KAPIL PARANJAPE

Abelian varieties associated to certain K3 surfaces

Compositio Mathematica, tome 68, nº 1 (1988), p. 11-22
<http://www.numdam.org/item?id=CM_1988__68_1_11_0>

Compositio Mathematica 68: 11–22 (1988)

Abelian varieties associated to certain $K3$ surfaces

KAPIL PARANJAPE
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400 005, India

Received 16 June 1987; accepted 17 March 1988

0. Introduction

In ([K–S]) Kuga and Satake associate an abelian variety to each polarized $K3$ surface. This is refined by Deligne in [D] where he shows that this is in fact an absolute Hodge correspondence, i.e. according to the Hodge conjecture there should exist an algebraic correspondence between the $K3$ surface and the associated Kuga–Satake abelian variety. The aim of this paper is to set up this correspondence in the particular case when the $K3$ surface is the desingularization of the double cover of the plane branched along six lines. It can be shown that when the six lines become tangent to a conic the $K3$ surface is the Kummer quartic surface. Further, the Kuga–Satake abelian variety in this case is just a number of copies of the jacobian of the hyperelliptic curve obtained as the double cover of the conic branched at the six points of tangency (see [M]). for the general set of six lines, however, cohomology computations show that the Kuga–Satake abelian variety is a number of copies of a four dimensional abelian variety. In fact, this four-dimensional variety turns out to be the Prym variety of a certain fourfold cover of an elliptic curve, the cover being of genus five. The problem thus reduces to setting up a correspondence between the $K3$ surface and this curve of genus five. The key to this is a construction of C. Schoen (in [S]) emulating which we construct a surface which we call the "Schoen surface". This surface is a quotient of the product of the curve with itself and has an involution on it, the quotient by which gives the $K3$ surface. Though we begin with the curve and then construct the $K3$ surface, we can show that every $K3$ surface of the above type is reached by the construction.

Let $Y1$ be the desingularization of the double cover of the plane branched along six lines. In Section 1 we show that the lattice T of transcendental cycles on $Y1$ is isomorphic to Hyp $\oplus$ Hyp $\oplus$ $[-2]$ $\oplus$ $[-2]$ as a quadratic space (where Hyp stands for the 2-dimensional hyperbolic space). The Hodge structure on this is given by a 2-dimensional real subspace of $T \otimes \mathbf{R}$ on which the quadratic form is positive definite.

Let C be a curve of genus five with an automorphism of order 4, such that the quotient is an elliptic curve and the isotropy of any point is atmost of order 2. In Section 2 we compute the Hodge structure of the Prym variety. The lattice F of the Prym variety is a hermitian lattice over $\mathbf{Q}[i]$ of the form Hyp $\oplus$ Hyp. The Hodge structure is given by a 2-dimensional complex subspace of $F \otimes \mathbf{R}$ on which the hermitian form is positive definite. There is a natural $\mathbf{Q}[i]$-antilinear involution on $\Lambda^2_{\mathbf{Q}[i]} F$ which preserves the Hodge structure. The invariants of this involution thus give a Hodge structure which we check is of the same type as the lattice of transcendental cycles on the $K3$ surface as described above. Further, it is easily seen that this is the Kuga–Satake–Deligne correspondence in the given case (upto isogeny and repetitions of the Prym). In fact, if $G1$ is the Shimura variety corresponding to SO(Hyp $\oplus$ Hyp $\oplus$ $[-2] \oplus [-2]$) which parametrizes Hodge structures of type T and $G2$ is the Shimura variety corresponding to U(Hyp $\oplus$ Hyp) which parametrizes Hodge structures of type F, then we see that we have a finite–finite correspondence between $G1$ and $G2$. Thus, the problem can be seen to be twofold. Given a curve of the above type, to construct a $K3$ surface and to represent the above Hodge correspondence by an algebraic cycle on the product of the Prym variety and the $K3$ surface.

The geometric construction in Section 3 exactly mirrors the above algebraic calculation. Reading between the lines in [S] there is a natural way to construct out of C a surface W whose lattice of transcendental cycles is isomorphic to $\Lambda^2_{\mathbf{Q}[i]} F$ as a Hodge structure. We then produce an involution on W which acts like the involution described above. The desingularization of the quotient of W by this involution gives a $K3$ surface of the required type. One then checks (in Section 4) that this construction represents the Hodge correspondence as required.

1. Cohomology of the $K3$ surface

Let $L_1, \ldots, L_6$ be lines in the projective plane $\mathbf{P}^2$ such that no three of these are coincident. We can then form the double cover of $\mathbf{P}^2$ branched precisely along these six lines. Let $Y1 \to Y$ be the desingularization. If $p_{i,j} = L_i \cap L_j$, then we have exceptional curves $E_{i,j}$ of $Y1 \to Y$ precisely over these points. Further, if $[H]$ is class of the inverse image of a line in $\mathbf{P}^2$, then for a general choice of $L_1, \ldots, L_6$, the Néron–Severi group NS of $Y1$ can be described over the rationals:

$$NS = \mathbf{Q}\text{-span of } \{[E_{i,j}] | i < j;\ i, j = 1, \ldots, 6\} \oplus \mathbf{Q}[H].$$

We wish to prove

LEMMA 1. *Let $Y1$ be the desingularization of the double cover Y of the plane branched along six lines. Let $E_{i,j}$'s be the exceptional curves of the morphism $Y1 \to Y$, and H the pull back to $Y1$ of a general line in the plane. If*

$$N1 = \mathbf{Q}\text{-span of } \{[E_{i,j}] | i < j;\ i, j = 1, \ldots, 6\} \oplus \mathbf{Q}[H]$$

then,

$$T = N1^{\perp} = \text{Hyp} \oplus \text{Hyp} \oplus [-2] \oplus [-2] \quad \textit{as a quadratic space.}$$

Further, T is a sub-Hodge structure of $H^2(Y1, \mathbf{Q})$, with this structure being given by a 2-dimensional linear subspace of $T \otimes \mathbf{R}$ on which the intersection pairing is positive definite.

Proof of Lemma 1. Looking at the Hodge structure on $H^2(Y1, \mathbf{C})$ we see that $H^0(Y1, K_{Y1}) \oplus H^2(Y1, \mathcal{O}_{Y1})$ is perpendicular to $NS \otimes \mathbf{C} \subset H^1(Y1, \Omega^1_{Y1})$. Thus, the second part of the statement is immediate. We note that the first part of the statement needs to be proved only when $Y1$ and Y specialize to $X1$ and X, where $X1$ and X are the surfaces obtained for the special choice of six lines tangent to a conic Q. Then it is known classically that $X1$ and X can be alternatively described as follows.

Let C be the hyperelliptic curve obtained as the double cover of the conic Q branched along the six points of tangency, and A be the Jacobian variety of C. Choosing as base point on C one of the six points of ramification we get a natural inclusion $C \to A$. This gives the theta divisor on A. The linear system $|2C|$ on A gives a morphism $A \to \mathbf{P}^3$, which factors through $X' = A/i$, where i is the involution on A given by the inverse in the group law.

X is obtained by blowing up the image in X' of the origin in A and the map to $\mathbf{P}^2$ is obtained as the projection from this point. $X1$ is got by blowing up all the sixteen nodes of X' corresponding to the points of order 2 on A. Equivalently, $X1 = A1/i$, where $A1$ is the blow up of A at the sixteen points of order 2 on A. If E_a denotes the exceptional curve in $A1$ corresponding to the point a in A, then

$$H^2(A1, \mathbf{Q}) = H^2(A, \mathbf{Q}) \oplus \mathbf{Q}\text{-span of } \{[E_a] | a \in_2 A\}.$$

Now, A is the Jacobian of C and thus

$$H^2(A, \mathbf{Q}) = \Lambda^2 H^1(C, \mathbf{Q}).$$

The action of i on $H^1(C, \mathbf{Q})$ is by $-$Identity, so i acts trivially on $H^2(A, \mathbf{Q})$. The natural homomorphism $H^2(X1, \mathbf{Q}) \to H^2(A1, \mathbf{Q})$ is therefore an isomorphism of vector spaces. The intersection pairing on $H^2(X1, \mathbf{Q})$ is however, 1/2 the intersection pairing on $H^2(A1, \mathbf{Q})$.

Choose a symplectic basis $\{e_1, f_1, e_2, f_2\}$ for $H^1(C, \mathbf{Q})$. This gives a basis for $H^2(A, \mathbf{Q})$:

$$\begin{aligned} a_1 &= e_1 \wedge f_1; & b_1 &= e_2 \wedge f_2 \\ a_2 &= e_1 \wedge e_2; & b_2 &= f_2 \wedge f_1 \\ a_3 &= e_1 \wedge f_2; & b_3 &= f_1 \wedge e_2 \end{aligned}$$

such that:

1. $(a_i \cdot b_i) = 1$ w.r.t. the intersection pairing on $H^2(A, \mathbf{Q})$.
2. $a_1 + b_1 = [C]$, with $[C]$ the class of C as a curve on A as above.

By this description, under the specialization from $Y1$ to $X1$, the class $[H]$ specializes to $2(a_1 + b_1 - [E_0])$ with 0 the origin in A, and the classes $[E_{i,j}]$ specialize to $[E_a]$'s for a in ${}_2A$ different from 0. Hence we may compute the lattice T in $H^2(X1, \mathbf{Q})$ as the perpendicular of,

$$N1 = \mathbf{Q}\text{-span of } \{[E_a] | a \neq 0\} \oplus \mathbf{Q}(a_1 + b_1 - [E_0]).$$

This is easily computed to be isomorphic to Hyp $\oplus$ Hyp $\oplus$ $[-2] \oplus [-2]$, where Hyp stands for the 2-dimensional hyperbolic space. Q.E.D. Lemma 1.

2. The Prym variety

LEMMA 2. *Let C be a curve of genus 5 with an automorphism of order 4, s.t. the quotient by this automorphism is an elliptic curve E and the isotropy of any point is atmost of order 2. The Prym variety for this cover is given by a Hodge structure F of the following type:*

1. *The action of the automorphism makes F a $\mathbf{Q}[i]$-vector space of rank 2.*
2. *The symplectic form of F gives a hermitian form w.r.t. this $\mathbf{Q}[i]$-structure which of type Hyp $\oplus$ Hyp.*
3. *The Hodge structure is determined by a 2-dimensional complex subspace of $F \otimes \mathbf{R}$ on which the hermitian form is positive definite.*

Proof of Lemma 2. Let $p_1 + p_2 + p_3 + p_4$ be the branch locus on E. We have a line bundle L on E of degree 2 such that,

$$L^4 \simeq \mathcal{O}_E(p_1 + \cdots + p_4)^2 \quad \text{but} \quad L^2 \not\simeq \mathcal{O}_E(p_1 + \cdots + p_4).$$

This gives a line bundle $M = L^{-2} \otimes \mathcal{O}_E(p_1 + \cdots + p_4)$ which is of order 2 in $\mathrm{Pic}(E)$. The isomorphisms,

$$L^2 \to M(p_1 + \cdots + p_4) \quad \text{and} \quad M^2 \to \mathcal{O}_E$$

give an algebra structure to $\mathcal{O}_E \oplus L^{-1} \oplus L^{-1} \otimes M \otimes M$, as well as to $\mathcal{O}_E \oplus M$.

It is easily seen that for a suitable choice of L,

$$C = \mathrm{Spec}_E \mathcal{O}_E \oplus L^{-1} \oplus L^{-1} \otimes M \otimes M.$$

Further, if E' is the quotient of C by the isotropy then,

$$E' = \mathrm{Spec}_E \mathcal{O}_E \oplus M.$$

The automorphism J of order 4 is described by $-i \cdot$ (Identity) on L and by ($-$Identity) on M. (Here i denotes the square root of -1).

Let P be the connected component of identity in the kernel of the natural homomorphism $\mathrm{Jac}(C) \to \mathrm{Pic}(E)$, i.e. the Prym variety. Then we have:

1. $H^2(C, Q) = H^1(E, Q) \oplus F$; where $F = H^1(P, \mathbf{Q})$ and
2. $H^1(C, \mathcal{O}_C) = H^1(E, \mathcal{O}_E) \oplus V \oplus W$; where $V = H^1(E, L^{-1})$ and $W = H^1(E, L^{-1} \otimes M)$.

The action of J is by $i \cdot$ (Identity) on V and by $i \cdot$ ($-$Identity) on W. There is a natural morphism $H^1(C, \mathbf{R}) \to H^1(C, \mathcal{O}_C)$ and $F \otimes \mathbf{R}$ goes isomorphically (as real vector spaces) to $V \oplus W$. This isomorphism is compatible with the J action.

The automorphism J of C gives a $\mathbf{Q}[i]$ action on F, and so the restriction to F of the symplectic structure on $H^1(C, \mathbf{Q})$ is the imaginary part of a Hermitian structure H. Since the order of the automorphism is a power of 2 the discriminant of H is also a power of 2 in $\mathbf{Q}$. But then $2 = \mathrm{Norm}(1 + i)$ so that $\mathrm{disc}(H) = 1$ in $\mathbf{Q}^x/\mathrm{Norm}(\mathbf{Q}[i]^x)$.

Now $V \oplus W$ is the tangent space to P and the symplectic structure on F is the restriction of the imaginary part of the positive definite Hermitian structure $H1$ which gives the polarization on P. This Hermitian structure is

J-invariant and so, for all v in V and w in W

$$H1(v, w) = H1(Jv, Jw) = H1(i \cdot v, -i \cdot w) = -H1(v, w) = 0.$$

Let v_1, v_2 in V and w_1, w_2 in W give an orthonormal basis of $V \oplus W$ with respect to the form $H1$. Then $\operatorname{Im} H1(J \cdot v_i, v_i) = -\operatorname{Im} H1(J \cdot w_i, w_i) = 1$ and the rest of the pairings are zero. Since $\operatorname{Im} H1 = \operatorname{Im} H$ we get that H is of signature (2, 2). By a result of Landherr (see [L]) the signature and discriminant of a hermitian form over $\mathbf{Q}[i]$ determine it completely (discriminant being taken as an element of $\mathbf{Q}^x/\mathrm{Norm}(\mathbf{Q}[i]^x)$) upto isomorphism. Thus F is isomorphic to Hyp $\oplus$ Hyp as a Hermitian space. Further, the Hodge structure on F is clearly determined by giving W (or V) as a quotient of $F \otimes R$. Q.E.D. Lemma 2.

Let $\{e_1, f_1, e_2, f_2\}$ be a basis for F over $\mathbf{Q}[i]$ such that $H(e_i, f_i) = 1$ and the rest of the pairings are zero. Then the lattice $U = \Lambda^2_{\mathbf{Q}[i]}F$ has a basis,

$$\begin{aligned} a_1 &= e_1 \wedge f_1; & b_1 &= e_2 \wedge f_2 \\ a_2 &= e_1 \wedge e_2; & b_2 &= f_2 \wedge f_1 \\ a_3 &= e_1 \wedge f_2; & b_3 &= f_1 \wedge e_2 \end{aligned}$$

such that $a_i \wedge b_i = g$ a generator for $\Lambda^4_{\mathbf{Q}[i]}F$. By extending the Hermitian form H to U we have,

$$H(a_1, a_1) = H(b_1, b_1) = H(a_2, b_2) = -H(a_3, b_3) = 1,$$

while the rest of the pairings are zero. Similarly, extending H to $\Lambda^4_{\mathbf{Q}[i]}F$ we have $H(g, g) = 1$. Let t: $U \to U$ be the $\mathbf{Q}[i]$-antilinear homomorphism defined by,

$$\begin{aligned} a_1 &\to -b_1; & b_1 &\to -a_1 \\ a_2 &\to -a_2; & b_2 &\to -b_2 \\ a_3 &\to a_3; & b_3 &\to b_3. \end{aligned}$$

Then $H(u_1, u_2)g = u_1 \wedge t(u_2)$ for all u_1, u_2 in U, and this condition determines the involution t. The isomorphism $F \otimes \mathbf{R} \simeq V \oplus W$ gives rise to an isomorphism $U \otimes \mathbf{R} \simeq \Lambda^2_{\mathbf{C}}V \oplus \Lambda^2_{\mathbf{C}}W \oplus V \otimes_{\mathbf{C}} W$. Thus U is a polarized Hodge structure of weight 2. It is easy to see that t is an automorphism of this polarized Hodge structure. Let T be the invariant subspace of U under the involution t. The set $\{a_1 - b_1, J \cdot (a_1 + b_1), J \cdot a_2, J \cdot b_2, a_3, b_3\}$ gives a basis for T and it follows that the quadratic form on T induced by H on U is isomorphic to Hyp $\oplus$ Hyp $\oplus$ $[-2] \oplus [-2]$. In fact the Hodge structure on T is given by the t-invariants in $\Lambda^2_{\mathbf{C}} V \oplus \Lambda^2_{\mathbf{C}} W$ which form a 2-dimensional real subspace of $T \otimes \mathbf{R}$. Thus we may state as proved,

LEMMA 3. *Let F be a Hodge structure as in Lemma 2. Then there is a Hodge structure T as in Lemma 1, and a Hodge correspondence* $T \to \Lambda^2_{\mathbf{Q}[i]} F$.
Q.E.D. Lemma 3.

We note that as a consequence of the proof of Lemma 3 SU(Hyp $\oplus$ Hyp, $\mathbf{Q}[i]$) is a double cover of SO(Hyp $\oplus$ Hyp $\oplus$ $[-2] \oplus [-2]$, $\mathbf{Q}$). Hence it is the spin group of this orthogonal group. Since the 4-dim representation of SU(2, 2) is unique, F is perforce the spin representation associated with the orthogonal group above. As a consequence (see [D] Sects. 3 and 4, and [M] Lemma 7) the Kuga–Satake–Deligne correspondence for the Hodge structure T of Lemma 1 is precisely what has been constructed in Lemma 3, providing that we show that every such T is associated with some choice of F. This will be a consequence of the construction in Sections 3 and 4.

3. The Schoen surface

With notation as in Section 2 let us choose an origin o in E s.t. $\mathcal{O}_E(p_1 + \cdots + p_4) = \mathcal{O}_E(4.o)$. Also choose an origin o' in E' lying over o in E. Let p, q in E be points such that $L \simeq \mathcal{O}_E(2 \cdot p)$ and $M \simeq \mathcal{O}_E(2 \cdot p - 2 \cdot q)$. Then, we have equivalences on E:

1. $8 \cdot p \equiv 2 \cdot (p_1 + \cdots + p_4)$ but $4 \cdot p \not\equiv (p_1 + \cdots + p_4)$.
2. $4 \cdot p \equiv 4 \cdot q$.
3. $2 \cdot p + 2 \cdot q \equiv (p_1 + \cdots + p_4)$.

Let G be the group acting on $C \times C$ generated by automorphisms (J, J^{-1}) and the flipping of the two factors. Following C. Schoen we define the "Schoen surface" W as the quotient of $C \times C$ by the action of G. Similarly we define the surfaces S' and S as the quotients by G of $E' \times E'$ and $E \times E$ respectively (note that J acts as an involution on E' and acts trivially on E). The addition morphisms $E' \times E' \to E'$ and $E \times E \to E$ give rise to morphisms $S' \to E'$ and $S \to E$, making them ruled surfaces. In fact, S' is the fibre product of $E' \to E$ and $S \to E$. The morphisms

$C \to E' \to E$ of Section 2 give rise to morphisms $W \to S' \to S$.

$$\begin{array}{ccccc} C \times C & \to & W & \to & E' \\ \downarrow & & \downarrow & & \downarrow \\ E' \times E' & \to & S' & \to & E' \\ \downarrow & & \downarrow & & \downarrow \\ E \times E & \to & S & \to & E \end{array}$$

For each point e in E we have a section D_e of $S \to E$ given by the image of $E \times \{e\}$ in S. Let D'_e denote the corresponding divisor in S'. Let L_e denote the line bundle on S corresponding to the divisor D_e.

W has an automorphism of order 4 induced by $(J, 1)$ on $C \times C$, s.t. the isotropy is of order at most 2. The quotient by the isotropy is S' and the quotient by the automorphism of order 4 is S. The branch locus in S is the union of the divisors $D_{p_1}, \ldots, D_{p_4}$. As in Section 2 we get $W = \mathrm{Spec}_S \mathcal{O}_S \oplus L_p^{-2} \oplus L_q^{-2} \oplus M$. W has twelve ordinary double points corresponding pairwise to the six points of intersection of the D_{p_i}. We may blow up these points on S, S' and W to get $S1$, $S1'$ and $W1$.

Let $E_{i,j}$ denote the exceptional curve in $S1$ lying over the point of intersection of D_{p_i} and D_{p_j}, and let $F_{i,j}$ denote the strict transform of the fibre of $S \to E$ through this point. Let D_i denote the strict transform of the section D_{p_i} of $S \to E$. Then D_i meets $E_{i,j}$ for every $i \neq j$, and $F_{n,k}$ for every $k \neq n \neq i \neq k$. This shows that $S1 \to E$ is a family of stable 4-pointed curves of genus zero in the sense of Harris and Mumford (see [H–M], Section 4). They show that there is a fine moduli space for 4-pointed stable curves of genus 0. By examining the morphism $S1 \to E$ we see that the fibres over $p_1 + p_2$ and $p_3 + p_4$ are pairs of lines which meet the sections D_i in the pairs $\{D_1, D_2\}$ and $\{D_3, D_4\}$. By similarly examining the other singular fibres one concludes that $S1$ is in fact a double cover of the universal family and the corresponding involution on E acts as inversion in the group law. Hence there is an involution s: $S1 \to S1$ such that:

1. s carries the section D_i into itself for each i.
2. s carries $E_{i,j}$ onto $F_{k,l}$; $\{i, j, k, l\} = \{1, 2, 3, 4\}$.
3. s lies over the involution $e \to -e$ of E.

Thus by lifting this inversion to $e' \to -e'$ on E' we also get a natural lift of the involution to $S1'$. This also fixes an action of s on M.

Let b: $S1 \to S$ denote the natural morphism. Then it is easily checked that:

$$W1 = \mathrm{Spec}_{S1} \mathcal{O}_{S1} \oplus b^* L_p^{-2} \otimes \mathcal{O}_{S1}(\Sigma E_{i,j}) \oplus b^* L_q^{-2} \otimes \mathcal{O}_{S1}(\Sigma E_{i,j}) \oplus M.$$

Further, the fact that $\mathcal{O}_{S1}(D_1) = b^*L_{p_1} \otimes \mathcal{O}_{S1}(-\Sigma E_{1,j})$ is invariant under s shows that:

$$s^*(b^*L_p^{-2} \otimes \mathcal{O}_{S1}(\Sigma E_{i,j})) = b^*L_q^{-2} \otimes \mathcal{O}_{S1}(\Sigma E_{i,j})$$

Hence we have two possible choices of a lift of s to an involution on $W1$.

The composite $W1 \to S1' \to E'$ makes $W1$ an elliptic fibration over E' with level two structure. The morphism $W1 \to S1'$ is the quotient by the inversion in the group law on the fibre. The involution s': $S1' \to S1'$ as described above acts as identity on the fibre of $S1' \to E'$ over o' since it fixes the four points of intersection with the four chosen sections. Thus the two choices of lifts of s to an involution on $W1$ restrict to identity and the inverse in the group law on the fibre of $W1 \to E'$ over o'. Notice that the action of these lifts is by multiplication by the section of M over E' which gives the algebra structure to $\mathcal{O}_E \oplus M$. But then, if p' is the other point of E' lying over o, the section has opposite signs at o' and p'. Thus the action of the lift can be chosen to be identity on the fibre of $W1$ over o', in which case this lift will act as inverse in the group law on the fibre over p'. Similarly, if q' and r' are the other two fixed points of the inversion on E', this lift will act as identity on the fibre over one of these (say q') and by inversion on the other fibre. Thus, this choice of lift has 8 isolated fixed points and two fixed fibres.

Now blow up these 8 fixed points to get $W2$ and form the quotient $Y2$. Then $Y2$ is a smooth surface with a morphism to the quotient of E' by the involution, i.e. $\mathbf{P}^1$. The four sections of $W1 \to E'$ also descend since they are invariant under the involution. Thus $Y2 \to \mathbf{P}^1$ is an elliptic fibration with level two structure. There are 6 singular fibres of type I_2 corresponding to the twelve singular fibres of $W1 \to E'$ and two singular fibres of type I_0^* corresponding to the fibres of $W1$ fixed under the involution (notation for elliptic fibrations as in Kodaira [K]).

Let $S2'$ be the blow up of $S1'$ at the images of the 8 points on $W1$. Then we can also form the quotient of $S2'$ by the involution to get Z. The natural morphism $Y2 \to Z$ is a double covering. Further, the morphism $Y2 \to \mathbf{P}^1$ factors through Z and $Z \to \mathbf{P}^1$ is a genus zero fibration with 8 singular fibres and four disjoint sections $C_1, \ldots, C_4$. Six of these singular fibres are of the following type:

1. Each fibre has 2 components with each component meeting 2 of the sections.
2. For each pair of sections there are two such fibres where these sections meet the same fibre component.

The remaining two singular fibres are of the following type:

3. Each fibre consists of five components. There are four exceptional curves each meeting exactly one of the sections, and the remaining component meets each of these four components exactly once.

Looking back to Section 1 we see that if we choose one pair of lines say L_5 and L_6 out of the six lines in $\mathbf{P}^2$, then we have a morphism from $Y1$ to $\mathbf{P}^1$ making it an elliptic fibration like $Y2$ above. Further, blowing up the $p_{i,j}$ in $\mathbf{P}^2$ to get $Z1$ we also have a morphism from $Z1$ to $\mathbf{P}^1$ and $Y1$ is a double cover of $Z1$. The morphism $Z1 \to \mathbf{P}^1$ has the same description as $Z \to \mathbf{P}^1$ above (with L_5 and L_6 corresponding to the fifth component in the fibres as in (3) above and $L_1, \ldots, L_4$ being the four sections).

Blowing down the four exceptional curves in the fibres of type (3) (as above) in Z and $Z1$ we get families of stable four pointed curves of genus zero which are characterized by the morphisms to the moduli space (the lambda functions), $z\colon \mathbf{P}^1 \to \mathbf{P}^1$ and $z1\colon \mathbf{P}^1 \to \mathbf{P}^1$ (see [H–M] *loc cit.*). Thus, $Z \to \mathbf{P}^1$ (and $Z1 \to \mathbf{P}^1$) is characterized by the lambda function on the base $\mathbf{P}^1$ and a choice of two points on the base over which the fibres are of type (3) as above.

Under different choices of E and $\{p_1, \ldots, p_4\} \subset E$ such that $p_1 + \cdots + p_4 = o$ in E, the above construction will give any $z\colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree two and any pair of points in $\mathbf{P}^1 = E'/s'$ that we choose. Again, by varying the choice of lines in $\mathbf{P}^2$ we can get any $z1\colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree two and any pair of points in $\mathbf{P}^1$. Thus, for a suitable choice we get an isomorphism between Z and $Z1$, hence also an isomorphism between $Y2$ and $Y1$.

4. Cohomology computations

We now state,

MAIN THEOREM. *Let $Y1$ be the desingularization of the double cover of the plane branched along six lines no* 3 *of which are coincident. Then there is an abelian variety P and an element of $CH^2(P \times Y1)$ such that the corresponding homomorphism $H^2(Y1, \mathbf{Q}) \to H^2(P, \mathbf{Q})$ is an inclusion on the lattice of transcendental cycles.*

Proof of Theorem. Let $\mathrm{Sym}^2(C)$ be the second symmetric power of C. Then the addition morphism $\mathrm{Sym}^2(C) \times \mathrm{Sym}^2(C) \to \mathrm{Jac}(C)$ can be composed with the morphism $\mathrm{Jac}(C) \to P$ of Section 2 (P is the Prym variety of Section 2), to give $\mathrm{Sym}^2(C) \times \mathrm{Sym}^2(C) \to P$. Further, use the projection to the first factor of $\mathrm{Sym}^2(C) \times \mathrm{Sym}^2(C)$ to get a class in $CH^2(\mathrm{Sym}^2(C) \times P)$. This class represents the homomorphism $H^2(\mathrm{Sym}^2(C), \mathbf{Q}) = \Lambda^2_{\mathbf{Q}} H^1(C, \mathbf{Q}) \to \Lambda^2_{\mathbf{Q}} H^1(P, \mathbf{Q})$.

Similarly, the morphism $C \times C \to W$ descends to $\mathrm{Sym}^2(C) \to W$ giving us a class in $CH^2(\mathrm{Sym}^2(C) \times W)$ which represents the inclusion $\Lambda^2_{\mathbf{Q}[i]} F \to \Lambda^2_{\mathbf{Q}} H^1(C, \mathbf{Q})$.

Now compose the above correspondences with $W2 \to W \times Y2$. This gives a class in $CH^2(P \times Y2)$.

Since we have already shown in Section 3 that our construction reaches every $Y1$, we only need to show that the above construction actually gives a correspondence of the required type.

The following claims are easily checked:

1. $H^2(W1, \mathbf{Q}) = H^2(W, \mathbf{Q}) \oplus \mathbf{Q}$-span of 1-diml. fibres of $W1 \to W$.
2. $H^2(W, \mathbf{Q}) = G$-invariants in $H^2(C \times C, \mathbf{Q})$.
3. The action of J on $H^0(C, \mathbf{Q})$ and $H^2(C, \mathbf{Q})$ is trivial, so the G-invariants in $H^0(C, \mathbf{Q}) \otimes H^2(C, \mathbf{Q}) \oplus H^2(C, \mathbf{Q}) \otimes H^0(C, \mathbf{Q})$ form a 1-diml. vector space spanned by the class of the image in W of $C \times \{c\}$, for some general point c in C.
4. The flipping operates as the alternating action on $H^1(C, \mathbf{Q}) \otimes H^1(C, \mathbf{Q})$ so that the G-invariants in $H^1(C, \mathbf{Q}) \otimes H^1(C, \mathbf{Q}))$ are just $\Lambda^2_{\mathbf{Q}[i]} F \oplus \Lambda^2_{\mathbf{Q}} \mathrm{H}^1(E', \mathbf{Q})$. Further, $\Lambda^2_{\mathbf{Q}} H^2(E', \mathbf{Q})$ is spanned by the class of a general fibre of $W \to E'$.
5. The involution on $W1$ interchanges the $(J, 1)$ action with the $(1, J^{-1})$ action. Thus the action on $\Lambda^2_{\mathbf{Q}[i]} F$ of this involution is $\mathbf{Q}[i]$ antilinear.
6. The real part of the Hermitian form on $\Lambda^2_{\mathbf{Q}[i]} F$ is a constant multiple of the intersection pairing on $\Lambda^2_{\mathbf{Q}[i]} F \subset H^2(W1, \mathbf{Q})$. Hence, this real part is invariant under the involution.

As a consequence of (1)–(4) we see that the lattice of transcendental cycles on $W1$ is $\Lambda^2_{\mathbf{Q}[i]} F$. The invariants under the involution is the lattice of transcendental cycles on $Y2$. Since (5) and (6) show that the involution has the required type of action on $\Lambda^2_{\mathbf{Q}[i]} F$ we have the required correspondence.

Acknowledgements

I am greatly indebted to Madhav Nori who started me off on this problem by showing me how to do the calculations in Sections 1 and 2. Without his continued guidance and support this paper would never have been possible. I also wish to thank T.N. Venkataramana, for many discussions on arithmetic groups relating to Section 2. The calculations in Section 4 were greatly aided by discussions with Mangala Nori and A.R. Shastri on their results about elliptic surfaces (see [MN] and [G–S]).

References

[D] P. Deligne: La conjecture de Weil pour les surfaces K3 *Invent. Math.* 15 (1972) 206–226.

[G–H] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*. John Wiley & Sons (1978).

[G–S] R. Gurjar and A.R. Shastri: Covering spaces of an elliptic surface. *Comp. Math.* 54 (1985) 95–104.

[H–M] J. Harris and D. Mumford: On the Kodaira dimension of the moduli space of curves. *Invent. Math.* 67 (1982) 23–86.

[K] K. Kodaira: On compact analytic surfaces. *Ann. Math.* 77 (1973) 563–636.

[K–S] M. Kuga and I. Satake: Abelian varieties attached to polarized K3 surfaces. *Math. Ann.* 169 (1967) 239–242.

[L] W. Landherr: Aquiv. Hermitischen Formen über algebraischen Zahlkorper. *Abh. Math. Sem. Hamburg Univ.* 11 (1936) 245-248.

[M] D. Morrison: The Kuga-Satake variety of an Abelian surface. *J. Algebra* 92 (1985) 454–476.

[MN] Mangala Nori: On the lattice of transcendental cycles. *Math. Zeit.* 193 (1986) 105–112.

[S] C. Schoen: Hodge cycles on self products on a variety with an automorphism. *Comp. Math.* 65 (1988) 3–32.