V. B. MEHTA
A. RAMANATHAN

Schubert varieties in $G/B \times G/B$


<http://www.numdam.org/item?id=CM_1988__67_3_355_0>
Schubert varieties in $G/B \times G/B$

V.B. MEHTA & A. RAMANATHAN
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400 005, India

Received 1 December 1987; accepted 11 February 1988

Introduction

Let $G$ be a semi-simple, simply connected algebraic group defined over an algebraically closed field of characteristic $p > 0$. Let $T \subset G$ be a maximal torus, $B \supset T$ a Borel subgroup and $W = N(T)/T$ the Weyl group. $G$ acts on the homogeneous space $G/B$ and also on $G/B \times G/B$ by the diagonal action: for $g, x_1, x_2, \in G$, $g(x_1 B, x_2 B) = (gx_1 B, gx_2 B)$. By Schubert Varieties in $G/B \times G/B$ we mean the closures of the $G$-orbits in $G/B \times G/B$. It is known ([11, 12]) that these orbit closures are in 1-1 correspondence with the elements of $W$, the element $w \in W$ corresponding to the closure of the orbit of $(eB, wB)$, where $e \in G$ is the identity element. In particular, taking $w = e$, $G/B$ gets imbedded diagonally in $G/B \times G/B$.

In this paper we prove that these Schubert Varieties are Frobenius-split in the sense of [4, Def. 2]. Our method is as follows: fix $w \in W$ with $l(w) = i$ and denote the Schubert variety in $G/B$ corresponding to $w$ by $X_i$. Then $B$ acts on $X_i$ on the left and one may form the associated fibre-space $\tilde{X}_i = G \times \mathbb{A}^8 X_i$. The map $f: \tilde{X}_i \rightarrow G/B \times G/B$ given by $f(g, x) = (gB, gxB)$ is an isomorphism onto the $G$-orbit closure of $(eB, wB)$ (cf. [11]). Hence we may work with $\tilde{X}_i$ instead. Express $w$ as a product of reflections associated to the simple roots, $w = s_{a_1}s_{a_2} \ldots s_{a_k}$ and $Z_i \rightarrow X_i$ be the corresponding Demazure desingularization of $X_i$ (cf. [2, 3]) and let $\psi_i: Z_i \rightarrow X_i$ be the birational map. $B$ acts on $Z_i$ on the left and we may construct the associated fibre-space $\tilde{Z}_i = G \times \mathbb{A}^8 Z_i$. The map $\psi_i$ is $B$-equivariant and descends to a birational map $\tilde{\psi}_i: \tilde{Z}_i \rightarrow \tilde{X}_i$. Since $X_i$ is normal [1, 5, 7, 10] and $\tilde{X}_i \rightarrow G/B$ is a fibre-space with fibre $X_i$ it follows that $\tilde{X}_i$ is also normal and that $\tilde{\psi}_i_*(\mathcal{O}_{\tilde{Z}_i}) = \mathcal{O}_{\tilde{X}_i}$. So to prove that $\tilde{X}_i$ is Frobenius-split, it is sufficient to prove that $\tilde{Z}_i$ is Frobenius-split. We calculate the canonical bundle $K_{\tilde{Z}_i}$ of $\tilde{Z}_i$ (this has been done, without detail, in [11]). From this description of $K_{\tilde{Z}_i}$ it follows from [4 prop. 8] that $\tilde{Z}_i$ is Frobenius-split. It also follows from [6, 8] that $\tilde{X}_i$ is Cohen-Macaulay and has rational singularities. We first recall the basic
facts about the standard resolutions of Schubert Varieties in $G/B$ and Frobenius-splitting from [4, 8] and then we prove the main result. Our result should prove useful in the study of the decomposition of the $G$-module $H^0(G/B, L) \times H^0(G/B, M)$, where $L$ and $M$ are line bundles on $G/B$, see [11].

Section 1

Let $G$, $B$ and $W$ be as in the introduction, let $w \in W$ with $l(w) = i$ and denote by $X_i$ the Schubert variety in $G/B$ corresponding to $w$. Then according to [2, 3, 8] there exists a smooth projective variety $Z_i$, and a map $\psi_i : Z_i \to X_i$ with the following properties:

1. $\psi_i$ is birational.
2. There exists $i$ smooth subvarieties of codim 1 in $Z_i$, denoted by $Z_{i,1}, \ldots, Z_{i,i}$ intersecting transversally. Further if we denote $\bigcup_{j=1}^i Z_{i,j}$ by $\partial Z_i$, then $\psi_i^{-1}(\partial X_i) = \partial Z_i$, where $\partial X_i$ is the union of the codim 1 Schubert varieties in $X_i$.
3. Put $v = w s_0 B$ and $X_{i-1} = B v B/B$. Then there exists a map $f_i : Z_i \to Z_{i-1}$ such that $f_i$ is a locally trivial $\mathbb{P}^1$ fibration with a section $\sigma_i : Z_{i-1} \to Z_i$. Further, $\partial Z_i = f_i^{-1}(\partial Z_{i-1}) \cup \sigma_i(Z_{i-1})$.
4. The canonical bundle $K_{Z_i}$ is given by the formula $K_{Z_i} = \mathcal{O}_{Z_i}(-\partial Z_i) \times \psi_i^* L_{\psi_i}^{-1}$ is the line bundle on $X_i \subset G/B$ associated to half sum of the positive roots.

The varieties $Z_i$ and the morphisms $\psi_i$ are constructed by induction on $l(w)$, see [3, 8] for more details. We recall one proposition from [8].

**Proposition 1.** $Z_i$ is Frobenius-split and any sub-intersection of the divisors in $\partial Z_i$ is compatibly split in $Z_i$.

**Proof.** This is [8, Remark 2.5].

Now consider the varieties $\tilde{Z}_i = G \times B Z_i$ as in the introduction. The maps $f_i : Z_i \to Z_{i-1}$ and $\sigma_i : Z_{i-1} \to Z_i$ are $B$-equivariant, hence we get maps $\tilde{f}_i : \tilde{Z}_i \to \tilde{Z}_{i-1}$ and $\tilde{\sigma}_i : \tilde{Z}_{i-1} \to \tilde{Z}_i$. It follows that there exist $i$ smooth subvarieties of $\tilde{Z}_i$ denoted by $\tilde{Z}_{i,1}, \ldots, \tilde{Z}_{i,i}$, intersecting transversally, whose union we denote by $\partial \tilde{Z}_i$. Let $p_1$ and $p_2$ denote the two projections of $G/B \times G/B$ and for any pair of line bundles $L$, $M$ on $G/B$, denote $(p_1^* L \times p_2^* M)$ by $(L, M)$.

**Proposition 2.** The canonical bundle $K_{\tilde{Z}_i}$ is given by

$$K_{\tilde{Z}_i} = \mathcal{O}_{\tilde{Z}_i}(-\partial \tilde{Z}_i) \times \tilde{\psi}_i^*(L_{\psi_i}^{-1}, L_{\psi_i}^{-1}).$$
Proof. (See also [11]). We prove this by induction on \( l(w) \). If \( l(w) = 0 \) then \( Z_0 = G/B \) and \( \partial Z_0 = \emptyset \). So \( \mathcal{O}_{Z_0}(\partial Z_0) \times \hat{\psi}_*(L^{-1}_{\emptyset}, L^{-1}_{\emptyset}) \) is the line-bundle \( L^{-1}_{\emptyset} \) on \( G/B \), as \( \hat{\psi}_0: G/B \to G/B \times G/B \) is the diagonal imbedding. Assume the result for \( l(w) = i - 1 \). Now it follows from [8, Lemma 3] that \( K_{Z_{i-1}} = \mathcal{O}_{Z_i}(\sigma(\hat{Z}_{i-1})) \times \hat{\psi}_i(1, L^{-1}_{\emptyset}, L^{-1}_{\emptyset}) \times \hat{f}_i^* \sigma^* \hat{\psi}_i^*(1, L_0) \). Denote this line bundle on \( Z_i \) by \( A \). Then \( K_{Z_i} = A \times \hat{f}_i^* (K_{Z_{i-1}}) = A \times \hat{f}_i^* \left( \mathcal{O}_{Z_i}(\partial Z_i) \times \hat{\psi}_i(1, L^{-1}_{\emptyset}, L^{-1}_{\emptyset}) \times \hat{f}_i^* \sigma^* \hat{\psi}_i^*(1, L_0) \right) \). But \( \hat{\psi}_i^*: Z_{i-1} \to G/B \times G/B = \hat{\psi}_i: Z_{i-1} \to G/B \times G/B \). Hence we get \( K_{Z_i} = \mathcal{O}_{Z_i}(\partial Z_i) \times \hat{\psi}_i(L^{-1}_{\emptyset}, L^{-1}_{\emptyset}) \).

**Theorem 1.** \( Z_i \) is Frobenius-split and any sub-intersection of the divisors in \( \partial Z_i \) is compatibly split in \( Z_i \).

Proof. From Prop. 2, we know that \( K_{Z_{i-1}} = \mathcal{O}_{Z_i}(\partial Z_i) \times \hat{\psi}_i^*(L_0, L_0) \). From [8, Remark 2], we know that \( \sigma = D + \hat{D} \) is an element of \( H^0(G/B, L^2_0) \) such that \( \sigma^{p-1} \) splits \( G/B \). Consider the section \( t = \partial Z_i + \hat{\psi}_i^*(D, \hat{D}) \) of \( K_{Z_i} \). It follows from [4, Prop. 8] that \( t^{p-1} \) splits \( Z_i \) and that any sub-intersection of the divisors in \( \partial Z_i \) is compatibly split in \( Z_i \) by \( t^{p-1} \).

**Corollary 1.** Let \( N \) be the length of the maximal element \( w_0 \in W \). Then by the above, \( Z_0 \) is compatibly-split in \( Z_N \). So it follows from [4, Prop. 4] that \( \hat{\psi}_0(\hat{Z}_0) = G/B \) is compatibly-split in \( \hat{\psi}_N(\hat{Z}_N) = G/B \times G/B \), where \( G/B \) is imbedded diagonally in \( G/B \times G/B \).

This was first proved by the second author by other methods (cf. [9]).

**Corollary 2.** From Corollary 1 and [9, Cor. 2.3] it follows that any imbedding of \( G/B \) by a complete linear system is projectively normal.

This was first proved in [7] (see also [9]).

**Corollary 3.** It follows from [6, 8] that the Schubert varieties \( \tilde{X}_i \) in \( G/B \times G/B \) are Cohen–Macaulay and have rational singularities.

**Remark.** It can be proved, using the methods of [9], that these Schubert varieties in \( G/B \times G/B \) are scheme-theoretically defined by quadrics. This will be taken up in a later paper. Analogues follow for \( G/P_1 \times G/P_2 \).

**References**