = \prod_{m: \text{even}} \vartheta[m]^2,$$

$$\psi_{12} = \frac{1}{288} \sum_{M: \Gamma_1/\Gamma_1(2)} M(\theta[0010]\theta[0101]^2 \theta[0101]^2 \theta[111]^2\theta[101]^4)$$

(note that we are considering only modular forms of even weight). Let $\alpha_4, \alpha_6, \alpha_{10}, \alpha_{12}, \alpha_{20}, \alpha_{30} \in A$ be as in Section 5, and let $\alpha'_{20} = (\alpha_{20} - 5\alpha_{10})/7$, $\alpha'_{30} = (7\alpha_{30} - 313\alpha_{10}\alpha_{20} + 865\alpha_{10}^2)/7$. By [19], Section 23, we have

$$\Psi\alpha_4 = \psi_4 \otimes j_4, \quad \Psi\alpha_6 = \psi_6 \otimes j_6, \quad \Psi\alpha_{12} = 3^{-3}\psi_{12} \otimes (-j_6^2 + 4j_4^3),$$

$$\Psi\alpha'_{12} = 2^43^{-3}\psi_4^3 \otimes (-j_6^2 + 4j_4^3)$$

$$- 3^{-3}\psi_6^2 \otimes (-j_6^2 + 4j_4^3) + 3^2\psi_{12} \otimes (j_6^2 + 8j_4^3),$$

$$\Psi\alpha'_{20} = \psi_{10} \otimes j_6^5, \quad \Psi\alpha'_{30} = \psi_{10}^3 \otimes j_6^5.$$

**Lemma 3.** Let $\Lambda$ denote a graded $\mathbb{C}$-algebra generated by $\Psi$-images of i) $\alpha_4, \alpha_6, \alpha_{12}, \alpha_{20}, \alpha_{30}$, or ii) $\alpha_4, \alpha_6, \alpha_{12}, \alpha_{12}', \alpha_{20}, \alpha_{30}, k$ being any fixed positive integer. Then $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\Lambda$, and $F_0(\Lambda)$ equals $F_0(A(\Gamma_2) \otimes A(\Gamma_1))$.

**Proof.** The first assertion follows from the fact that $\Psi\alpha_4, \Psi\alpha_6, \Psi\alpha_{12}, \Psi\alpha_{12}', \Psi\alpha_{20}, \Psi\alpha_{30}$ do not vanish simultaneously at any point of the projective variety $(H_2/\Gamma_2)^* \times (H_1/\Gamma_1)^*, (H_n/\Gamma_n)^*$ denoting the Satake compactification,
which is not difficult to see. We treat only the case ii), because the similar argument is applicable to the case i). Put \( s = (j''_6/j''_3)(\zeta_3) \). Then \( s \) is an element of degree five over \( \mathbb{C}[\Psi \zeta_{30}^2/\Psi \zeta_{20}^3] \). As easily seen, \( F_0(\Lambda(\Gamma_2) \otimes A(\Gamma_1)) \) is an extension over \( F_0(\mathbb{C}[\Psi \zeta_4, \Psi \zeta_6, \Psi \zeta_{12}, \Psi \zeta_{20}, \Psi \zeta_{30}]) \) of degree five, since the former is obtained from the latter by adding \( s \). In particular, the extension is simple. Since an element \( \Psi(\zeta_{12}^k/\zeta_{3}^k) \) is not contained in the latter one, \( F_0(\Lambda) \) equals \( F_0(\Lambda(\Gamma_2) \otimes A(\Gamma_1)) \). Q.E.D.

**Theorem 1.** Let \( \lambda \) be any modular form of weight twenty with \( \nu(\lambda) = 2 \), and let \( c \) be a constant. Let us put \( \Lambda := \mathbb{C}[\zeta_4, \zeta_6, \zeta_{12}, \zeta_{12}', (\zeta_{20} - 5\zeta_{10}^2)/7 + c\lambda, (7\zeta_{30} - 313\zeta_{10}\zeta_{20} + 865\zeta_{10}^3)/7, \chi_{28}, \chi_{18}] \). Then the modular function field of degree three is given by \( F_0(\Lambda) \), except at most one value of \( c \). (See Sect 5 for the definition of modular forms.)

**Remark.** Our argument will show that the assertion of Theorem 1 holds even if we replace \( \Lambda \) by other rings such as \( \mathbb{C}[\zeta_4, \zeta_6, \zeta_{12}, \zeta_{12}', \zeta_{30}, \chi_{28}, \chi_{18}], \mathbb{C}[\zeta_4, \zeta_6, \zeta_{12}, \zeta_{12}', \zeta_{20}, \zeta_{30} + c\lambda, \chi_{28}, \chi_{18}] \) and so on, \( \lambda \) being a modular form of appropriate weight with \( \nu(\lambda) = 2 \).

**Proof.** Let us find an algebraic relation among \( \Psi \zeta_4, \Psi \zeta_6, \Psi \zeta_{12}, \Psi \zeta_{12}', \Psi \zeta_{20} = \Psi(\zeta_{20} + c\lambda), \Psi \zeta_{30} \). Let \( s \) be as in the proof of Lemma 3. If we put

\[
\begin{align*}
 p_0 &= 16\zeta_4^3, \\
 p_1 &= -128\zeta_4^3 - \zeta_6^3 + 243\zeta_{12} + 27\zeta_{12}', \\
 p_2 &= 256\zeta_4^3 + 8\zeta_6^2 + 1944\zeta_{12} - 108\zeta_{12}', \\
 p_3 &= -16\zeta_6^2,
\end{align*}
\]

then we have

\[
(\Psi p_0)s^3 + (\Psi p_1)s^2 + (\Psi p_2)s + \Psi p_3 = 0 \tag{6}
\]

by a direct computation. For an indeterminate \( X \), we put

\[
L(X) = p_3^5(\zeta_{20} + X)^0 + (p_2^5 + 5p_0p_2p_3^2 + 5p_1^2p_2p_3^2 - 5p_0p_1p_3^3 - 5p_1p_2p_3)\zeta_{30}^3(\zeta_{20} + X)^0 + (p_1^5 + 5p_0^2p_1p_3^3 + 5p_0^2p_1p_3^3 - 5p_0^3p_2p_3 - 5p_0^2p_1p_2)(\zeta_{20} + X)^3 + p_0^5\zeta_{30}^6.
\]

\(1 \) Such a detail is not necessary to prove merely Theorem 1. However, it (or \( L(X) \)) will be used for other purposes later.
Then $\Psi L(0)/\Psi_{2 \alpha}$ = 0 is a minimal algebraic relation among $\Psi_4$, $\Psi_6$, $\Psi_{12}$, $\Psi_{14}$ and $(j_2/j_3)^5 (= s^4)$, given by eliminating $s$ from (6). Hence $\Psi L(0) = 0$, which is an algebraic relation among $\Psi_4, \ldots, \Psi_{20}$.

By Lemma 3 $\Lambda$ satisfies the condition in Corollary to Proposition 1. $\beta := L(c)\lambda$ is a modular form contain in $\Lambda$, which equals $L(0) + cL'(0)\lambda$ up to $\Lambda(4)$ where $L'$ is the derivative of $L$ in terms of $X$. Since $L(0), \lambda \in A(2)$, $\beta$ is a modular form of order at least two. Since $L'(0) \in A(0) - A(2)$, we have, except for at most one value of $c$

$$\Psi(2)L(0) + c\Psi L'(0)\lambda \neq 0,$$

i.e., $v(\beta) = 2$. Then by the Corollary to Proposition 1, the modular function field is given by $F_0(\Lambda[\beta]) = F_0(\Lambda)$. Q.E.D.

Let us make $c\lambda$ explicit for which the assertion of Theorem 1 holds. By the above proof it is enough to find $c\lambda$ satisfying (7). From the definition, $\alpha_4$, $2^{-3}\alpha_6, 23^2\alpha_{12}, 2^4 3^{-1} \alpha_{14}, 2^9 3^2 5 \cdot 7 \cdot 11 \alpha_20, 2^{12} 3^3 5^2 7^2 11^2 \alpha_30$ are easily checked to have integral Fourier coefficients. By the way, $30\chi_{28}, \chi_{18}$ have too. Let $N$ be the rational number given in the last part of Section 2. Since $2^{106} 3^{24} 5^{16} 7^{16} 11^{24} L(0)$ has integral Fourier coefficients, also $2N$ times its $\Psi(2)$-image does. So (7) holds if $2N \cdot 2^{106} 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0) \Psi(2)\lambda$ has a non-integral Fourier coefficient.

We take as $\lambda, \alpha_{6} \beta_{14}$ where $\beta_{14}$ is a cusp form of weight 14 and of order two which is defined in Section 5 (or, also in [19], Sect. 24). $\Psi(2)\alpha_{6} \beta_{14}$ equals $\Psi_{24} \Psi(2) \beta_{14}$. Now we must find a rational number $c$ such that $2^{106} 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0) \Psi_{24} \beta_{14}$ has a non-integral Fourier coefficient, $F_2$ being the one given for $f = \beta_{14}$ in (4), which implies (7). $\alpha_6$ has the Fourier expansion starting from the constant term 8, and a direct calculation shows that $\Psi L'(0)$ has the Fourier expansion starting from

$$-2^{34} 3^7 5^2 \{2e(\text{tr}(Z_1)) - e(\text{tr}(\pm_1 Z_1))\}^{16} e(2z_3).$$

Let $a$ be the Fourier coefficient of $\beta_{14}$ for $e(\text{tr}(Z))$. Combining the above calculation with that of the last part of Section 2, $2^{106} 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0) \Psi_{24} \beta_{14}$ is shown to have $2^{34} 3^{17} 5^7 7^{16} 11^{24} ac$ as a Fourier coefficient. Here we give a rough estimate of $a$. $\beta_{14}$ is written as a sum of 2160 products with sign, of 28 theta constants, where each of products has the Fourier expansion starting from the terms corresponding to positive semi-integral ternary matrices with their diagonal components $\geq 1$. From this, $a \in Z$, and a rough estimate shows $|a| < 2160 \times 2^{3 \times 8} = 2^{28} 3^4 5$. On the other hand $a \neq 0$ is shown in the following way. So if $c$ is a rational number such that
Let us prove $a \neq 0$. Let $E_{k,n}$ denote the normalized Eisenstein series of degree $n$ and of weight $k$, where 'normalized' implies that its constant term is one. By the structure theorem of $A(\Gamma_2)$ (Igusa [3], [4], [5]) and by the formulas for the Fourier coefficients of Eisenstein series of degree two in Maass [7], Satz 1, the identity $3\cdot 7 \cdot 11 \cdot 659E_{4,2}^2E_{6,2} - 2^2269 \cdot 43867E_{4,2}E_{10,2} + 53 \cdot 657931E_{14,2} = 0$ follows. Hence

$$3^37 \cdot 11 \cdot 659E_{4,3}^2E_{6,3} - 2^2269 \cdot 43867E_{4,3}E_{10,3} + 53 \cdot 657931E_{14,3}$$

(8)
is a cusp form of weight fourteen where $E_{4,3}$ is well-defined by Raghavan [10]. By virtue of Ozeki and Washio [8], [9], the Fourier coefficient of (8) for $e(tr(Z))$ can be calculated, namely $-2^33^5\cdot 7^2 \cdot 11 \cdot 79973$. By [19], the vector space of cusp forms of weight 14 is one-dimensional, and hence (8) and $\beta_{14}$ are proportional. Thus $a \neq 0$. We have proved the following theorem.

**Theorem 2.** The Siegel modular function field $K_3$ of degree three over $\mathbb{Q}$ is generated by the following seven modular functions; $\alpha_{12}/\alpha_3$, $\alpha_{12}/\alpha_2$, $\alpha_{12}/\alpha_1$, $(\alpha_{20} - 5\alpha_{10}^2 + 7c\alpha_6\beta_{14})/\alpha_4$, $(7\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^3)/\alpha_6\alpha_4$, $\chi_{28}/\alpha_4$, $\chi_{18}/\alpha_2\alpha_6$ where $c$ is any rational number exclusive of at most one value. If $c$ is such that $2^{364}3^{31}5^{17}7^{16}11^{24}ac \notin \mathbb{Z}$ for any positive integer $a$ less than $2^{28}3^45$, then our assertion holds. (see Sect. 5 for the definition of modular forms).

**Remark**

i) In Theorem 2, we may replace $\alpha_{12}/\alpha_3$ or $\alpha_{12}/\alpha_2$ by its power for general $c \in \mathbb{Q}$. This implies for example, that $K_3$ is not a cyclic extension of $\mathbb{Q}(\alpha_6/\alpha_3, \alpha_{12}/\alpha_4, (\alpha_{20} - 5\alpha_{10}^2 + 7c\alpha_6\beta_{14})/\alpha_4, (7\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^3)/(\alpha_4^2\alpha_6, \chi_{28}/\alpha_4, \chi_{18}/\alpha_2\alpha_6)$ unless the extension is trivial.

ii) In Theorem 2 we can replace $\beta_{14}$ by the cusp form (8). Then $c$ is taken to be a rational number such that $2^{371}3^{39}5^{19}7^{18}11^{25}79973c \notin \mathbb{Z}$, e.g., $c = 1/13$.

### 5. Modular forms

We give definition of modular forms $\alpha_6$, $\alpha_{10}$, $\alpha_{12}$, $\alpha_{20}$, $\alpha_{30}$, $\beta_{14}$, $\chi_{28}$ with their subscripts as their weight. We denote by $E_k$, the Eisenstein series of degree three and of weight $k$. 
i) \( \alpha_4 = 2^{-3} \sum_m \theta[m]^8 \), \( m \) running over the set of all even theta characteristics (mod 2). \( \alpha_4 \) is equal to the Eisenstein series \( E_4 \).

ii) \( \alpha_6 = 2^{-6} 3^{-1} 7^{-1} \Sigma_{M: \Gamma_3(2)} M(\theta[000000] \theta[100001] \theta[100011] \theta[101011] \theta[101111] \theta[111111]) \), which is equal to \( 8E_6 \).

iii) \( \alpha_{10} = -2^{-4} 3^{-2} 5^{-1} 11^{-1} \Sigma_{M: \Gamma_3(2)} M(\theta[010101] \theta[100101] \theta[101011] \theta[101111] \theta[100000] \theta[100010] \theta[100011] \theta[100100] \theta[100101] \theta[100111] \theta[110011] \theta[110111]) \), which is proportional to \( E_4 E_6 - E_{10} \).

iv) \( \alpha_{12} = 2^{-3} 3^{-2} \sum (\theta[m_1] \cdots \theta[m_6])^4 \) where \( \{m_1, \ldots, m_6\} \) runs through all the maximal azygetic sequences of even theta characteristics. Such an azygetic sequence is characterized by the property that a sum of any distinct three elements is odd (cf. Igusa [5]). \( \alpha_{12} \) cannot be written as a polynomial of Eisenstein series. Indeed \( \alpha_{12} \) is a cusp form, however, no non-trivial elements of the vector space spanned by \( E_4, E_6, E_{12} \) are cusp forms.

v) \( \alpha'_{2} = 2^{-8} 3^{-5} 5^{-1} \Sigma_{M: \Gamma_3(2)} M(\theta[101000] \theta[100100] \theta[100101] \theta[101011] \theta[101111] \theta[100010] \theta[100001] \theta[100111] \theta[100001]). \) In the summation, the same term appears \( 2^{23} 3 \cdot 7 \) times, so \( \beta_{14} \) is actually a sum of \( 2^{-5} 3^{-1} 7^{-1} \Gamma_3: \Gamma_3(2) \) (= 2160) terms. \( \beta_{14} \) is proportional to the cusp form \( \chi_8 \).

vi) Let \( P \) denote the product \( \theta[101111] \theta[101101] \theta[101110] \theta[101100] \theta[101011] \theta[101010] \theta[101001] \theta[101000] \theta[100111] \theta[100110] \theta[100011] \theta[100010] \theta[100001] \theta[100000] \theta[100101] \theta[100100] \theta[100010] \theta[100001] \theta[100000]). \) Then \( \alpha_{20} = 2^{-3} 3 \cdot 5^{-1} \Sigma_{M: \Gamma_3(2)} M(\chi_{18}/P^2) \), \( \chi_8 \) denoting as before the product of all theta constant with even characteristics.

vii) \( \alpha_{30} = 2^{-8} 3^{4} 5^{-1} \Sigma_{M: \Gamma_3(2)} M(\theta[000000] \chi_{18}/\theta[100000])^2 P^3). \)

viii) \( \beta_{14} = 2^{-5} 3^{-1} 7^{-1} \Sigma_{M: \Gamma_3(2)} M(\theta[101111] \chi_{18}/\theta[100101] \theta[100100] \theta[101011] \theta[101111] \theta[100011] \theta[100010] \theta[100100] \theta[100101] \theta[100111] \theta[100001]). \) In the summation, the same term appears \( 2^3 3 \cdot 7 \) times, so \( \beta_{14} \) is actually a sum of \( 2^{-5} 3^{-1} 7^{-1} \Gamma_3: \Gamma_3(2) \) (= 2160) terms. \( \beta_{14} \) is proportional to the cusp form \( \chi_8 \).

ix) \( \chi_{28} = 2^{-10} 3^{-2} 5^{-1} 7^{-1} \Sigma_{M: \Gamma_3(2)} M(\chi_{18}/\theta[100000] \theta[100010] \theta[100011] \theta[100001] \theta[100000] \theta[100010] \theta[100100] \theta[100101] \theta[100111] \theta[100001]). \) In the summation, the same term appears \( 2^2 3 \cdot 7 \) times.

Correction to [19]

- p. 802 line 1 should be read as \( \psi_{10} = \Pi_{k: \text{even}} \theta[k]^2 \).
- Sect. 23, (1) should be read as follows:

\[
\alpha_4 = \frac{1}{8} \sum_{k: \text{even}} \theta[k]^8 = \sum_{i=1}^{135} ((i)) = \frac{1}{21504} \sum_{M: \Gamma_3(2)} M((131) \cap (132)).
\]

\[
\Sigma(1234, 5678)^2 = 8 \Sigma D^{1/2}/(12)(34)(56)(78) = \frac{8}{7} \Sigma D^{1/2}/(12)(36)(45)(78)
\]

\[
+\frac{8}{7} \Sigma (34)(56) D^{1/2}/(12)(78)(35)(46)(36)(45).
\]

- p. 847 line 7 should be read as \( +8 \Sigma_{M: \theta[\Gamma_3(2)]} M(((115))^2((135))^2/214^2 24^4). \)
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References