COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 68, nº 1 (1988), p. 41-57
<http://www.numdam.org/item?id=CM_1988__68_1_41_0>

Compositio Mathematica 68: 41–57 (1988)

Congruence properties of coefficients of certain algebraic power series

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Received 4 September 1987; accepted in revised form 17 March 1988

Abstract. Let $\Sigma_{i=1}^{\infty} u_n X^n$ denote the power series expansion around $X = 0$ of the algebraic function $(1 + \Sigma_{i=1}^{e} \alpha_i X^i)^{-1/e}$. In this paper we show some congruences for the coefficients u_n. Furthermore we give some lower bounds for the number of factors of an arbitrary prime $p \geqslant 3$ in u_n, if $p \equiv 1 \bmod e$ and $p|\alpha_j$ for at least one j.

1. Introduction

Let $f(X) = \Sigma_{n=0}^{\infty} u_n X^n$ be a power series with rational coefficients which satisfies an equation of the form

$$P(X, f(X)) = 0 \quad \text{where } P(X, Y) \in \mathbb{Z}[X, Y] \text{ and } P(X, Y) \not\equiv 0.$$

Such power series are called algebraic power series. It follows from a theorem of Eisenstein that the set of primes which divide the denominator of some coefficients, is finite. Let us call this set of primes S.

Let p be a prime, $p \notin S$. Christol, Kamae, Mendès-France and Rauzy [1] showed that the sequence $\{u_n \bmod p\}_{n=0}^{\infty}$ is p-recognisable. This means that the sequence $\{u_n \bmod p\}_{n=0}^{\infty}$ can be generated by a p-automaton. Denef and Lipshitz [2] showed that the sequence $\{u_n \bmod p^s\}_{n=0}^{\infty}$ is p^s-recognisable for each $s \in \mathbb{N}$. They reformulate this property in the following way:

$\forall s \in \mathbb{N}, \exists r \in \mathbb{N}, \forall i \in \mathbb{Z}$ *with* $0 \leqslant i < p^r$ *we can find*
$r' \in \mathbb{N}$ *with* $r' < r$ *and* $i' \in \mathbb{Z}$ *with* $0 \leqslant i' < p^{r'}$
such that $\forall m \in \mathbb{N}$ *we have* $u_{mp^r+i} \equiv u_{mp^{r'}+i'} \bmod p^s$.

In special cases this congruence takes on a simple form. In this paper we consider algebraic power series of a special form

$$\left(1 + \sum_{i=1}^{e} \alpha_i X^i\right)^{-1/e} = \sum_{n=0}^{\infty} u_n X^n, \quad \text{where } e \geqslant 2, \alpha_i \in \mathbb{Z}, \text{ for } i = 1, 2, \ldots, e. \tag{1}$$

One of the results in this paper is

THEOREM A. *Let p be a prime,* $p \equiv 1$ *mod e. Then we have*

$$u_{mp^r} \equiv u_{mp^{r-1}} \bmod p^r \textit{ for all } m, r \in \mathbb{N}.$$

The second result in this paper is quite different. It provides a lower bound for the number of factors p in u_n in the case $e = p - 1$. It is based on the following identity mod p which is known as Frobenius factorisation (cf. [3]).

$$\left(1 + \sum_{i=1}^{p-1} \alpha_i X^i\right)^{1/(1-p)} \equiv \left(1 + \sum_{i=1}^{p-1} \alpha_i X^i\right)^{1+p+p^2+\cdots} \equiv \prod_{j=0}^{\infty} \left(1 + \sum_{i=1}^{p-1} \alpha_i X^i\right)^{p^j}$$

$$\equiv \prod_{j=0}^{\infty} \left(1 + \sum_{i=1}^{p-1} \alpha_i X^{ip^j}\right) \bmod p.$$

It follows from a simple calculation that

$$u_n \equiv \prod_i \alpha_{n_i} \bmod p,$$

where $n = n_0 + n_1 p + \cdots + n_t p^t$, $0 \leqslant n_i < p$ is the p-adic representation of n. In particular we have $u_n \equiv 0 \bmod p$ if $p|\alpha_j$ and $n_i = j$ for some i. The following theorem gives a stronger law.

THEOREM B. *Let p be a prime,* $p \geqslant 3$. *Let* $\Sigma_{n=0}^{\infty} u_n X^n$ *be the power series expansion of* $(1 + \Sigma_{i=1}^{p-1} \alpha_i X^i)^{-1/(p-1)}$ *where* $\alpha_i \in \mathbb{Z}$ *for* $i = 1, \ldots, p-1$. *Let n be a positive integer with p-adic representation* $\Sigma_{i=0}^{t} n_i p^i$. *Let* $J = \{1 \leqslant j \leqslant p-1 : p|\alpha_j\}$ *and* $S = \{k \in \mathbb{N} : n_k \in J\}$. *Then*

$$\mathrm{ord}_p u_n \geqslant [\tfrac{1}{2}(|S| + 1)].$$

This phenomenon appears also in the case that the Taylor series does not represent an algebraic function, but satisfies a linear differential equation. We finish the introduction with a conjecture of F. Beukers.

Let $b_n = \Sigma_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. Let $J_5 = \{1, 3\}$ and $J_{11} = \{5\}$. Let $S_5 = \{k \in \mathbb{N} \mid n_k \in J_5$, where $\Sigma_j n_j 5^j$ is the 5-adic representation of $n\}$ and $S_{11} = \{k \in \mathbb{N} \mid n_k \in J_{11}$, where $\Sigma_j n_j 11^j$ is the 11-adic representation of $n\}$. Beukers conjectures that

(i) $\mathrm{ord}_5(b_n) \geqslant |S_5|$,
(ii) $\mathrm{ord}_{11}(b_n) \geqslant |S_{11}|$,

cf. [4] and [5].

2. Some preliminaries

We use the following *notation*:
- For a finite set S we denote the cardinality of S by $|S|$,
- $[X]$ is the largest integer not exceeding X, $\{X\} = X - [X]$,
- p is a fixed prime, $p \geqslant 3$,
- $\mathrm{ord}_p(r)$ = multiplicity of the prime factor p in r, for $r \in \mathbb{Z}\backslash\{0\}$,
- $r^* = r \cdot p^{-\mathrm{ord}_p(r)}$ is the p-free part of the rational number $r \neq 0$,
- for $\alpha \in \mathbb{Q}$, $m_1, \ldots, m_n \in \mathbb{Z}_{\geqslant 0}$ we define the multinomial coefficient

$$\binom{\alpha}{m_1 \ldots m_n} \text{ by } \frac{\alpha(\alpha - 1) \ldots \left(\alpha + 1 - \sum_{i=1}^{n} m_i\right)}{m_1! m_2! \ldots m_n!}.$$

- We denote by $\mathbb{Z}_p$ the set of p-adic integers.

For any $\alpha \in \mathbb{Z}_p$ we have its p-adic representation $\Sigma_{n=0}^{\infty} a_n p^n$ with $a_n \in \mathbb{Z}$ and $0 \leqslant a_n < p$ for all n. For $k \in \mathbb{N}$ we denote its truncation $\Sigma_{n=0}^{k-1} a_n p^n$ by $[\alpha]_k$.
- Let n be a positive integer. Let $\{b_1, \ldots, b_e\}$ be any partition of non-negative integers such that

$$\sum_{i=1}^{e} ib_i = n. \tag{2}$$

We denote the p-adic representation of b_i by

$$b_i = b_{i0} + b_{i1}p + \cdots + b_{it}p^t \quad (i = 1, \ldots e). \tag{3}$$

Further we define integers c_k, T_k and rationals d_k for $k = 0, \ldots, t$ by

$$c_k = \sum_{i=1}^{e} b_{ik}, \tag{4}$$

$$d_k = p \sum_{i=1}^{e} \left\{\frac{b_i}{p^{k+1}}\right\} \quad \text{for } k \geqslant 0, \text{ and } d_{-1} = d_{-2} = 0, \tag{5}$$

$$T_k = \sum_{j=0}^{k} \sum_{i=1}^{e} ib_{ij}p^j. \tag{6}$$

LEMMA 2.1. *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{Z}_p$. Then*

$$\operatorname{ord}_p\binom{\alpha}{n} = \sum_{k=1}^{\infty}\left(-\left[\frac{[\alpha]_k}{p^k} - \left\{\frac{n}{p^k}\right\}\right]\right).$$

Proof. We have

$$\binom{\alpha}{n} = \frac{1}{n!}\cdot\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1).$$

We define u_k as the number of the factors among $\alpha, \alpha-1, \ldots, \alpha-n+1$ which are divisible by p^k. Then

$$\operatorname{ord}_p\binom{\alpha}{n} = \sum_{k=1}^{\infty}\left(u_k - \left[\frac{n}{p^k}\right]\right).$$

We have to calculate u_k. To do so, we define v_k as the largest integer not exceeding 0 such that $\operatorname{ord}_p(\alpha + v_k) \geq k$ and w_k as the largest integer not exceeding $-n$ such that $\operatorname{ord}_p(\alpha + w_k) \geq k$. Then $u_k = (v_k - w_k)/p^k$. It is clear that $v_k = -[\alpha]_k$ and $w_k = -[\alpha]_k + [([\alpha]_k - n)p^k]\cdot p^k$. Hence $u_k = -[([\alpha]_k - n)/p^k]\cdot p^k$. By $n/p^k = [n/p^k] + \{n/p^k\}$, we have

$$\begin{aligned}\operatorname{ord}_p\binom{\alpha}{n} &= \sum_{k=1}^{\infty}\left(u_k - \left[\frac{n}{p^k}\right]\right)\\ &= \sum_{k=1}^{\infty}\left(-\left[\frac{[\alpha]_k}{p^k} - \left[\frac{n}{p^k}\right] - \left\{\frac{n}{p^k}\right\}\right] - \left[\frac{n}{p^k}\right]\right).\end{aligned}$$ □

COROLLARY 2.2. *Let $M, N, r \in \mathbb{Z}_{\geq 0}$, $N \leq M < p^{t+1}$ and let e be an integer, $e \geq 2$, which divides $p-1$. Put $N_k = \{N/p^k\}$, $M_k = \{M/p^k\}$, and let $b_1, \ldots, b_e, d_k$ be defined as in* (2) *and* (5). *Then*

(i) $$\operatorname{ord}_p\binom{Mp^r}{Np^r} = \operatorname{ord}_p\binom{M}{N} = \sum_{k=1}^{t+1} -[M_k - N_k],$$

(ii) $$\begin{aligned}\operatorname{ord}_p\binom{-1/e}{Np^r} &= \sum_{k=1}^{t+1}\left[N_k + \frac{e-1}{e}\right]\\ &= \sum_{k=1}^{t+1}\left(\left[\frac{N}{p^k} + \frac{e-1}{e}\right] - \left[\frac{N}{p^k}\right]\right),\end{aligned}$$

(iii) $\operatorname{ord}_p \begin{pmatrix} -1/e \\ b_1 p^r \ldots b_e p^r \end{pmatrix} = \sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{e-1}{e} \right].$

Proof. (i) The first equality follows by induction on r. Apply Lemma 2.1 with $\alpha = M$ for proving the case $r = 0$.

(ii) Let $a = (p-1)/e$. Then $-1/e = a/(1-p) = a + ap + ap^2 + \cdots \in \mathbb{Z}_p$. We use Lemma 2.1 with $\alpha = -1/e$. Since

$$[\alpha]_k = \sum_{j=0}^{k-1} ap^j = a \cdot \frac{p^k - 1}{p - 1} = \frac{p^k - 1}{e}$$

and

$$\left[\frac{p^l - 1}{ep^l} - \left\{ \frac{Np^r}{p^l} \right\} \right] = 0 \text{ for } 0 \leqslant l \leqslant r,$$

we have

$$\operatorname{ord}_p \begin{pmatrix} -1/e \\ Np^r \end{pmatrix} = \sum_{l=1}^{r+t+1} \left(-\left[\frac{p^l - 1}{ep^l} - \left\{ \frac{Np^r}{p^l} \right\} \right] \right.$$

$$= \sum_{k=1}^{t+1} \left(-\left[\frac{p^k - 1}{ep^k} - \left\{ \frac{N}{p^k} \right\} \right] \right).$$

Since for any rational integer f

$$\left[\frac{1}{e} - \frac{1}{ep^k} + \frac{f}{p^k} \right] = \left[\frac{1}{e} + \frac{f}{p^k} \right],$$

we obtain

$$\operatorname{ord}_p \begin{pmatrix} -1/e \\ Np^r \end{pmatrix} = \sum_{k=1}^{t+1} -\left[\frac{1}{e} - N_k \right].$$

A simple calculation shows that

$$-[1/e - N_k] = \left[\frac{e-1}{e} + N_k \right].$$

(iii) Put $N = \Sigma_{i1}^{e} b_i$. We have

$$\binom{-1/e}{b_1p^r \dots b_ep^r} = \binom{-1/e}{Np^r} \cdot \binom{Np^r}{b_1p^r \dots b_ep^r}.$$

Hence

$$\mathrm{ord}_p \binom{-1/e}{b_1p^r \dots b_ep^r} = \mathrm{ord}_p \binom{-1/e}{Np^r} + \mathrm{ord}_p \binom{Np^r}{b_1p^r \dots b_ep^r}.$$

Since

$$\mathrm{ord}_p \binom{-1/e}{Np^r} = \sum_{k=1}^{t+1} \left[N_k + \frac{e-1}{e} \right],$$

$$\mathrm{ord}_p \binom{Np^r}{b_1p^r \dots b_ep^r} = \mathrm{ord}_p \binom{N}{b_1 \dots b_e} = \sum_{k=1}^{t+1} \left(\left[\frac{N}{p^k} \right] - \left[\frac{b_1}{p^k} \right] - \dots - \left[\frac{b_e}{p^k} \right] \right) = \sum_{k=1}^{t+1} \left(\frac{N}{p^k} - N_k - \sum_{i=1}^{e} \left[\frac{b_i}{p^k} \right] \right)$$

and

$$\sum_{i=1}^{e} \left[\frac{b_i}{p^k} \right] = \sum_{i=1}^{e} \left(\frac{b_i}{p^k} - \left\{ \frac{b_i}{p^k} \right\} \right) = \frac{N}{p^k} - \frac{d_{k-1}}{p},$$

we obtain

$$\mathrm{ord}_p \binom{-1/e}{b_1p^r \dots b_ep^r} = \sum_{k=1}^{t+1} \left[N_k + \frac{e-1}{e} \right] + \frac{d_{k-1}}{p} - N_k.$$

Now (iii) follows by noting that $d_{k-1}/p - N_k$ is an integer. □

LEMMA 2.3. *Let $n \in \mathbb{Z}_{\geqslant 0}$ and $n = n_0 + n_1 p + \dots + n_t p^t$ its p-adic representation. Let $\{b_1, \dots, b_e\}$ be an arbitrary partition, as in (2). Then we have with the notation of* (3)–(6)

(i) $T_k \equiv n \bmod p^{k+1}$ *for* $k \geqslant 0$,

(ii) $c_m p^m \leqslant T_k \leqslant ed_k p^k$ *for* $0 \leqslant m \leqslant k$,

(iii) $T_k = T_{k-1} + \sum_{i=1}^{e} ib_{ik}p^k$ *for* $k \geqslant 1$.

Proof. (i) We have, by using the definition of b_i, T_k and b_{ij},

$$n = \sum_{i=1}^{e} ib_i = \sum_{i=1}^{e}\sum_{j=0}^{t} ib_{ij}p^j \equiv \sum_{i=1}^{e}\sum_{j=0}^{k} ib_{ij}p^j = T_k \bmod p^{k+1}.$$

(ii) We prove the left inequality by

$$c_m p^m = \sum_{i=1}^{e} b_{im}p^m \leqslant \sum_{i=1}^{e} ib_{im}p^m \leqslant \sum_{i=1}^{e}\sum_{j=0}^{k} ib_{ij}p^j = T_k.$$

For the right inequality notice that

$$T_k = \sum_{i=1}^{e}\sum_{j=0}^{k} ib_{ij}p^j \leqslant \sum_{i=1}^{e}\sum_{j=0}^{k} eb_{ij}p^j = ed_k p^k.$$

(iii) follows immediately from definition (5). □

LEMMA 2.4. *Let* $\alpha_i \in \mathbb{Q}$, $e \in \mathbb{N}$. *Then*

$$\left(1 + \sum_{i=1}^{e} \alpha_i X^i\right)^{-1/e} = \sum_{n=0}^{\infty} u_n X^n,$$

where

$$u_n = \sum^{0} \binom{-1/e}{b_1 \ldots b_e} \prod_{i=1}^{e} \alpha_i^{b_i}$$

and 0 *indicates that the sum is taken over all partitions* $\{b_1, \ldots, b_e\}$ *such that* $\Sigma_{i=1}^{e} ib_i = n$.

Proof. We have

$$\left(1 + \sum_{i=1}^{e} \alpha_i X^i\right)^{-1/e} = \sum_{m=0}^{\infty} \binom{-1/e}{m} \cdot \left(\sum_i \alpha_i X^i\right)^m$$

$$= \sum_{m=0}^{\infty} \binom{-1/e}{m} \cdot \sum^{0} \binom{m}{b_1 \ldots b_e} \cdot \prod_i \alpha_i^{b_i} \cdot X^{(\Sigma_i i b_i)}$$

$$= \sum_{n=0}^{\infty} \sum^{0} \binom{-1/e}{b_1 + \cdots + b_e} \cdot \binom{b_1 + \cdots + b_e}{b_1 \ldots b_e} \cdot \prod_i \alpha_i^{b_i} \cdot X^n .$$

LEMMA 2.5. *Let* $n = np^r$ *and let* $\{b_1 \ldots b_e\}$ *be an arbitrary partition as in* (2). *For any non-negative integer* j *such that* $c_j > 0$ *we have*

$$\operatorname{ord}_p \binom{-1/e}{b_1 p^r \ldots b_e p^r} \geqslant r - j.$$

Proof. From Corollary 2.2 (iii) it follows that

$$\operatorname{ord}_p \binom{-1/e}{b_i p_1^r \ldots b_l p_e^r} = \sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{e-1}{e} \right].$$

It suffices to prove that

$$\left[\frac{d_k}{p} + \frac{e-1}{e} \right] \geqslant 1 \quad \text{for } j \leqslant k < r.$$

Suppose that

$$\left[\frac{d_k}{p} + \frac{e-1}{e} \right] = 0 \quad \text{for some } j \leqslant k < r.$$

Then $d_k < p/e$. From Lemma 2.3(ii) it follows that $T_k < p^{k+1}$. By using Lemma 2.3(i) we conclude that $T_k = 0$. But Lemma 2.3(ii) implies $c_j p^j \leqslant T_k$. Hence $c_j = 0$ which contradicts $c_j > 0$. □

LEMMA 2.6. *Let* $e \geqslant 2$ *be an integer which divides* $p - 1$. *Let* $r \geqslant 1$ *be an integer. Then*

$$\binom{-1/e}{b_1 p^r \ldots b_e p^r}^* \equiv \binom{-1/e}{b_1 p^{r-1} \ldots b_e p^{r-1}}^* \bmod p^r .$$

Proof. Put $m = \Sigma_{i=1}^{e} b_i$. Then we have

$$\binom{-1/e}{b_1p^r \ldots b_ep^r} = (-1/e)^{mp^r} \cdot \frac{1 \cdot (1+e) \ldots (1+mep^r-e)}{(b_1p^r)! \cdot (b_2p^r)! \ldots (b_ep^r)!}$$

$$= (-1/e)^{mp^r} \cdot \frac{p \cdot (p+ep) \ldots (p+mep^r-ep)}{(p \cdot 2p \ldots b_1p^r) \ldots (p \cdot 2p \ldots b_ep^r)}$$

$$\times \frac{1 \cdot (1+e) \ldots (1+mep^r-e)}{p \cdot (p+ep) \ldots (p+mep^r-ep)}$$

$$\times \frac{(p \cdot 2p \ldots b_1p^r) \ldots (p \cdot 2p \ldots b_ep^r)}{(b_1p^r)! \cdot (b_2p^r)! \ldots (b_ep^r)!}$$

$$= (-1/e)^{mp^r - mp^{r-1}} \cdot \binom{-1/e}{b_1p^{r-1} \ldots b_ep^{r-1}}$$

$$\times \frac{1 \cdot (1+e) \ldots (1+mep^r-e)}{p \cdot (p+ep) \ldots (p+mep^r-ep)}$$

$$\times \frac{(p \cdot 2p \ldots b_1p^r) \ldots (p \cdot 2p \ldots b_ep^r)}{(b_1p^r)! \cdot (b_2p^r)! \ldots (b_ep^r)!}.$$

By Corollary 2.2(iii) we have

$$\mathrm{ord}_p \binom{-1/e}{b_1p^r \ldots b_ep^r} = \mathrm{ord}_p \binom{-1/e}{b_1p^{r-1} \ldots b_ep^{r-1}}.$$

Hence we have mod p^r

$$\binom{-1/e}{b_1p^r \ldots b_ep^r}^* \equiv \binom{-1/e}{b_1p^{r-1} \ldots b_ep^{r-1}}^* \cdot (-1/e)^{mp^r - mp^{r-1}}.$$

$$\times \frac{1 \cdot (1+e) \ldots (1+mep^r-e)}{p \cdot (p+ep) \ldots (p+mep^r-ep)}$$

$$\times \frac{(p \cdot 2p \ldots b_1p^r) \ldots (p \cdot 2p \ldots b_ep^r)}{(b_1p^r)! \cdot (b_2p^r)! \ldots (b_ep^r)!}. \tag{7}$$

Note that $(-1/e)^{mp^r} \equiv (-1/e)^{mp^{r-1}} \bmod p^r$ by a theorem of Fermat–Euler. Furthermore by $e|(p-1)$,

$$\left(\frac{1\cdot(1+e)\ldots(1+mep^r-e)}{p\cdot(p+ep)\ldots(p+mep^r-ep)}\right)$$

$$\text{and}\quad\left(\frac{(b_1p^r)!\cdot(b_2p^r)!\ldots(b_ep^r)!}{(p\cdot 2p\ldots(p\cdot 2p\ldots b_ep^r)}\right)$$

are rational integers. It now follows that

$$\left(\frac{1\cdot(1+e)\ldots(1+mep^r-e)}{p\cdot(p+ep)\ldots(p+mep^r-ep)}\right)^* \equiv \left(\sum_{a=1,\,p\nmid a}^{p^r} a\right)^m \tag{8}$$

$$\equiv\left(\frac{(b_1p^r)!\cdot(b_2p^r)!\ldots(b_ep^r)!}{(p\cdot 2p\ldots b_1p^r)\ldots(p\cdot 2p\ldots b_ep^r)}\right)^* \bmod p^r.$$

The substitution of these congruences in (7) completes the proof of the lemma. □

COROLLARY 2.7. *With r and e as in Lemma 2.6 we have*

$$\binom{-1/e}{b_1p^r\ldots b_ep^r} \equiv \binom{-1/e}{b_1p^{r-1}\ldots b_ep^{r-1}} \bmod p^{r+\mu}$$

where $\mu = \operatorname{ord}_p\binom{-1/e}{b_1\ldots b_e}$.

Proof. This is obvious since

$$\binom{-1/e}{b_1p^m\ldots b_ep^m} = \binom{-1/e}{b_1p^m\ldots b_ep^m}^*\cdot p^\mu \quad \text{for all } m \geqslant 0.$$ □

3. Congruences

THEOREM A. *Let*

$$\left(1+\sum_{i=1}^{e}\alpha_iX^i\right)^{-1/e} = \sum_{n=0}^{\infty} u_nX^n, \text{ where } \alpha_i \in \mathbb{Z} \text{ for } i=1\ldots e \text{ and } e\in\mathbb{Z},\ e\geqslant 2.$$

Let p be a prime such that $p \equiv 1 \bmod e$. Let $r, m \in \mathbb{N}$. Then

$$u_{mp^r} \equiv u_{mp^{r-1}} \bmod p^r.$$

Proof. Put $n = mp^r$. We may assume $p \nmid m$. Take an arbitrary partition $\{b_1 \ldots b_e\}$ as defined in (2). Define j with $0 \leqslant j \leqslant r$ by $c_0 = c_1 = \cdots = c_{j-1} = 0$, $c_j > 0$. If $j = 0$ then Lemma 2.5 implies that

$$\binom{-1/e}{b_1 \ldots b_e} \equiv 0 \bmod p^r. \tag{9}$$

Now suppose that $j > 0$. Since $c_k = \Sigma_{i=1}^{e} b_{ik}, b_{ik} \geqslant 0$ and $c_k = 0$ for $k < j$, we have $p^j | b_i$ for $i = 1 \ldots e$. Substitute $b = b_i' p_i^j$. By Lemma 2.6 we have

$$\binom{-1/e}{b_1'p^j \ldots b_e'p^j}^* \equiv \binom{-1/e}{b_1'p^{j-1} \ldots b_e'p^{j-1}}^* \bmod p^j.$$

Since $\alpha_i^{p^j} \equiv \alpha_i^{p^{j-1}} \bmod p^j$, by Fermat–Euler, we have

$$\binom{-1/e}{b_1'p^j \ldots b_e'p^j}^* \prod_i \alpha_i^{b_i'p^j} \equiv \binom{-1/e}{b_1'p^{j-1} \ldots b_e'p^{j-1}}^* \prod_i \alpha_i^{b_i'p^{j-1}} \bmod p^j.$$

Since $c_j > 0$ we find, using Corollary 2.2(iii) and Lemma 2.5,

$$\binom{-1/e}{b_1'p^j \ldots b_e'p^j} \prod_i \alpha_i^{b_i'p^j} \equiv \binom{-1/e}{b_1'p^{j-1} \ldots b_e'p^{j-1}} \prod_i \alpha_i^{b_i'p^{j-1}} \bmod p^r. \tag{10}$$

We recall Lemma 2.4,

$$u_n = \sum^{0} \binom{-1/e}{b_1 \ldots b_e} \cdot \prod_{i=1}^{e} \alpha_i^{b_i}.$$

For $n = mp^r$ we split this sum into two parts: One part for which $p \nmid b_i$ for some i, the other part for which $p | b_i$ for all i. Congruence (9) implies that the first part vanishes mod p^r. Hence

$$u_{mp^r} \equiv \sum^{\wedge} \binom{-1/e}{b_1 \ldots b_e} \cdot \prod_{i=1}^{e} \alpha_i^{b_i} \bmod p^r,$$

where ^ denotes the sum taken over all partitions $\{b_1, \ldots, b_e\}$ such that $\Sigma_{i=1}^{e} ib_i = mp^r$ and $p|b_i$ for $i = 1, \ldots, e$. According to (10) the right side of this congruence equals

$$\sum^{0} \binom{-1/e}{b_1 \ldots b_e} \cdot \prod_{i=1}^{e} \alpha_i^{b_i} \equiv u_{mp^{r-1}} \bmod p^r,$$

here 0 denotes the sum is taken over all partitions $\{b_1, \ldots, b_e\}$ such that $\Sigma_{i=1}^{e} ib_i = mp^{r-1}$. □

4. Prime factors p of the algebraic power series $(1 + \Sigma_{i=1}^{p-1} \alpha_i X^i)^{-1/(p-1)}$

THEOREM B. *Let p be a prime, $p \geqslant 3$, and $\alpha_i \in \mathbb{Z}$ for $i = 1, \ldots, p - 1$. Put*

$$\left(1 + \sum_{i=1}^{p-1} \alpha_i X^i\right)^{-1/(p-1)} = \sum_{n=0}^{\infty} u_n X^n.$$

Let n be a positive integer with p-adic representation $n_0 + n_1 p + \cdots + n_t p^t$. Let $J = \{1 \leqslant j \leqslant p - 1: p|\alpha_j\}$, $S = \{k \in \mathbb{N}: n_k \in J\}$ and let R be a subset of S such that for each pair of successive numbers m and $m + 1$, at most one of the numbers n_m and n_{m+1} belongs to R. Put $\sigma = |S|$ and $\varrho = |R|$. Then

(i) $\mathrm{ord}_p u_n \geqslant \varrho$,

(ii) $\mathrm{ord}_p u_n \geqslant [(\sigma + 1)/2]$,

(iii) *if $J = \{p - s, p - s + 1, \ldots, p - 1\}$ for some s, then* $\mathrm{ord}_p u_n \geqslant \sigma$.

Proof. Let $\{b_1 \ldots b_e\}$ be an arbitrary partition, as defined in (2). We need the following notation in this proof:

$$B = \left\{k \in \mathbb{N}: \sum_{j \in J} b_{jk} > 0\right\},$$

$$K_i = \left\{k \in \mathbb{N}: \left[\frac{d_k}{p} + \frac{p-2}{p-1}\right] = i\right\}, \quad \text{for } i = 0, 1, 2, \ldots.$$

$$\bar{K}_i = \{k + j: k \in K_i, 0 \leqslant j \leqslant i - 1\},$$

$$\bar{K} = \bigcup_{i=1}^{\infty} \bar{K}_i,$$

$$\beta = |B|, \quad \tau = \sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{p-2}{p-1}\right].$$

Notice that

$$\tau = \sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{p-2}{p-1} \right] = \sum_{i=1}^{t} i \cdot |K_i| \geqslant |\bar{K}|.$$

We prove the theorem by use of the two following lemmas.

LEMMA 4.1.

$$\mathrm{Ord}_p(u_n) \geqslant \min_{\Sigma i b_i = n} (\beta + \tau).$$

Proof. Lemma 2.4 implies that

$$u_n = \sum^{0} \binom{-1/(p-1)}{b_1 \ldots b_{p-1}} \cdot \prod_{i=1}^{p-1} \alpha_i^{b_i}.$$

Hence

$$\mathrm{ord}_p(u_n) \geqslant \min_{\Sigma i b_i = n} \left(\sum_{i=1}^{p-1} b_i \cdot \mathrm{ord}_p(\alpha_i) + \mathrm{ord}_p \binom{-1/(p-1)}{b_1 \ldots b_{p-1}} \right).$$

It now follows from Corollary 2.2 that

$$\mathrm{ord}_p(u_n \geqslant \min_{\Sigma i b_i = n} \left(\sum_{i=1}^{p-1} b_i \cdot \mathrm{ord}_p(\alpha_i) + \sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{p-2}{p-1} \right] \right).$$

Since

$$\sum_{i=1}^{p-1} b_i \cdot \mathrm{ord}_p(\alpha_i) \geqslant \sum_{i \in J} b_i \cdot \mathrm{ord}_p(\alpha_i) \geqslant |B| = \beta$$

and

$$\sum_{k=0}^{t} \left[\frac{d_k}{p} + \frac{p-2}{p-1} \right] = \tau,$$

the lemma is proved. □

LEMMA 4.2. *If $d_{k-1} < p/(p-1)$ and $d_k < p/(p-1)$ then either*

$$c_k = n_k = 0$$

or

$$c_k = 1, n_k = j, b_{jk} = 1 \textit{ for some } j \in \{1, \ldots, p-1\} \textit{ and } b_{ik} = 0 \textit{ for all } i \neq j.$$

Proof. By Lemma 2.3(ii) the conditions $d_{k-1} < p/(p-1)$ and $d_k < p/(p-1)$ imply that $T_{k-1} < p^k$ and $T_k < p^{k+1}$. Furthermore we have, by Lemma 2.3(iii), $T_k = T_{k-1} + \Sigma_i ib_{ik}p^k$ and finally we have, by Lemma 2.3(i), $T_k \equiv n \bmod p^{k+1}$. By combining this we obtain $n_k = \Sigma_i ib_{ik}$. Note that $d_k < p/(p-1)$ implies $c_k \leqslant 1$. Hence either $c_k = 0$ or $c_k = 1$. If $c_k = 0$ then $\Sigma_i ib_{ik} = 0$ and $n_k = 0$. If $c_k = 0$. If $c_k = 1$ then $\Sigma_i b_{ik} = 1$. Hence there exists a j such that $b_{jk} = 1$ and $b_{ik} = 0$ for all $i \neq j$. Here we conclude $n_k = j$. □

Proof of Theorem B (i). Let $\{b_1 \ldots b_{p-1}\}$ be an arbitrary partition, as defined in (2). We will construct a set $K \subset \mathbb{Z}_{\geqslant 0}$ with the properties:

(i) $|K| \leqslant \tau$,
(ii) $R \subset B \cup K$.

For any such set K we have

$$\beta + \tau = |B| + |K| \geqslant |B \cup K| \geqslant |R| = \varrho.$$

We can complete the proof of Theorem B(i) by applying Lemma 4.1 which yields

$$\mathrm{ord}_p(u_n) \geqslant \min(\beta + \tau) \geqslant \varrho.$$

We shall now construct K satisfying properties (i) and (ii). Let M be the set of all k such that $k \in \bar{K}$, $k+1 \notin \bar{K}$ and $k \notin R$. Put $N = \{k+1 : k \in M\}$ and take $K = (\bar{K} \backslash M) \cup N$. Then K satisfies property (i) because $|K| \leqslant |\bar{K}| \leqslant \tau$. We shall prove property (ii) by showing that $k \in R$, $k \notin B \cup K$ leads to a contradiction. Note that $k \notin K$ implies $k \notin K_i$ for any $i \geqslant 1$. Hence

$$\left[\frac{d_k}{p} + \frac{p-2}{p-1}\right] = 0.$$

We conclude that $d_k < p/(p-1)$. By definition of R, we have $k-1 \notin R$. If $k-1 \in \bar{K}$ then our construction of K would imply $k \in K$, which contradicts the supposition that $k \notin B \cup K$. Hence $k-1 \notin K_i$ for any $i \geqslant 1$. This implies $d_{k-1} < p/(p-1)$. Thus by Lemma 4.2 we have either $n_k = 0$ or $n_k = j$ and $b_{jk} = 1$ for some j. Since $n_k = 0$ implies $k \notin R$, the first case of Lemma 4.2 is excluded. However $k \in R$ implies $j = n_k \in J$. The second case therefore implies $k \in B$, which is also excluded. This yields the desired contradiction.

Proof of Theorem B(ii). Choose $R \subset S$ such that ϱ is maximal. Then at least $\varrho \geqslant \frac{1}{2}\sigma$.

Proof of Theorem B(iii). Let $\{b_1 \ldots b_{p-1}\}$ be an arbitrary partition, as defined in (2). We will construct a set $K \subset Z_{\geqslant 0}$ with the properties:

(i) $|K| \leqslant \tau$,
(ii) $S \subset B \cup K$.

The construction of K is more complicated than in the first part. Put

$$M_1 = \{k \in \bar{K}: k \notin S,\ k+1 \notin \bar{K}\}, \quad N_1 = \{k+1 \in \mathbb{N}: k \in M_1\},$$

$$M_2 = \{k \in \bar{K}: k \in \bar{K}_i \cap \bar{K}_j \text{ for some distinct positive integers } i, j\},$$

$$N_2 = \{k+1 \in \mathbb{N}: k \in M_2\},$$

$$M_3 = \{k \in \bar{K} \cap B\}, \quad N_3 = \{k+1 \in \mathbb{N}: k \in M_3\}.$$

Take $K = (\bar{K}\backslash(M_1 \cup M_3)) \cup N_1 \cup N_2 \cup N_3$. Note that $|M_i| = |N_i|$ for $i = 1, 2, 3$, and $|M_1 \cup M_3| = |N_1 \cup N_3|$ and $|\bar{K}| + |N_2| \leqslant \Sigma_i|\bar{K}_i|$. We conclude $|K| \leqslant \Sigma_i\, |\bar{K}_i| \leqslant \tau$ and K satisfies property (i). K also satisfies property (ii). to see this, suppose $k \in S$ and $k \notin B \cup K$. This will lead to a contradiction. $k \in \bar{K}$ implies that $k \in M_1 \cup M_3$, since $k \notin K$. But $k \in M_1$ implies $k \notin S$ which contradicts $k \in S$, while $k \in M_3$ implies $k \in B$ which contradicts $k \notin B \cup K$. Therefore $k \notin \bar{K}$, hence

$$d_k < \frac{p}{p-1} \quad \text{and} \quad d_{k-1} < \frac{p^2}{p-1}.$$

We distinguish five cases:

(a) $d_{k-1} < p/(p-1)$. This leads to a contradiction, just as in the proof Theorem B(i).
(b) $d_{k-1} \geqslant p/(p-1)$ and $k-1 \notin S$. These imply that $k-1 \in \bar{K}$. Hence $k \in N_1$, contradicting $k \notin K$.

(c) $d_{k-1} \geqslant p/(p-1)$ and $d_{k-2} \geqslant p^2/(p-1)$. These imply that $k-1 \in K_i$ for some $i \geqslant 1$, and $k-2 \in K_j$ for some $j \geqslant 2$. Hence $k-1 \in \bar{K}_i \cap \bar{K}_j$. If $i \neq j$ then $k \in N_2$, which contradicts $k \notin K$. If $i = j$ then $i \geqslant 2$. This implies $k \in \bar{K}_i$, which also contradicts $k \notin K$.

(d) $d_{k-1} \geqslant p/(p-1)$ and $k-1 \in B$. These imply that $k-1 \in \bar{K} \cap B$. Hence $k \in N_3$, contradicting $k \notin K$.

(e) The remaining case reads

$$d_k < \frac{p}{p-1} \leqslant d_{k-1} < \frac{p^2}{p-1}, \quad d_{k-2} < \frac{p^2}{p-1}, \quad k-1 \in S, \quad k-1 \notin B.$$

Then $d_{k-2} < p^2/(p-1)$ implies that $T_{k-2} < p^k$ by Lemma 2.3(ii). Further $d_{k-1} < p^2/(p-1)$ implies that $c_{k-1} \leqslant p+1$. Since $k-1 \notin B$, we have

$$\sum_{i=1}^{p-1} ib_{i(k-1)} = \sum_{i=1}^{p-s-1} ib_{i(k-1)} \leqslant (p-s-1) \cdot c_{k-1} \leqslant (p-s-1) \cdot (p+1).$$

These arguments imply that

$$\begin{aligned} T_{k-1} &= T_{k-2} + \sum_i ib_{i(k-1)} p^{k-1} < p^k + (p+1) \cdot (p-s-1) \cdot p^{k-1} \\ &= p^{k+1} - (s-1) \cdot p^k - (s+1) \cdot p^{k-1} \\ &= (p-s) \cdot p^k + (p-s-1) \cdot p^{k-1}. \end{aligned}$$

Since $d_k < p/(p-1)$, $d_k = c_k + d_{k-1}/p$ and $p/(p-1) \leqslant d_{k-1}$, we have $c_k = 0$. Hence by use of Lemma 2.3(iii) we have

$$T_k = T_{k-1} < (p-s) \cdot p^k + (p-s-1) \cdot p^{k-1}. \tag{11}$$

On the other hand we have $k, k-1 \in S$, which implies $n_k \geqslant p-s$ and $n_{k-1} \geqslant p-s$ and thus

$$\begin{aligned} T_k &= \sum_{j=0}^{k} \sum_{i=1}^{e} ib_{ij} p^j \geqslant \sum_{j=0}^{k} n_j p^j \geqslant n_{k-1} p^{k-1} + n_k p^k \\ &\geqslant (p-s)p^k + (p-s)p^{k-1}, \end{aligned}$$

which contradicts (11). □

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