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Limiting subcontinua and Whitney maps of tree-like continua

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Abstract. It is known that if X is a tree-like continuum and ω is any Whitney map for $C(X)$, then the Whitney continuum $\omega^{-1}(t)$ is an FAR for each $0 \leqslant t \leqslant \omega(X)$ (see [5] or [17]). In this paper, we define limiting subcontinua of a continuum and we prove the following: Let X be a tree-like continuum and let ω be any Whitney map for $C(X)$. Then for each $0 \leqslant t \leqslant \omega(X)$, the following are equivalent:

(1) $\omega^{-1}(t)$ is an absolute retract (= AR).
(2) $\omega^{-1}(t)$ is a Peano continuum
(3) $t \geqslant \sup \{\omega(L) | L$ is a limiting subcontinuum of $X\}$.

1. Introduction

Let X be a continuum and let ω be any Whitney map for $C(X)$. It is known that if X is a tree-like continuum, then the Whitney continuum $\omega^{-1}(t)$ is an FAR for each $0 \leqslant t \leqslant \omega(X)$ (see [5] or [17]). Also, if X is a dendrite (= locally connected tree-like continuum), then $\omega^{-1}(t)$ is an AR for each $0 \leqslant t \leqslant \omega(X)$ (see [19]). In this paper, we consider the following question: Let X be a tree-like continuum and let ω be any Whitney map for $C(X)$. What is the smallest number $I(\omega) \geqslant 0$ such that $\omega^{-1}(I(\omega))$ is an AR? Note that $\omega^{-1}(\omega(X)) = \{X\}$ is an AR. If X is a hereditarily indecomposable tree-like continuum, then $\omega^{-1}(t)$ is also a hereditarily indecomposable tree-like continuum for each $0 \leqslant t < \omega(X)$ (see [13]), hence $I(\omega) = \omega(X)$. On the other hand, it is easily seen that there is a tree-like continuum X such that $0 < I(\omega) < \omega(X)$. For example, consider the following set X in the plane E^2:

$$X = \{(X, \sin 1/x) \in E^2 | 0 < x \leqslant 1\} \cup \{(0, y) \in E^2 | -1 \leqslant y \leqslant 1\}.$$

Then $0 < I(\omega) = \omega(\{(0, y) \in E^2 | -1 \leqslant y \leqslant 1\}) < \omega(X)$.

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In this paper, we define limiting subcontinua of a continuum and we prove the following: Let X be a tree-like continuum and let ω be any Whitney map for $C(X)$. Then for each $0 \leqslant t \leqslant \omega(X)$, the following are equivalent:

(1) $\omega^{-1}(t)$ is an AR.
(2) $\omega^{-1}(t)$ is a Peano continuum.
(3) $t \geqslant \sup \{\omega(L) | L$ is a limiting subcontinuum of $X\}$.

Hence $I(\omega) = \sup \{\omega(L) | L$ is a limiting subcontinuum of $X\}$.

All spaces considered in this paper are assumed to be metric spaces. A *continuum* is a compact connected space. We denote by $C(X)$ the *hyperspace* of all nonempty subcontinua of a continuum X with the Hausdorff metric ϱ_H. Given a continuum X, a *Whitney map* ω for $C(X)$ (see [18] and [21]) is a map from $C(X)$ into $[0, \infty)$ satisfying $\omega(\{x\}) = 0$ for each $x \in X$ and $\omega(A) < \omega(B)$ if $A, B \in C(X)$, $A \subset B$ and $A \neq B$. It is well-known that such a map ω: $C(X) \to [0, \omega(X)]$ is a monotone map. Then the continua $\omega^{-1}(t)$ $(0 \leqslant t < \omega(X))$ are called *Whitney continua*. A continuum X is a *tree-like continuum* if for any $\varepsilon > 0$, there is an onto map f: $X \to T$ such that T is a (polyhedral) tree and diam $f^{-1}(y) < \varepsilon$ for each $y \in T$.

We refer readers to [18] for hyperspace theory.

2. Limiting subcontinua of a continuum

Let X be a continuum. A subcontinuum L of X is said to be a *limiting subcontinuum* of X provided that one of the following conditions (1) and (2) holds.

(1) L is a one point set.
(2) There is an open set $U \supset L$ of X and a sequence $\{L_n\}$ of subcontinua in U such that $\lim L_n = L$ and $A_n \cap A_m = \phi$ $(n \neq m)$, where A_n is the component of $Cl\ U$ containing L_n for each n.

Set $L(X) = \{L \in C(X) | L$ is a limiting subcontinuum of $X\}$. Note that $L(X) \supset F_1(X) = \{\{x\} | x \in X\}$ and $L(X)$ does not contain X.

The following propositions are easily seen. Hence we omit the proofs.

(2.1) Proposition. *If $L \in L(X)$ and L is nondegenerate, then there is $L' \in L(X)$ such that $L \subset L'$ and $L \neq L'$.*

(2.2) Proposition. *A continuum X is a Peano continuum if and only if $L(X) = F_1(X)$.*

Now, we prove the following

(2.3) PROPOSITION. *Let X be a continuum and let ω be any Whitney map for $C(X)$. Then for each $0 \leqslant t \leqslant \omega(X)$, the following are equivalent.*

(1) $\omega^{-1}(t)$ *is a Peano continuum.*
(2) $\omega^{-1}([t, \omega(X)])$ *is a Peano continuum.*
(3) $t \geqslant \sup \{\omega(L) | L \in L(X)\}$.

To prove (2.3), we need the following (cf. [14, (2.3)]).

(2.4) LEMMA. *Let $A \in \omega^{-1}(t)$ $(0 \leqslant t \leqslant \omega(X))$ and $\varepsilon > 0$. Then there are a neighborhood $U(t)$ of A in X and numbers t_0 and t_1 such that $t_0 < t < t_1$ and if $B \in U^*(t) \cap \omega^{-1}([t_0, t_1])$, then $\varrho_H(A, B) < \varepsilon$, where $U^*(t) = \{D \in C(X) | D \subset U(t)\}$.*

Proof of (2.3). We shall show that (3) implies (1). Let $A \in \omega^{-1}(t)$ and $\varepsilon > 0$. By (2.4), there is a neighborhood $U(t)$ of A in X satisfying the condition of (2.4). Let A' be the component of Cl $U(t)$ which contains A. Set $W(A) = C(A') \cap \omega^{-1}(t)$. We shall show that $W(A)$ is a neighborhood of A in $\omega^{-1}(t)$. Suppose, on the contrary, that there is a sequence $\{L_n\}$ of points of $\omega^{-1}(t)$ such that L_n is not contained in $W(A)$ for each n and $\lim L_n = A$. Since $U^*(t) \cap \omega^{-1}(t)$ is an open set in $\omega^{-1}(t)$, we may assume that $L_n \subset U(t)$ for all n. Note that $L_n \cap A' = \phi$ for each n. Let A_n be the component of *Cl* $U(t)$ containing L_n. Since $L_n \cap A' = \phi$, $A_n \cap A' = \phi$. Hence $A_n \cap A_m = \phi$ $(n \neq m)$. Then $A \in L(X)$. By (2.1), there is $L \in L(X)$ such that $A \subset L$ and $A \neq L$, which implies that $t = \omega(A) < \omega(L) < \sup \{\omega(L) | L \in L(X)\} \leqslant t$. This is a contradiction. Hence $W(A)$ is a neighborhood of A in $\omega^{-1}(t)$. Note that $W(A)$ is a continuum and diam $W(A) < 2\varepsilon$ (see (2.4)). This implies that $\omega^{-1}(t)$ is a Peano continuum.

Next, we shall show that (1) implies (3). Suppose, on the contrary, that $0 \leqslant t < \sup \{\omega(L) | L \in L(X)\}$. Then there is $L \in L(X)$ such that $\omega(L) > t$. Then there are a neighborhood U of L in X and a sequence $\{L_n\}$ of subcontinua of X such that $\lim L_n = L$, $L_n \subset U$ and $A_n \cap A_m = \phi$ $(n \neq m)$, where A_n denotes the component of *Cl* U containing L_n. We may assume that $\omega(L_n) > t$ for all n. Choose $B_n \in \omega^{-1}(t)$ with $B_n \subset L_n$ for each n. We may assume that $\lim B_n = B \subset L$. Since $\omega^{-1}(t)$ is locally connected, there are continua α_n $(n \geqslant n_0)$ of $\omega^{-1}(t)$ such that B, $B_n \in \alpha_n$ and $D_n = \cup \{E \in \alpha_n) \subset U$. Then D_n is a continuum containing B_n and B $(n \geqslant n_0)$ (see [13]). Hence $A_n \cap A_m \neq \phi$ $(m, n \geqslant n_0)$. This is a contradiction. The remainder of the proof is similar and will be omitted.

(2.5) COROLLARY. *Let X be a chainable continuum (resp. a proper circle-like continuum) and let ω be any Whitney map for $C(X)$. Then for any $t > 0$, the following are equivalent.*

(1) $\omega^{-1}(t)$ *is an arc* or a one point set (*resp. a circle* or a one point set).
(2) $t \geqslant \sup\{\omega(L) | L \in L(X)\}$.

Proof. By J. Krasinkiewicz [14], for any $0 \leqslant t' < \omega(X)$, $\omega^{-1}(t)$ is a chainable (resp. circle-like) continuum. Hence (2.5) follows from (2.3).

In [15], J. Krasinkiewicz and S.B. Nadler proved that the property of being an indecomposable chainable continuum is a Whitney property, and if X is a decomposable chainable continuum, then there is $t_0 < \omega(X)$ such that $\omega^{-1}(t)$ is an arc for each $t_0 \leqslant t < \omega(X)$. Also, they proved that if X is a decomposable proper circle-like continuum, then there is $t_0 < \omega(X)$ such that $\omega^{-1}(t)$ is a circle for each $t_0 \leqslant t < \omega(X)$. Hence we have

(2.6) Corollary. (1) *Let X be a chainable continuum. Then X is decomposable if and only if X is not contained in the closure of $L(X)$ in $C(X)$.* (2) *If X is a decomposable circle-like continuum, then X is not contained in the closure of $L(X)$ in $C(X)$.*

(2.7) Example. There is a decomposable tree-like continuum X such that X is contained in the closure of $L(X)$ in $C(X)$. Let P be a pseudo-arc from p to q in the plane E^2 and let U be an open set of P such that $\dim \mathrm{Fr}_X U = 0$, $p \in U$ and $q \in \mathrm{Int}_X (P - U)$. Set $X = P \cup (\mathrm{Fr}_X U \times [-1, 1]) \subset E^3$. Then X is a decomposable tree-like continuum. We can check that X is contained in the closure of $L(X)$ in $C(X)$.

3. Whitney continua of a tree-like continuum which are ARs

In this section, we prove the following main result in this paper.

(3.1) Theorem. *Let X be a tree-like continuum and let ω be any Whitney map for $C(X)$. Then for each $0 \leqslant t \leqslant \omega(X)$, the following are equivalent.*

(1) $\omega^{-1}(t)$ *is an* AR.
(2) $\omega^{-1}([t, \omega(X)])$ *is an* AR.
(3) $\omega^{-1}(t)$ *is a Peano continuum.*
(4) $\omega^{-1}([t, \omega(X)])$ *is a Peano continuum.*
(5) $t \geqslant \sup\{\omega(L) | L \in L(X)\}$.

Let X be a continuum contained in a metric space M. Then X is *weak homotopically trivial within small neighborhoods of M* provided that if f: $S^n \to X$ is any map from the n-sphere S^n ($n \geqslant 0$) to X, f is null-homotopic in any neighborhood of X in M. Note that if X is an FAR (see [1] for the definition of FAR), then X is weak homotopically trivial within small neighborhoods of any ANR M which contains X. Let X be a continuum contained in a metric space M. We may assume that diam $X < 1$. Then we consider the following property; (∗) there exists a sequence $\{\mathscr{V}_n\}_{n=0,1,2,\ldots}$ of finite closed coverings of X such that (i) $\mathscr{V}_0 = \{X\}$, and $X = \cup \{\mathrm{Int}_X V | V \in \mathscr{V}_n\}$ for each n, (ii) mesh $\mathscr{V}_n < 1/2^n$ for each n, and (iii) if $V_\alpha \in \mathscr{V} = \cup \mathscr{V}_n$ and $\cap V_\alpha \neq \phi$, then $\cap V_\alpha$ is weak homotopically trivial within small neighborhoods of M (cf. [16]). Note that if $\cap V_\alpha \neq \phi$, then $\cap V_\alpha$ has the property (∗).

The key lemma is the following:

(3.2) LEMMA. *Let X be a continuum contained in a metric space M. If X has the property* (∗), *then X is an* AR.

Proof. Let $\{\mathscr{V}_n\}$ be a sequence of closed coverings of X satisfying the property (∗). First, we shall prove that X is k-connected for each $k = 0, 1, 2, \ldots$. Since each intersection W of V's of $\mathscr{V} = \cup \mathscr{V}_n$ has the property (∗), the fact that X is k-connected implies that W is k-connected. We will show that X is 0-connected. Since each element V of $\mathscr{V}$ is connected, the conditions (i) and (ii) implies that X is a Peano continuum. Hence X is 0-connected. Next, we assume that X is $(k-1)$-connected ($k \geqslant 1$). Then each intersection of V's of $\mathscr{V}$ is also $(k-1)$-connected. We must show that X is k-connected. Let f: $\dot{\Delta} \to X$ be a map, where Δ denotes a $(k+1)$-simplex and $\dot{\Delta}$ denotes the boundary of Δ. Now, we will construct a sequence $\{f_n\}_{n=0,1,2,\ldots}$ of maps from Δ to M and a sequence $\{\mathscr{T}_n\}_{n=0,1,2,\ldots}$ of triangulations of Δ such that

(1) $\mathscr{T}_0$ is the standard triangulation of Δ and $\mathscr{T}_{n+1}$ is a subdivision of $\mathscr{T}_n$,
(2) $f_n(L_n) \subset X$, where L_n denotes the k-skeleton of $\mathscr{T}_n$, i.e., $L_n = |\mathscr{T}_n^k|$,
(3) $f_0|\dot{\Delta} = f$ and $f_{n+1}|L_n = f_n|L_n$ for each n,
(4) $(f_n, \mathscr{T}_n)$ is normed by $\mathscr{V}_n$, i.e., for any $(k+1)$-simplex σ of $\mathscr{T}_n$, there is some $V \in \mathscr{V}_n$ such that $f_n(\dot{\sigma}) \subset V$ and $f_n(\sigma) \subset N(V)$, where $N(V)$ is a neighborhood of V in M such that diam $N(V) < 1/2^n$ (see (ii)), and
(5) if σ is a $(k+1)$-simplex of $\mathscr{T}_n$ and $V \in \mathscr{V}_n$ is as in (4), then for any $(k+1)$-simplex σ' of $\mathscr{T}_{n+1}$ with $\sigma' \subset \sigma, f_{n+1}(\sigma') \subset N(V)$.

Note that $L_0 = \dot{\Delta}$. Since X is weak homotopically trivial within small neighborhoods of M, we have an extension f_0: $\Delta \to N(X)$ of f, where $N(X)$ is a neighborhood of X in M such that diam $N(X) < 1/2^0$. Clearly, f_0 satisfies the conditions (1)–(5). Suppose that we have maps $f_0, f_1, \ldots, f_{n-1}$

which satisfy the conditions (1)–(5). We will construct the desired map f_n as follows: For each $(k+1)$-simplex σ of $\mathscr{T}_{n-1}$, there is some $V \in \mathscr{V}_{n-1}$ satisfying the condition (4), i.e., $f_{n-1}(\dot{\sigma}) \subset V$ and $f_{n-1}(\sigma) \subset N(V)$.

Consider the following set

$$\mathscr{V}_n(\sigma) = \{V' \cap V \mid V' \cap V \neq \phi,\ V' \in \mathscr{V}_n\}.$$

For each $W = V' \cap V \in \mathscr{V}_n(\sigma)$, choose a closed subset $N(W)$ of M such that $N(W) \cap V = W$, diam $N(W) < 1/2^n$ and $\cup\{\mathrm{Int}_M N(W) \mid W \in \mathscr{V}_n(\sigma)\}$ is a neighborhood of V in $N(V)$. We may assume that $N(W_1) \cap \ldots \cap N(W_i) \neq \phi$ if and only if $W_1 \cap \ldots \cap W_i \neq \phi$ for $W_1, \ldots, W_i \in \mathscr{V}_n(\sigma)$. Since V is weak homotopically trival within small neighborhoods of M, there is an extension $g_\sigma\colon \sigma \to \cup\{\mathrm{Int}_M N(W) \mid W \in \mathscr{V}_n(\sigma)\}$ of $f_{n-1}|\dot{\sigma}$. Choose a subdivision $\mathscr{T}_n$ of $\mathscr{T}_{n-1}$ such that if σ' is a $(k+1)$-simplex of $\mathscr{T}_n$ and $\sigma' \subset \sigma \in \mathscr{T}_{n-1}$, then $g_\sigma(\sigma') \subset N(W)$ for some $W \in \mathscr{V}_n(\sigma)$. If P is a vertex of $\mathscr{T}_n$ and $P \in \sigma - \dot{\sigma}$ ($\sigma \in \mathscr{T}_{n-1}$), we choose a point $h(P) \in \cap\{W \in \mathscr{V}_n(\sigma) \mid g_\sigma(P) \in N(W)\}$. Hence we have a map $h\colon L_{n-1} \cup |\mathscr{T}_n^0| \to M$ such that $h|L_{n-1} = f_{n-1}|L_{n-1}$. Since $\cap\{W \in \mathscr{V}_n(\sigma) \mid h((L_{n-1} \cap \tau) \cup \tau^0) \subset W\}$ is $(k-1)$-connected for any $\tau \in \mathscr{T}_n^k$ with $\tau \subset \sigma$ (where σ is a $(k+1)$-simplex of $\mathscr{T}_{n-1}$ and τ^0 denotes the 0-skeleton of τ), by induction we can easily see that there is an extension $h'\colon L_n \to X$ of h such that if σ' is a $(k+1)$-simplex of $\mathscr{T}_n$ and $\sigma' \subset \sigma \in \mathscr{T}_{n-1}$, then $h'(\dot{\sigma}') \subset W$, where $W \in \mathscr{V}_n(\sigma)$ with $g_\sigma(\sigma') \subset N(W)$. Since $W \in \mathscr{V}_n(\sigma)$ is weak homotopically trivial within small neighborhoods of M, there is a map $f_n\colon \Delta \to M$ such that $f_n|L_n = h'$ and if σ' is $(k+1)$-simplex of $\mathscr{T}_n$ and $\sigma' \subset \sigma \in \mathscr{T}_{n-1}$, then $f_n(\dot{\sigma}') \subset W$ and $f_n(\sigma') \subset N(W)$ for some $W \in \mathscr{V}_n(\sigma)$ with $g_\sigma(\sigma') \subset N(W)$. Clearly, $(f_n, \mathscr{T}_n)$ is normed by $\mathscr{V}_n$. Also, f_n satisfies the desired conditions. Hence we obtain a sequence $\{f_n\}$ of maps from Δ to M such that $\{f_n\}$ satisfies the conditions (1)–(5). By (4) and (5), we see that $\{f_n\}$ is a Cauchy sequence of maps. Set $F = \lim f_n$. By (4), we can conclude that $F(\Delta) \subset X$. Also, by (3) $F|\dot{\Delta} = f$. Hence X is k-connected, which implies that each intersection of V's of $\mathscr{V} = \cup\mathscr{V}_n$ is k-connected. Finally, we shall show that X is an AR. Let $\mathscr{U}$ be any open covering of X. By (ii), we may assume that $\mathscr{V}_1$ is a refinement of $\mathscr{U}$. Let K be a simplicial complex and let L be a subcomplex of K such that $K^0 \subset L$. Let f: $|L| \to X$ be a partial realization of K in X relative $\mathscr{V}_1$, i.e., for each simplex σ of K, there is some $V \in \mathscr{V}_1$ such that $f(|L| \cap \sigma) \subset V$ (see [2] and [3]). By using the fact that each intersection of V's of $\mathscr{V}_1$ is k-connected for all $k = 0, 1, 2, \ldots$, we can easily see that there is a full realization $F\colon |K| \to X$ of f in X relative $\mathscr{V}_1$ such that if $\sigma \in K$, then $F(\sigma) \subset \cap\{V \in \mathscr{V}_1 \mid f(|L| \cap \sigma) \subset V\}$. By [2] or [3], X is an ANR. Since

X is k-connected for all $k = 0, 1, 2, \ldots$, X is an AR. This completes the proof.

Proof of (3.1). We shall prove that (5) implies (1). Suppose that $t \geqslant \sup \{\omega(L) | L \in L(X)\}$. We shall show that $\omega^{-1}(t)$ has the property (*). Note that $\omega^{-1}(t)$ is an FAR (see [5] or [17]). Hence $\omega^{-1}(t)$ is weak homotopically trivial within small neighborhoods of Q, where Q is the Hilbert cube which contains $\omega^{-1}(t)$. Let $\varepsilon > 0$. As in the proof of (2.3), if $A \in \omega^{-1}(t)$, then $W(A) = C(A') \cap \omega^{-1}(t)$ is a closed neighborhood of A in $\omega^{-1}(t)$ such that diam $W(A) < 2\varepsilon$. Suppose that $A_\alpha \in \omega^{-1}(t)$ and $\cap W(A_\alpha) \neq \phi$. Note that

$$\cap W(A_\alpha) = \cap C(A'_\alpha) \cap \omega^{-1}(t) = C(\cap A'_\alpha) \cap \omega^{-1}(t).$$

Since X is a tree-like continuum, $\cap A'_\alpha$ is also a tree-like continuum. Hence $\cap W(A_\alpha)$ is an FAR (see [5] or [17]). Hence we can conclude that $\omega^{-1}(t)$ has the property (*). By (3.2), $\omega^{-1}(t)$ is an AR. In a similar way, we can see that (5) implies (2). The remainder of the proof follows from (2.3).

(3.3) COROLLARY. *If X is a tree-like continuum and ω is any Whitney map for $C(X)$, then $\omega^{-1}(t)$ is contractible for $t \geqslant$* $\sup \{\omega(L) | L \in L(X)\}$.

(3.4) EXAMPLE. Consider the following points in the plane E^2. $p = (3, 0)$, $q = (-2, 0)$, $p' = (-1, 0)$, $q' = (1, 0)$, $p_n = (-1, -1/n)$ and $q_n = (1, 1/n)$ $(n = 1, 2, \ldots)$. Let $X = [p, q] \cup \bigcup_{n=1}^{\infty} [p, p_n] \cup \bigcup_{n=1}^{\infty} [q, q_n]$, where $[x, y]$ denotes the segment from x to y in E^2, $x, y \in E^2$. Then X is a dendroid (= path-connected tree-like continuum).

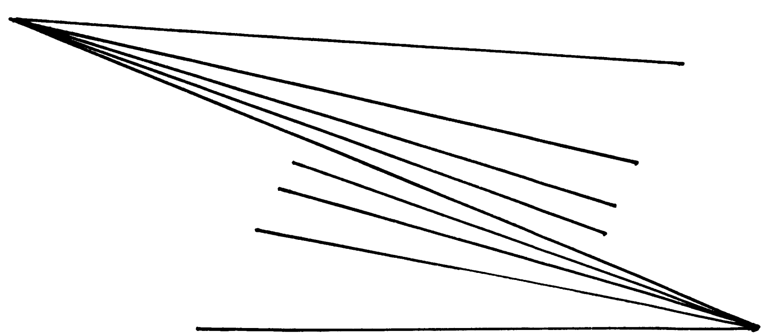

Fig. 1.

Let ω be any Whitney map for $C(X)$. It is easily seen that $\sup\{\omega(L)|L \in L(X)\} = \max\{\omega([p, p']), \omega([q, q'])\}$.

(a) $0 < t \leqslant \omega([p', q'])$.

$\omega^{-1}(t)$:

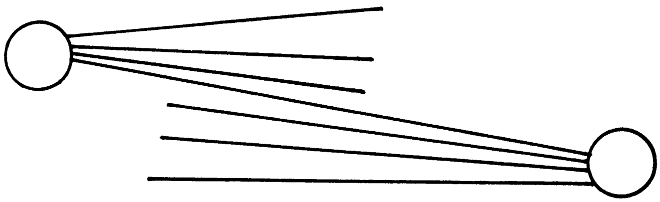

Fig. 2.

(b) $\omega([p', q']) < t \leqslant \min\{\omega([p, p']), \omega([q, q'])\}$.

$\omega^{-1}(t)$:

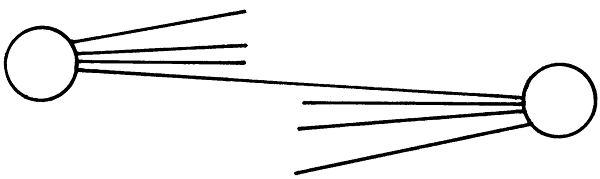

Fig. 3.

(c) $\min\{\omega([p, p']), \omega([q, q'])\} \leqslant t < \max\{\omega([p, p']), \omega([q, q'])\}$.

$\omega^{-1}(t)$:

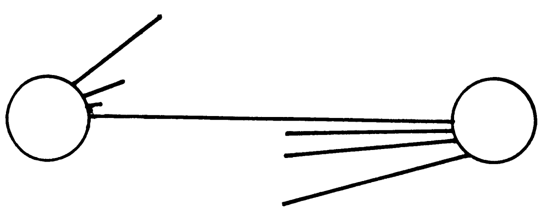

Fig. 4.

(d) $\max\{\omega([p, p']), \omega([q, q'])\} \leqslant t < \omega([p, q])$.

$\omega^{-1}(t)$:

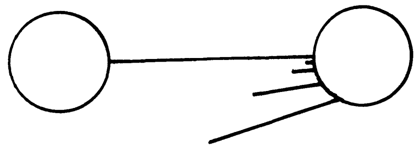

Fig. 5.

(e) $\omega([p, q]) \leqslant t < \omega(X)$.

$\omega^{-1}(t)$:

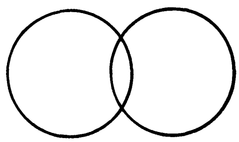

Fig. 6.

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