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***K*-theory, λ -rings, and formal groups**

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Introduction

In [C] the author showed how one can compute the algebraic *K*-group $K_2(R, I)$ when R has a structure of λ -ring; this relied on a presentation of that group found by H. Maazen and J. Stienstra [Ms] and F. Keune [K]. The purpose of this paper is to show that the same technique works for a more general group $K_{2,F}(R, I)$ in which also a formal group F is involved. An instance of this more general group occurs in §17 of [S], which inspired the present paper.

DEFINITION 0.1. Let F be a one-dimensional formal group over \mathbf{Z} . Let R be a commutative ring and let I be a nilpotent ideal. Then $K_{2,F}(R, I)$ is the abelian group with as generators the symbols $\langle a, b \rangle_F$ for $(a, b) \in I \times R \cup R \times I$ and relations

$$\begin{aligned} \langle a, 1 \rangle_F & & \text{for } a \in I \\ \langle a, b \rangle_F + \langle a, c \rangle_F - \langle a, a^{-1}F(ab, ac) \rangle_F & & \text{for } (a, b, c) \in I \times R \times R \\ & & \cup R \times I \times I \\ \langle a, bc \rangle_F + \langle b, ac \rangle_F - \langle ab, c \rangle_F & & \text{for } (a, b, c) \in I \times R \times R \\ & & \cup R \times I \times R \cup R \times R \times I \end{aligned}$$

where $a^{-1}F(ab, ac)$ has to be interpreted in the obvious way.

If F is the multiplicative formal group defined by $F(X, Y) = X + Y - XY$ then $K_{2,F}(R, I)$ coincides with the group $K_2(R, I)$ from algebraic *K*-theory. If F is the additive formal group defined by $F(X, Y) = X + Y$ then $K_{2,F}(R, I)$ coincides with the group $K_{2,L}(R, I)$ considered in [C] and [LQ]. As explained there it is a cyclic homology group of (R, I) ; it is isomorphic to $\Omega_{R,I}/\delta I$ if the natural projection $R \rightarrow R/I$ splits.

To formulate the main theorem we introduce the notation $G(R, I)^{\text{top}}$ for the inverse limit $\lim G(R/J^N, (I + J^N)/J^N)$ if G is a functor from pairs (R, I) as above to abelian groups and J is an ideal describing a topology on R .

THEOREM 0.2. *Let F be a one dimensional formal group over \mathbf{Z} which is strongly isomorphic to a special one. Let R be a λ -ring and let I and J be ideals such that (R, J, I) is admissible. Then there is a homomorphism $L_F: K_{2,F}(R, I)^{\text{top}} \rightarrow K_{2,L}(R, I)^{\text{top}}$ such that*

$$L_F \langle a, b \rangle_F = \langle a, b \rangle_L + \text{higher order terms.}$$

The notion of special formal group and of strong isomorphism are defined in the next section. For the definition of admissible and other notions connected with λ -rings we refer to [C].

§1. Generalities about formal groups

Let A be a commutative ring. A one-dimensional commutative formal group F over A is a formal power series $F(X, Y) = \sum f_{ij} X^i Y^j$ with $f_{ij} \in A$ such that

$$F(X, Y) = X + Y + \text{terms of degree } > 1$$

$$F(X, Y) = F(Y, X),$$

$$F(X, F(Y, Z)) = F(F(X, Y), Z).$$

By substitution into F one can define a structure of abelian group on any topologically nilpotent ideal in a complete commutative A -algebra. In particular this applies to the formal power series over A with vanishing constant term. In this paper we use the words “formal group” in the understanding that we always mean a one-dimensional commutative one.

If F and \tilde{F} are both formal groups then an isomorphism G from \tilde{F} to F means a formal power series $G(X) = \sum g_n X^n$ with $g_n \in A$ such that

$$G(X) = X + \text{terms of degree } > 1,$$

$$G\tilde{F}(X, Y) = F(G(X), G(Y)).$$

In the case that $A = \mathbf{Z}$ one can find integers γ_n such that G is an infinite sum of terms X^n , each occurring γ_n times; the sum being taken in the F sense.

DEFINITION 1.1. The isomorphism G is called a strong isomorphism if n divides γ_n for every n .

PROPOSITION 1.2. Let \tilde{F} and F be formal groups over \mathbf{Z} , and let $G: \tilde{F} \rightarrow F$ be a strong isomorphism. Then G induces an equivalence $G_*: K_{2,\tilde{F}} \rightarrow K_{2,F}$ such that

$$G_* \langle a, b \rangle_{\tilde{F}} = \langle a, a^{-1}G(ab) \rangle_F$$

Proof. We show that the three relations are satisfied.

- 1) $G_* \langle a, 1 \rangle_{\tilde{F}} = \langle a, a^{-1}G(a) \rangle_F = \sum \gamma_n \langle a, a^{-1}a^n \rangle_F = \sum n^{-1} \gamma_n \langle a^n, 1 \rangle_F = 0$ since it follows from the third relation that $\langle a^n, 1 \rangle_F = n \langle a, a^{n-1} \rangle_F$.
- 2) $G_* \langle a, b \rangle_{\tilde{F}} + G_* \langle a, c \rangle_{\tilde{F}} = \langle a, a^{-1}G(ab) \rangle_F + \langle a, a^{-1}G(ac) \rangle_F = \langle a, a^{-1}F(G(ab), G(ac)) \rangle_F = G_* \langle a, a^{-1}G^{-1}F(G(ab), G(ac)) \rangle_{\tilde{F}} = G_* \langle a, a^{-1}\tilde{F}(ab, ac) \rangle_{\tilde{F}}$
- 3) $G_* \langle a, bc \rangle_{\tilde{F}} + G_* \langle b, ac \rangle_{\tilde{F}} = \langle a, a^{-1}G(abc) \rangle_F + \langle b, b^{-1}G(abc) \rangle_F = \langle ab, a^{-1}b^{-1}G(abc) \rangle_F = G_* \langle ab, c \rangle_{\tilde{F}} \quad \bullet$

For any formal group over \mathbf{Q} there is an isomorphism f from F to the additive formal group; this is called the logarithm of the formal group. Let p be a prime number and let F be a formal group over \mathbf{Q} with logarithm $F(X) = \sum f_n X^n$; then the formal group with logarithm $\sum f_{p^n} X^{p^n}$ is called the p -typical formal group associated to F . If F is defined over \mathbf{Z} then its p -typification is (for these facts see [H]).

If $f = \sum f_n X^n$ is a logarithm for F and $\tilde{f} = \sum \tilde{f}_n X^n$ is one for \tilde{F} and if G is an isomorphism as above then we have $\tilde{f} = fG = \sum \gamma_n f(X^n)$. Writing this out yields $\tilde{f}_n = \sum_{mk=n} \gamma_m f_k$ hence $(\sum \tilde{f}_n n^s) = (\sum \gamma_m m^s)(\sum f_k k^s)$ in the language of formal Dirichlet series introduced in [C]. So if F and \tilde{F} are strongly isomorphic then their p -typifications are.

DEFINITION 1.3. A formal group F is called special if its logarithm $f(X) = \sum f_n X^n$ satisfies $f_{mk} = f_m f_k$ for every m, k .

So the p -typification of a special formal group is again special. Therefore the p -typification of a formal group satisfying the conditions of Theorem 0.2 again satisfies those conditions.

§2. The λ -operations associated to F

In [C] we introduced certain λ -operations λ^n, θ^n and η^n . In this § we introduce F -twisted versions λ^n_F, θ^n_F and η^n_F of these operations.

Recall that the element a of a λ -ring is called one-dimensional if $\lambda^n(a) = 0$ for $n > 1$.

PROPOSITION 2.1. *Let F be a formal group over \mathbf{Z} . There exists a unique sequence $\{\lambda_F^n\}$ of λ -operations such that $\lambda_i^F = \sum_{n=1}^\infty t^n \lambda_F^n$ satisfies*

- 1) $\lambda_i^F(a + b) = F(\lambda_i^F(a), \lambda_i^F(b))$
- 2) $\lambda_i^F(a) = ta$ if a is one-dimensional.

Proof. Recall that the universal λ -ring U can be embedded in the inverse limit of polynomial rings $\mathbf{Z}[s_1, s_2, \dots, s_n]$ so that the canonical element $u \in U$ corresponds to $\sum_{n=1}^\infty s_n$. Now $F(ts_1, ts_2, \dots)$ is the required element λ_i^F . ●

PROPOSITION 2.2. *Let F be a formal group over \mathbf{Q} with logarithm f . Let R be a λ -ring containing \mathbf{Q} , and let $a \in R$. Then*

$$f(\lambda_i^F(a)) = \sum_{n=1}^\infty f_n t^n \psi^n(a).$$

Proof. From the definition of logarithm and the first property of λ_i^F it follows that $f\lambda_i^F$ is additive. Moreover the statement is true on one-dimensional elements. Therefore it is true on $\mathbf{Q}[s_1, s_2, \dots]$ and thus on $U \otimes \mathbf{Q}$. ●

Now we define the F -twisted version of the operations η^n in §3 of [C].

LEMMA 2.3. *If R is a λ -ring and $a \in R$ is not a zero-divisor then*

$$\lambda_{ta}^F = \sum_{n=1}^\infty t^n a^n \lambda_F^n: R[[t]] \rightarrow taR[[t]] \text{ is a bijection.}$$

Proof. This follows easily from the first property of λ_i^F together with the fact that

$$\lambda_F^n(t^m R[[t]]) \subseteq t^{nm} R[[t]].$$
 ●

DEFINITION 2.4. Let F be a formal group over \mathbf{Z} . The λ -operations η_F^n in two variables are defined by the condition that

$$\lambda_{ta}^F \left(\sum_{n=1}^\infty t^{n-1} \eta_F^n(a, b) \right) = tab.$$

We write $\theta_F^n(b)$ for $\eta_F^n(1, b)$.

By applying f on both sides in this definition and using Proposition 2.3 we get the following generalisation of Proposition 3.3 of [C]:

OBSERVATION 2.5. For elements a, b in a λ -ring R containing \mathcal{Q} one has

$$f_n a^{n-1} b^n = \sum_{m|n} f_m a^{m-1} \psi^m(\eta_F^{n/m}(a, b)).$$

From this it is clear that $\eta_F^n(a, b)$ is of degree $n - 1$ in a and of degree n in b . Therefore we can rewrite the formula 2.4 as

$$\lambda_{ta}^F(\eta_F(ta, b)) = tab \quad \text{where} \quad \eta_F(ta, b) = \sum_{n=1}^{\infty} \eta_F^n(ta, b).$$

PROPOSITION 2.6. *If F is a formal group over Z and a, b, c are elements of a λ -ring then*

$$\eta_F(ta, b) + \eta_F(ta, c) = \eta_F(ta, (ta)^{-1}F(tab, tac)).$$

Proof. This follows from Lemma 2.3 since both sides have the same image under λ_{ta}^F by Definition 2.4 and the first property of λ^F . ●

Using the formalism of formal Dirichlet series introduced in §8 of [C] we can rewrite some of the foregoing identities as relations between such series.

DEFINITION 2.7. Let F be a formal group over Z . Then $Y_F \in DS(U)$ and $H_F \in DS(U_2)$ are defined by the formulas

$$Y_F(a) = \sum n f_n a^{n-1} n^s, \quad H_F(a, b) = \sum \eta_F^n(a, b) n^s$$

Now Observation 2.5 can be reformulated as $b Y_F(ab) = Y_F(a) \cdot TH_F(a, b)$. The formula at the end of §1 can be rewritten as $Y_{\tilde{F}}(1) = T\Gamma \cdot Y_F(1)$. Henceforth we write Y_F for $Y_{\tilde{F}}(1)$. ●

Since we have taken a generalisation of Proposition 3.8 of [C] as definition of the η_F^n , the generalisation of Definition 3.1 of [C] becomes a proposition:

PROPOSITION 2.8. *If F is a formal group over Z and a, b are elements of a λ -ring then*

$$\eta_F^n(a, b) = \sum_m \eta_F^m(a, a^{(n-m)/m}) \psi^m \theta_F^{n/m}(b) \quad \text{for } n > 1.$$

Here the sum extends over all $m \neq n$ dividing n .

Proof. Property 2.5 determines the operations η_F^n uniquely: we may assume that a, b are the canonical elements in U_2 , and the right hand side is of the form $\eta_F^n(a, b) + \text{terms involving } \eta_F^k(a, b) \text{ with } k < n$.

Therefore it is sufficient to show that the right hand side of the above statement satisfies the same identity. This follows easily by rearranging sums and using the induction hypothesis. ●

COROLLARY 2.9. *The operations η_F^n are elements of the subring V_2 of the ring U_2 of λ -operations in two variables as defined in §2 in [C].*

Therefore the theory of §5 of [C] tells us the following. If (R, J, I) is admissible then $\eta_F(a, b) = \sum_{n=1}^\infty \eta_F^n(a, b)$ converges in the J -topology for $a \in I$ or $b \in I$. Moreover Proposition 2.6 implies that in that situation one has

$$\eta_F(a, b) + \eta_F(a, c) = \eta_F(a, a^{-1}F(ab, ac)).$$

§3. The relation between η_F and η

In this § the operations η_F^n are expressed in terms of the η^m .

DEFINITION 3.1. Let F be a formal group over \mathbf{Q} with logarithm f . Then the λ -operations $C_F^{n,m} \in U \otimes \mathbf{Q}$ are defined by

$$\sum_{k|m} f_k a^{k-1} \psi^k(C_F^{n,m/k}(a)) = n f_{nm} a^{m-1}$$

In particular $C_F^{n,1}(a) = n f_n$; and if F is special then $C_F^{n,m} = 0$ for $m > 1$.

PROPOSITION 3.2. *Let F be a formal group over \mathbf{Q} and let a, b be elements of a λ -ring containing \mathbf{Q} . Then*

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a) \psi^m(\eta^{n/m}(a, b)).$$

Proof. Again it is sufficient to show that the right hand side satisfies identity 2.5. This follows easily by rearranging the sum and applying Definition 3.1 and Proposition 2.5 for the η^m . ●

We now show that in fact $C_F^{n,m} \in U$ if F is a formal group over \mathbf{Z} .

LEMMA 3.3

- a) Suppose that $\xi^m \in U \otimes \mathbf{Q}$ and that the operation $(a, b) \rightarrow \sum_{m|n} \xi^m(a)\psi^m(\theta^{n/m}(b))$ is element of U_2 ; then in fact $\xi^m \in U$.
- b) Suppose that $\xi \in U \otimes \mathbf{Q}$ and that the operation $a \rightarrow \xi(a)\psi^k(a')$ is in U . Then in fact $\xi \in U$.

Proof

- a) If the operations ξ^m are integral for $m < M$ then we may suppose that they vanish in that range. Now apply the above operation to the ring $U[t_1, t_2]/(t_1^{M+1})$ and take for a the canonical element $u \in U$ and $b = t_1 + t_2$ or t_2 respectively, and take the difference of the results. Then the m th term vanishes also for $m > M$, and the M th term is

$$\xi^M(u)\{\theta^{n/M}(t_1^M + t_2^M) - \theta^{n/M}(t_2^M)\} = \xi^M(u)t_1^M t_2^{n-M}.$$

If this is integral then $\xi^M(u)$ must be.

- b) The universal λ -ring U is the polynomial ring over \mathbf{Z} freely generated by the $\lambda^n(u)$. Therefore a product of two elements can only be a multiple of an integer > 1 if one of the factors is. But the element of U corresponding to the operation $a \rightarrow \psi^k(a')$ is not divisible by any integer > 1 as is easily seen by applying this operation on the ring $\mathbf{Z}[t]$ and the element t . ●

PROPOSITION 3.4. Let F be a formal group over \mathbf{Z} . Then the operations $C_F^{n,m}$ are in U_2 .

Proof. By substituting Definition 3.1 of [C] into Proposition 3.2 and rearranging terms we get

$$\eta_F^n(a, b) = \sum_{m|n} \xi^m(a)\psi^m(\theta^{n/m}(b))$$

where

$$\xi^m(a) = \begin{cases} \sum_{k|m} C_F^{n/k,k}(a)\psi^k(\eta^{m/k}(a, a^{n-m/m})) & \text{for } m < n \\ C_F^{1,m} & \text{for } m = n \end{cases}$$

So by Lemma 3.3a these operations are integral. From this it follows by induction on n and Lemma 3.3b that the operations $C_F^{n,m}$ are integral ●

COROLLARY 3.5. Let F be a formal group over \mathbf{Z} and let a, b be elements of a λ -ring. Then again

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a)\psi^m(\eta^{n/m}(a, b)).$$

REMARK 3.6. Proposition 3.4 implies that the numbers $C_F^{n,m}(1) \in \mathcal{Q}$ are integers. We will abbreviate these to $C_F^{n,m}$. They are determined by the formula $\sum_{k|m} f_k C_F^{n,m/k} = n f_{nm}$ which is exactly the formula in [D] expressing that $n f_n$ is a “lexoid function”. The meaning of these numbers for us becomes clear by putting $a = 1$ in Proposition 3.2; that yields the formula

$$\theta_F^n(b) = \sum_{m|n} C_F^{n/m,m} \psi^m(\theta^{n/m}(b)).$$

§4. Some identities for the numbers C_F

In this § the prime p is fixed. First we prove some general relations between the numbers C_F^{p',p^j} ; then we use these to draw some consequences for these numbers from the hypothesis of Theorem 0.2.

LEMMA 4.1 (see [D]). *Let F be a formal group over \mathcal{Q} . Then*

$$C_F^{p',p^j} = p C_F^{p'-1,p^{j+1}} + C_F^{p'-1,1} C_F^{p,p^j}$$

for every (i, j) with $i > 0$.

Proof. Let $f(X) = \sum_n f_n X^n$ be the logarithm of F . If we substitute the identities

$$C_F^{p',p^k} = p C_F^{p'-1,p^{k+1}} + C_F^{p'-1,1} C_F^{p,p^k} \text{ for } 0 \leq k \leq j - 1$$

into the identity

$$\sum_{k=0}^j C_F^{p',p^k} f_{p^j-k} = p^j f_{p^j} = p C_F^{p'-1,1} f_{p^{j+1}} + p \sum_{k=0}^j C_F^{p'-1,p^{k+1}} f_{p^j-k}$$

then we get

$$C_F^{p',p^j} + C_F^{p'-1,1} \sum_{k=0}^{j-1} C_F^{p,p^k} f_{p^j-k} = p C_F^{p'-1,p^{j+1}} + p C_F^{p'-1,1} f_{p^{j+1}}.$$

Now the statement follows by comparing this with the identity $p f_{p^{j+1}} = \sum_{k=0}^j C_F^{p,p^k} f_{p^j-k}$ ●

LEMMA 4.2. *Let F be a formal group over \mathcal{Q} . Then*

$$C_F^{p^i, p^j} = \sum_{k=0}^{i-1} p^k C_F^{p^i, p^{j+k}} C_F^{p^{i-k-1}, 1} \text{ for every } (i, j).$$

Proof. Immediate from Lemma 4.1 by induction on i . ●

LEMMA 4.3. *Let F be a formal group over \mathcal{Q} and let p be prime. Then for every (i, j, k) with $k \leq i$ one has*

$$C_F^{p^i, p^j} - \sum_{m=0}^k p^m C_F^{p^i-k, p^m} C_F^{k-m, p^j} = \begin{cases} p^k C_F^{p^i-k, p^{k+j}} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

Proof. We use induction on k ; for $k = 0$ the statement is empty. For $k > 0$ we apply Lemma 4.1 for (i, j) , the induction hypothesis for $(i - 1, 0, k - 1)$, Lemma 4.1 for $(k - m, j)$ and the induction hypothesis for $(i - 1, j + 1, k - 1)$. ●

Now we deduce from the condition on F in theorem 0.2. a relation for the integers $C_F^{n, m}$. First we recall a few notations from §9 of [C].

DEFINITION 4.4. If $\Xi = \sum \zeta_n n^s$ is a Dirichlet series then we write

$$\Xi_p = \sum \zeta_{p^e} (p^e)^s, \quad T_p \Xi = \sum p^{v_p(n)} \zeta_n n^s.$$

PROPOSITION 4.5. *Let F be a p -typical formal group over \mathcal{Z} strongly isomorphic to a special one. Then p^h divides $C_F^{p^i, p^h}$ for every h .*

Proof. By Remark 3.6 one has $p^i f_{p^{e+i}} = \sum_{d=0}^e C_F^{p^i, p^{e-d}} f_{p^d}$. Therefore

$$Y_F - \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s = Y_F \sum_h p^h C_F^{p^i, p^h} (p^{h+i})^s,$$

so

$$\sum_h p^h C_F^{p^i, p^h} (p^{h+i})^s = 1 - Y_F^{-1} \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s.$$

If F is strongly isomorphic to a special formal group with logarithm \tilde{f} then one can rewrite Y_F^{-1} as

$$T\Gamma_p \cdot Y_{\tilde{F}}^{-1} = T\Gamma_p \cdot (1 - p\tilde{f}_p p^s)$$

and the right hand side is of the form

$$1 - \sum_{k=0}^{\infty} p^k \gamma_{p^k} (p^k)^s \cdot \sum_{m=0}^i \xi_m (p^m)^s$$

where

$$\sum_{m=0}^i \xi_m (p^m)^s = (1 - p\tilde{f}_p p^s) \cdot \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s$$

has integral coefficients. Now the statement follows from the fact that $p^k \gamma_{p^k}$ is divisible by p^{2k} and thus by p^{2h} if $k + m = h + i$. ●

LEMMA 4.6. *Let F be any formal group, and a element of a λ -ring. Then*

$$Y_F(a) = \sum a^{p^h q^{-1}} Y_{F,P}(a^{p^h q}) p^h C_F^{q,p^h} (p^h q)^s.$$

In particular $Y_F = Y_{F,P} \cdot Y_{F,C} = Y_{F,C} \cdot Y_{F,P}$ where $Y_{F,C} = \sum p^h C_F^{q,p^h} (p^h q)^s$. Here the sums are over all q indivisible by p and over all h .

Proof. According to Remark 3.6 one has $qf_{p^e q} = \sum_{d=0}^e C_F^{q,p^{e-d}} f_{p^d}$. Substituting this into the formula $Y_F(a) = \sum p^e q f_{p^e q} a^{p^e q^{-1}} (p^e q)^s$ and writing h for $e - d$ gives the result. ●

PROPOSITION 4.7. *Let F be a formal group over \mathbf{Z} strongly isomorphic to a special one. Then p^h divides C_F^{q,p^h} if q is indivisible by p .*

Proof. Suppose that F is strongly isomorphic to \tilde{F} and that \tilde{F} is special. Then according to 2.7 one has $Y_{\tilde{F}} = T\Gamma \cdot Y_F$ where $T\Gamma$ is in the image of T_p^2 . This implies that $Y_{\tilde{F},P} = T\Gamma_p \cdot Y_{F,P}$ where $T\Gamma_p$ is in the image of T_p^2 . By lemma 4.6 one has $Y_{F,C} = Y_F \cdot Y_{F,P}^{-1} = T\Gamma^{-1} \cdot Y_{\tilde{F}} \cdot T\Gamma_p \cdot Y_{\tilde{F},P}^{-1} = T\Gamma^{-1} \cdot \Gamma_p \cdot Y_{\tilde{F},C}$. However $Y_{\tilde{F},C}$ is of the form $\sum_q C_{\tilde{F}}^{q,1} q^s$ and thus in the image of T_p^2 . Therefore $Y_{F,C}$ is in the image of T_p^2 . Writing this out one gets the statement. ●

§5. Some p -primary congruences

Now we use Proposition 4.5 to generalise the contents of §10 of [C] to the F -twisted case.

DEFINITION 5.1. Let F be a formal group over \mathbf{Z} . If $i \leq e$ then we write $\chi_F^{p^e, p^i}$ for the λ -operation defined by

$$\chi_F^{p^e, p^i}(a) = \theta_F^{p^e}(a) - \sum_{m=0}^i p^m C_F^{p^e-i, p^m} a^{p^e-p^i-m} \theta_F^{p^i-m}(a)$$

PROPOSITION 5.2. Let F be a p -typical formal group over \mathbf{Z} which is strongly isomorphic to a special one. Let a be the canonical element of U . Then

$$\chi_F^{p^e, p^i}(a) \equiv \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^e-i, p^m} C_F^{p^i-j-m, p^j} \psi^{p^j}(\chi_F^{p^e-i, p^i-j-m}(a)) \text{ modulo } p^i V.$$

Proof. If one expresses the θ_F in terms of the θ as in Remark 3.6 then the right hand side of Definition 5.1 becomes

$$\begin{aligned} & \sum_{j=i+1}^e C_F^{p^e-i, p^j} \psi^{p^j}(\theta^{p^e-i}(a)) \\ & + \sum_{j=0}^i \left\{ C_F^{p^e-i, p^j} - \sum_{m=0}^{i-j} p^m C_F^{p^e-i, p^m} C_F^{p^i-j-m, p^j} \right\} \psi^{p^j}(\theta^{p^e-i}(a)) \\ & - \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^e-i, p^m} C_F^{p^i-j-m, p^j} \{ a^{p^e-p^i-m} - \psi^{p^j}(a^{p^e-i-p^i-j-m}) \} \psi^{p^j}(\theta^{p^i-j-m}(a)) \\ & + \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^e-i, p^m} C_F^{p^i-j-m, p^j} \psi^{p^j} \{ \theta^{p^e-i}(a) - a^{p^e-i-p^i-j-m} \theta^{p^i-j-m}(a) \}. \end{aligned}$$

The last sum is the right hand side of the statement. From Propositions 4.3 and 4.5 and the fact that the λ -operation $x \rightarrow x^{p^i+k} - \psi^{p^i}(x^{p^k})$ is in $p^{k+1}V$ it follows that the first three terms are in $p^i V$. ●

COROLLARY 5.3. In the situation of Proposition 5.2 one has

$$\chi_F^{p^e, p^i}(a) \equiv \begin{cases} 0 & \text{if } p \text{ is odd} \\ C_F^{2^e-i, 1} \sum_{j=0}^i C_F^{2^i-j, 2^j} \psi^{2^j}(\chi^{2^e-i, 2^i-j}(a)) & \text{if } p = 2 \end{cases}$$

Proof. It follows from Lemma 10.5 of [C] that χ^{p^e, p^i} is in $p^i V$ if p is odd and in $p^{i-1} V$ if $p = 2$. ●

DEFINITION 5.4. Let F be a formal group over \mathbf{Z} . Then the λ -operations τ_F^n are defined by $\sum \tau_F^n(a, b)n^s = H_F(1, a) \cdot TH_F(a, b)$. In other words

$$\tau_F^n(a, b) = \sum_{m|n} \frac{n}{m} \theta_F^m(a) \psi^m \eta_F^{n/m}(a, b).$$

From this one deduces easily that

$$\tau_F^n(a, b) = \theta_F^n(a) \psi^n(b) + \sum \frac{n}{m} \tau_F^m(a, a^{(n-m)/m}) \psi^m \theta_F^{n/m}(b)$$

where the sum extends over all $m < n$ dividing n .

DEFINITION 5.5. Let p be a prime, and let F be a formal group over \mathbf{Z} . Then the operation $\varepsilon_F^{p^e}$ is defined by

$$\varepsilon_F^{p^e}(a, b) = \sum_{i=0}^e C_F^{p^{e-i} p^i} \psi^{p^i}(\varepsilon_F^{p^{e-i}}(a, b))$$

where ε is defined as in Definition 10.7 of [C] i.e.

$$\varepsilon^{p^e}(a, b) = \begin{cases} 0 & \text{if } p > 2 \text{ or } e \leq 1 \\ 2^{e-1}(ab)^{2^e-4} \theta^2(a)^2 \theta^2(b)^2 & \text{if } p = 2 \text{ and } e \geq 2 \end{cases}$$

PROPOSITION 5.6. Let p be a prime and let F be a p -typical formal group strongly isomorphic to a special one. Let a, b be the canonical elements of U_2 ; then one has

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a) b^{p^e} + \varepsilon_F^{p^e}(a, b) \text{ modulo } p^e W_2.$$

Proof. We use induction in e ; the case $e = 0$ is trivial so assume $e > 0$. We claim that

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a) b^{p^e} + \sum_{k=0}^e [C_F^{p^{e-k}, 1} \varepsilon_F^{p^k}(a, a^{p^{e-k}}) - \chi_F^{p^e, p^k}(a)] p^{e-k} \psi^{p^k} \theta_F^{p^{e-k}}(b).$$

To prove the claim we apply the induction hypothesis and get

$$\begin{aligned} \tau_F^{p^e}(a, b) &= \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i} \tau_F^{p^i}(a, a^{p^{e-i}-1})\psi^{p^i} \theta_F^{p^{e-i}}(b) \\ &= \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i} [\theta_F^{p^i}(a)a^{p^e-p^i} + \varepsilon_F^{p^i}(a, a^{p^{e-i}-1}) \text{ modulo } p^i W] \psi^{p^i} \theta_F^{p^{e-i}}(b) \\ &\equiv \sum_{i=0}^e p^{e-i} \theta_F^{p^i}(a)a^{p^e-p^i} \psi^{p^i} \theta_F^{p^{e-i}}(b) + \sum_{i=0}^{e-1} p^{e-i} \varepsilon_F^{p^i}(a, a^{p^{e-i}-1}) \psi^{p^i} \theta_F^{p^{e-i}}(b). \end{aligned}$$

The first sum can be rewritten as

$$\begin{aligned} &\sum_{i=0}^e \sum_{j=0}^{e-i} p^{e-i} \theta_F^{p^i}(a)a^{p^e-p^i} C_F^{p^{e-i-1}, p^j} \psi^{p^{i+j}} \theta^{p^{e-i-j}}(b) \\ &= \sum_{k=0}^e \left[\sum_{j=0}^k p^j \theta_F^{p^{k-j}}(a)a^{p^e-p^{k-j}} C_F^{p^{e-k}, p^j} \right] p^{e-k} \psi^{p^k} \theta^{p^{e-k}}(b) \\ &= \sum_{k=0}^e [\theta_F^{p^e}(a) - \chi_F^{p^e, p^k}(a)] p^{e-k} \psi^{p^k} \theta^{p^{e-k}}(b) \end{aligned}$$

and by Lemma 2.2 of [C] the terms involving $\theta_F^{p^e}(a)$ add up to $\theta_F^{p^e}(a)b^{p^e}$.

In the second sum the value of $\psi^{p^i} \theta_F^{p^{e-i}}(b)$ only matters modulo pW since its cofactor is in $p^{e-1}W$. So we may replace it by $C_F^{p^{e-i-1}, 1} \psi^{p^i} \theta^{p^{e-i}}(b)$. This establishes the claim.

Now consider the bracket expression in the claim, modulo $p^k W$. For $p > 2$ it vanishes according to Corollary 5.3 and the definition of ε_F , and the proof is finished. So let $p = 2$. For $k > 1$ the χ_F term yields

$$\begin{aligned} &- C_F^{2^e-k, 1} \left[C_F^{2, 2^{k-1}} \psi^{2^{k-1}} (a^{2^{e-k+1}-4} \theta^2(a)^2) \right. \\ &\quad \left. + \sum_{j=0}^{k-2} 2^{k-j-1} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} (a^{2^{e-j}-4} \theta^2(a)^2 + a^{2^{e-j}-8} \theta^2(a)^4) \right] \end{aligned}$$

and the ε_F term yields

$$\begin{aligned} &C_F^{2^e-k, 1} \sum_{j=0}^{k-2} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} \varepsilon_F^{2^{k-j}}(a, a^{2^{e-k}-1}) \\ &= C_F^{2^e-k, 1} \sum_{j=0}^{k-2} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} (2^{k-j-1} (a^{2^{e-k}})^{2^k-4} \theta^2(a)^2 \theta^2(a^{2^{e-k}-1})^2). \end{aligned}$$

Again the value of $\theta^2(a^{2^{e-k}-1})^2$ only matters modulo $2W$ so we may replace it by $a^{2^{e-k+2}-8}\theta^2(a)^2$. The expression thus becomes

$$- C_F^{2^{e-k},1} \sum_{j=0}^{k-1} C_F^{2^{k-j},2^j} 2^{k-j-1} \psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$$

For $k = 1$ one gets the same result; for $k = 0$ one gets zero. Finally we may replace $C_F^{2^{e-k},1} C_F^{2^{k-j},2^j}$ by $C_F^{2^{e-j},2^j}$ since it is the same modulo 2^{j+1} . We thus find the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\tau_F^{p^e}(a, b)$ to be

$$- 2^{e-j-1} C_F^{2^{e-j},2^j} \sum_{k=j+1}^{e-1} \psi^{2^k} \theta^{2^{e-k}}(b)$$

and according to Lemma 10.6 of [C] this may be rewritten as

$$- 2^{e-j-1} C_F^{2^{e-j},2^j} (b^{2^{e-j}-4}\theta^2(b)^2)$$

which is exactly the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\varepsilon_F^{2^e}(a, b)$. ●

§6. Proof of the theorem

LEMMA 6.1. *Let p be a prime. Let F be a p -typical formal group over \mathbf{Z} strongly isomorphic to a special one. If a, b are the canonical elements in U_2 then*

$$\theta_F^{p^e}(ab) \equiv \theta_F^{p^e}(a)b^{p^e} + a^{p^e}\theta_F^{p^e}(b) \text{ modulo } p^e W_2$$

Proof. In view of Proposition 4.5 and the fact that $\theta_F^{p^e} = \sum_{i=0}^e C_F^{p^{e-i},p^i} \psi^{p^i} \theta^{p^{e-i}}$ we only have to show that

$$\psi^{p^j}(\theta^{p^{e-j}}(ab)) \equiv \psi^{p^j}(\theta^{p^{e-j}}(a))b^{p^e} + a^{p^e}\psi^{p^j}(\theta^{p^{e-j}}(b)) \text{ modulo } p^{e-j} W_2$$

But this follows from the fact that

$$\theta^{p^j}(ab) = \theta^{p^j}(a)b^{p^j} + a^{p^j}\theta^{p^j}(b) - p^j \theta^{p^j}(a)\theta^{p^j}(b)$$

for all j . ●

PROPOSITION 6.2. *Let p be a prime. Let F be a p -typical formal group over \mathbf{Z} strongly isomorphic to a special one. Then for each natural number d there exist $B_F \in DS(W_d)$ such that*

$$TB_F(a_1, \dots, a_d) = \sum_{i=1}^d H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{i=1}^d a_i\right)$$

if a_1, \dots, a_d are elements of a λ -ring R .

Proof. For $d = 2$ we have to prove that there exists a sequence of operations $\beta_F^{p^e}$ of two variables such that $p^e \beta_F^{p^e}(a, b) = \tau_F^{p^e}(a, b) + \tau_F^{p^e}(b, a) - \theta_F^{p^e}(ab)$. But Proposition 5.6 and 6.1 accomplish just that.

The statement for $d > 2$ follows from the one for $d = 2$ as was proven in Proposition 9.1 of [C]. ●

To prove a similar result in the general case we need a generalisation of Lemma 9.2 of [C].

LEMMA 6.3. *Let F be a formal group over \mathbf{Z} . If R is a λ -ring and $a, b \in R$ then*

$$\begin{aligned}
 TH_F(1, a) &= Y_{F,C}^{-1} \cdot \sum TH_{F,P}(1, a^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s, \\
 TH_F(a, b) &= \left[\sum TH_{F,P}(a, a^{p^h q-1}) p^h C_F^{q,p^h}(p^h q)^s \right]^{-1} \\
 &\quad \times \left[\sum TH_{F,P}(a, a^{p^h q-1} b^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s \right]
 \end{aligned}$$

Here the sum is over all q prime to p and over all h .

Proof. One has $TH_F(1, a) = Y_F^{-1} \cdot a Y_F(a)$ and $TH_{F,P}(1, a^{p^h q}) = Y_{F,P}^{-1} \cdot a^{p^h q} Y_{F,P}(a^{p^h q})$ according to Remark 2.7. The substitution of these identities into Lemma 4.6 yields the first statement. According to 2.7 one has also

$$\begin{aligned}
 TH_F(1, ab) &= Y_F^{-1} \cdot ab Y_F(ab) = [Y_F^{-1} \cdot a Y_F(a)] \cdot [Y_F(a)^{-1} \cdot b Y_F(ab)] \\
 &= TH_F(1, a) \cdot TH_F(a, b).
 \end{aligned}$$

Similarly one has

$$TH_{F,P}(1, a^{p^h q} b^{p^h q}) = TH_{F,P}(1, a) \cdot TH_{F,P}(1, a^{p^h q-1} b^{p^h q}).$$

The combination of these identities with the first statement yields the second one. ●

PROPOSITION 6.4. *Let F be a formal group over \mathbf{Z} strongly isomorphic to a special one. Then there exist $B_F \in DS(W_d)$ such that*

$$TB_F(a_1, \dots, a_d) = \sum_{i=1}^d H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{j=1}^d a_j\right)$$

if a_1, \dots, a_d are elements of a λ -ring R .

Proof. (Essentially Proposition 9.3 of [C]). Let a, b be the canonical elements in U_2 . Let p be a prime. By Lemma 6.3 one has

$$\begin{aligned} & TH_F(1, a) \cdot T^2 H_F(a, b) - Y_{F,C}^{-1} \cdot TH_{F,P}(1, a) \\ & \quad \times \sum p^{2h} q C_F^{q,p^h} T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) (p^h q)^s \\ & = Y_{F,C}^{-1} \cdot \left[\sum [TH_{F,P}(1, a^{p^h q}) - p^h q TH_{F,P}(1, a) \right. \\ & \quad \times T^2 H_{F,P}(a, a^{p^h q - 1})] p^h C_F^{q,p^h} (p^h q)^s \left. \right] \\ & \quad \times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1}) p^{2h} q C_F^{q,p^h} (p^h q)^s \right]^{-1} \\ & \quad \times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) p^{2h} q C_F^{q,p^h} (p^h q)^s \right]. \end{aligned}$$

Here all four factors are in the image of T_p^2 ; for this one needs Proposition 4.7 for the first two factors and the $d = p^h q$ case of Proposition 6.2 for the second factor. So the whole difference is in the image of T_p^2 . Therefore modulo $T_p^2 DS(W_2)$ one has

$$\begin{aligned} & TH_F(1, a) \cdot T^2 H_F(a, b) + TH_F(1, b) \cdot T^2 H_F(b, a) - TH_F(1, ab) \\ & \equiv Y_{F,C}^{-1} \cdot \sum [p^h q TH_{F,P}(1, a) \cdot T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) + p^h q TH_{F,P}(1, b) \\ & \quad \times T^2 H_{F,P}(b, a^{p^h q} b^{p^h q - 1}) - TH_{F,P}(1, a^{p^h q} b^{p^h q})] p^h C_F^{q,p^h} (p^h q)^s \end{aligned}$$

and by using Proposition 6.2 for $d = 2p^h q$ one sees that this is in $T_p^2 DS(W_2)$. Thus the coefficient of n^s in this Dirichlet series is in $p^{2\gamma_r(n)} W_2$. Since that is

the case for every prime p the Chinese Remainder Theorem says that the coefficient is in $n^2 W_2$. In other words the Dirichlet series is in $T^2 DS(W_2)$. This finishes the proof for $d = 2$; the cases $d > 2$ follow from the case $d = 2$ as in Proposition 9.1 of [C]. ●

DEFINITION 6.5. Let F be a formal group over \mathbf{Z} . Then the operations v_F^n and the Dirichlet series N are defined by

$$\sum v_F^n(a, b)n^s = N_F(a, b) = H_F(1, a) \cdot \delta H_F(a, b).$$

In other words

$$v^n(a, b) = \sum_{mk=n} \theta_F^m(a) \phi^m(\delta \eta_F^k(a, b)).$$

Here the ϕ^m are the operations on differential forms introduced in §4 of [C].

PROPOSITION 6.6. Let F be a formal group over \mathbf{Z} which is strongly isomorphic to a special one, and let a_1, \dots, a_d be elements of a λ -ring R . Then

$$\sum_{i=1}^d N_F \left(a_i, \prod_{j \neq i} a_j \right) = \delta B_F(a_1, \dots, a_d)$$

Proof. This is an easy consequence of Proposition 6.4. For the details of the proof see Proposition 8.6 of [C]. ●

Now we prove the main theorem. Consider the expression $v_F(a, b) = \sum_{n=1}^\infty v_F^n(a, b)$. Using exactly the same arguments as in §6 of [C] one finds that it converges if $(a, b) \in I \times R \cup R \times I$ and that Corollary 2.9 and Proposition 6.6 imply that it maps the relations in Definition 0.1 to the zero element of $(\Omega_{R,I}/\delta I)^{\text{op}}$. By the same reasoning as in §7 of [C] one can lift the resulting map to $K_{2,L}(R, I)^{\text{op}}$.

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