COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 68, nº 2 (1988), p. 221-239

<http://www.numdam.org/item?id=CM_1988__68_2_221_0>

Compositio Mathematica 68: 221–239 (1988)

A stable manifold theorem for the gradient flow of geometric variational problems associated with quasi-linear parabolic equations

Dedicated to Professor Akihiko Morimoto on his 60th birthday

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Received 8 February 1988; accepted 14 April 1988

Key words: stable manifolds, quasi-linear parabolic equations, variational problems.

Abstract. In this paper we prove the stable manifold theorem for a class of quasi-linear parabolic equations, and discuss an asymptotic behavior of the gradient flow of geometric variational problems.

1. Introduction

One of the purpose of this paper is to prove existence of stable and unstable manifolds for a class of quasi-linear parabolic equations. In the case of semi-linear parabolic equations, some results of the stable manifold theorem can be found in D. Henry's monograph [9] or others (for example N. Chafee and E. Infant [2]). Recently, C.L. Epstein and M.I. Weinstein proved a stable manifold theorem for the curve shortening equation [6]. They treat the curve shortening equation as a quasi-linear parabolic equation from S^1 to $\mathbf{R}$ (c.f. M. Gage and R.S. Hamilton [8].)

In this paper, we consider, on a closed Riemannian manifold (M, g), a class of quasi-linear parabolic equations of the following type:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -J(u) + N(u) \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where J is an elliptic operator of $2k$-th order, $N(u)$ represents the non-linear part of the equation, and a map $u\colon M \times [0, \omega) \to H$ satisfies $\pi \circ u(x, t) = x$ for all $x \in M$ and $t \in [0, \omega)$, for a vector bundle H over M with the

1980 Mathematics Subject Classification (1985 Revision). Primary 35K45, 58G11, 58E20; Secondary 35B35, 35B40.

finite-dimensional fiber and the projection π. Moreover J and N are supposed to satisfy the following three conditions:

(C1) The zero is a stationary solution of (1.1), i.e.,

$$-J(0) + N(0) = 0.$$

(C2) $\langle J(u), u\rangle_{L^2} \geqslant \Lambda\langle u, u\rangle_{L^2}$ for some $\Lambda \in \mathbf{R}$, and J is self adjoint with respect to L^2.

(C3) For $m > \frac{1}{2}\dim M + 2k$ and for $u, v \in H^{m+k}(M, H)$ such that $\|u\|_{H^{m+k}}, \|v\|_{H^{m+k}} < 1$ we have

$$\|N(u) - N(v)\|_{H^{m-k}} \leqslant C[\|u\|_{H^m}\|u - v\|_{H^{m+k}} + \|u - v\|_{H^m}\|v\|_{H^{m+k}}], \tag{1.2}$$

and

$$N(0) = 0.$$

The main result on the existence of stable and unstable manifolds, briefly stated, is:

THEOREM A. *For the stationary solution 0 of (1.1), there exist*

(a) *a finite codimensional stable invariant manifold whose elements are close to 0,*

(b) *a finite dimensional unstable invariant manifold whose elements are close to 0.*

REMARK: In the above theorem,

(a) the codimension of the stable manifold is equal to the dimension of negative and zero eigenspaces of J,

(b) the dimension of the unstable manifold is equal to the dimension of negative eigenspaces of J.

Another purpose of this paper is to prove the asymptotic stability of the gradient flow of a variational problem in geometry. In our previous papers [13, 14], we prove that a strongly stable harmonic map is an asymptotically stable stationary solution of Eells-Sampson equation (the equation of the gradient flow for harmonic maps). In this paper, the above result is extended to weakly stable or unstable harmonic maps as an application of Theorem A. On the asymptotic behavior, the main result, briefly stated, is:

THEOREM B. *For the functional*

$$\mathscr{L}(s) = \int_M L(s)(x)\, \mathrm{d}\mu_M \qquad s \in C^\infty(E),$$

where E is a smooth fiber bundle over (M, g), we suppose that s_1 is a weakly stable critical point and that the connected component of critical set which contains s_1 is non-degenerate. Then the equation of the gradient flow of $\mathscr{L}$:

$$\begin{cases} \dfrac{\partial s}{\partial t} = -\mathscr{E}\mathscr{L}(s) \\ s(0) = s_0 \end{cases}$$

has unique solution provided that Euler–Lagrange operator of $\mathscr{L}$: $\mathscr{E}\mathscr{L}$ is elliptic and that s_1 is close to s_0. Moreover the solution tends to a critical point at $t \to \infty$ with exponential order.

(For more precise statement, see Section 5).

In the second section, we prepare some of linear analysis: the definition of the norm of Sobolev spaces, the spectral theory of J and some well known inequalities. In Section 3 and 4, we prove Theorem A via a contraction mapping argument. Finally, in Section 5, we present examples of quasi-linear parabolic equations which originated in differential geometry and apply our result to prove Theorem B.

Many ideas due to [6] are used in this paper.

2. Preliminaries

The operator $-J$ is self adjoint as an operator on $L^2(M, H)$ and is an elliptic operator of $2k$-th order. It has therefore a discrete spectrum $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ accumulating only at $-\infty$. The projection operators onto the positive, negative and zero eigenspaces will be denoted by π_+, π_- and π_0, respectively. We will renumber the eigenvalues so that the positive eigenvalues are $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ and the negative eigenvalues are $\{\lambda_{-1}, \lambda_{-2}, \ldots\}$. Let $\lambda := \min\{|\lambda_1|, |\lambda_{-1}|\}$. Here we remark that since $\mathrm{Im}(\pi_+ + \pi_0)$ is finite dimensional all norms on $\mathrm{Im}(\pi_+ + \pi_0)$ are equivalent.

The norm of the Sobolev spaces $H^m(M, H)$ are defined by

$$\|u\|^2_{H^m} := \|J^{m/2k}\pi_- u\|^2_{L^2} + \|\pi_0 u\|^2_{L^2} + \|\pi_+ u\|^2_{L^2}.$$

The norm $\|\cdot\|_{H^m}$ is well-defined since $\|\cdot\|_{H^m}$ is equivalent to the usual defined norm of the Sobolev space.

We define the norm on the space $L^2(\mathbf{R}_+;\, H^m(M, H))$ by

$$|||u|||_m^2 := \int_0^\infty \|u(t)\|_{H^m}^2 \,\mathrm{d}t.$$

Let $\mathscr{B}_{\mu,m}$ denote the subspace of $L^2(\mathbf{R}_+;\, H^{m+k}(M,H)) \cap L^\infty(\mathbf{R}_+;\, H^m(M,H))$ defined by the norm:

$$|u|_{\mu,m}^2 := |||u|||_{m+k}^2 + \sup_{t>0}\, [e^{2\mu t}\, \|u(t)\|_{H^m}^2].$$

These spaces are clearly Banach spaces. In Sections 3 and 4, a non-linear operator will define contraction on $\mathscr{B}_{\mu,m}$ for suitable data.

Some basic inequalities we needed are following lemmas.

LEMMA 2.1. *Let $u \in L^2(\mathbf{R}_+;\, H^{m+k}(M, H))$, $v \in L^2(\mathbf{R}_+;\, H^{m-k}(M, H))$ and assume that u and v satisfy*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -J(u) + \pi_- v \\ u(0) \in \operatorname{Im} \pi_-, \end{cases} \tag{2.1}$$

then

$$\int_0^\infty \|u(t)\|_{H^{m+k}}^2 \,\mathrm{d}t \leqslant \|u(0)\|_{H^m}^2 + \int_0^\infty \|v(t)\|_{H^{m-k}}^2 \,\mathrm{d}t.$$

Proof. Multiply (2.1) by $J^{m/k}(u)$ and integrate over M to obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\langle u, J^{m/k}(u)\rangle_{L^2} &= \langle -J(u), J^{m/k}(u)\rangle_{L^2} + \langle \pi_- v, J^{m/k}(u)\rangle_{L^2} \\ &\leqslant \tfrac{1}{2}\{-\langle u, J^{(m/k)+1}(u)\rangle_{L^2} + \langle \pi_- v, J^{(m/k)-1}(v)\rangle_{L^2}\}. \end{aligned}$$

Integration in t gives

$$\begin{aligned} &\int_0^T \| J^{(m-k)/2k}\pi_- v(t)\|_{L^2}^2 \,\mathrm{d}t - \langle u, J^{m/k}(u)\rangle_{L^2}(T) + \langle u, J^{m/k}(u)\rangle_{L^2}(0) \\ &\quad \geqslant \int_0^T \| J^{(m+k)/2k}(u)(t)\|_{L^2}^2 \,\mathrm{d}t. \end{aligned}$$

Since $u \in \operatorname{Im}\pi_-$ and $u \in L^2(\mathbf{R}_+; H^{m+k}(M, H))$, we can find a sequence $T_n \to \infty$ such that $\langle u, J^{m/k}(u)\rangle_{L^2}(T_n) \to 0$. This gives the assertion of this lemma. ■

LEMMA 2.2. *If $u \in H^m(M, H)$ then*
(a) $e^{-Jt}\pi_- u \in H^m(M, H)$,

$$\|e^{-Jt}\pi_- u\|_{H^m} \leqslant C_\alpha e^{-\lambda t} t^{-\alpha/2} \|\pi_- u\|_{H^{m-k\alpha}}$$

and $C_0 = 1$ for $t > 0$ and $0 \leqslant \alpha \leqslant 1$.
(b) $e^{-Jt}\pi_+ u \in H^m(M, H)$,

$$\|e^{-Jt}\pi_+ u\|_{H^m} \leqslant C_\alpha e^{-\lambda t}(-t)^{-\alpha/2} \|\pi_+ u\|_{H^{m-k\alpha}}$$

and $C_0 = 1$ for $t < 0$ and $0 \leqslant \alpha \leqslant 1$.
Proof. They are well known estimate that

$$\|e^{-Jt}\pi_- u\|_{H^m} \leqslant e^{-\lambda t} \|\pi_- u\|_{H^m},$$

and

$$\|e^{-Jt}\pi_- u\|_{H^m} \leqslant Ce^{-\lambda t} t^{-1} \|\pi_- u\|_{H^{m-k}}, \quad \text{for } t > 0.$$

The following estimates is called the interpolation property of Sobolev spaces:

$$\|w\|_{H^{m-\alpha k}} \leqslant C \|w\|_{H^m}^{1-\alpha} \|w\|_{H^{m-k}}^{\alpha}.$$

for $w \in H^m$ and $0 \leqslant \alpha \leqslant 1$. These three estimates imply the estimate (a).
In the similar manner of the above argument, we obtain the estimate (b). ■

LEMMA 2.3. *Under the hypothesis of Lemma 2.1, and for $0 < \mu < \lambda$ the following inequality holds*:

$$e^{2\mu t} \|u(t)\|_{H^m}^2 \leqslant \|u(0)\|_{H^m}^2 + C \int_0^\infty \|v(t)\|_{H^{m-k}}^2 \, dt,$$

where λ is the constant in Lemma 2.2.

Proof. Similarly as proof of Lemma 2.1, we get

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\langle J^{m/k}(u), u\rangle_{L^2} &= \langle J^{m/2k}(u), J^{m/2k}(u_t)\rangle_{L^2} \\
&= \langle J^{m/k}(u), u_t\rangle_{L^2} \\
&= \langle J^{m/k}(u), (-J)(u)\rangle_{L^2} + \langle J^{m/k}(u), v\rangle_{L^2} \\
&= -\| J^{(m+k)/2k}(u)\|^2_{L^2} + \langle J^{(m+k)/2k}(u), J^{(m-k)/2k}(v)\rangle_{L^2} \\
&\leqslant -(1-\varepsilon)\| J^{(m+k)/2k}(u)\|^2_{L^2} + \frac{1}{2\varepsilon}\| J^{(m-k)/2k}(v)\|^2_{L^2}, \\
&\quad \text{for } 0 < \varepsilon < 1. \qquad (2.2)
\end{aligned}
$$

From Lemma 2.2, we have the estimate:

$$\| J^{(m+k)/2k}(u)\|^2_{L^2} \geqslant \lambda \| J^{m/2k}(u)\|^2_{L^2}. \tag{2.3}$$

Put $\mu = \lambda(1-\varepsilon)$, and combining (2.2) and (2.3) we get

$$\frac{1}{2}\frac{\partial}{\partial t}\| J^{m/2k}(u)\|^2_{L^2} \leqslant \mu \| J^{m/2k}(u)\|^2_{L^2} + C\| J^{(m-k)/2k}(v)\|^2_{L^2}. \tag{2.4}$$

An inequality (2.4) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\mathrm{e}^{2\mu t}\| J^{m/2k}(u)\|^2_{L^2}\} \leqslant C\| J^{(m-k)/2k}(v)\|^2_{L^2}. \tag{2.5}$$

Integration in t gives

$$\mathrm{e}^{2\mu t}\| J^{m/2k}(u)\|^2_{L^2}(t) \leqslant \| J^{m/2k}(u)\|^2_{L^2}(0) + C\int_0^t \| J^{(m-k)/2k}(v)\|^2_{L^2}(s)\,\mathrm{d}s.$$

This inequality asserts this lemma. ■

The above lemma was suggested by Y. Yamada.

3. An apriori estimate for the non-linear equation

We replace non-linear parabolic equation (1.1) by an integral equation:

$$
\begin{aligned}
Tu(t) := \mathrm{e}^{-Jt}\pi_- u_0 + \int_0^t \mathrm{e}^{-J(t-s)}\pi_- N(u)(s)\,\mathrm{d}s - \int_t^\infty \pi_0 N(u)(s)\,\mathrm{d}s \\
- \int_t^\infty \mathrm{e}^{-J(t-s)}\pi_+ N(u)(s)\,\mathrm{d}s. \qquad (3.1)
\end{aligned}
$$

In Sections 3 and 4, let $m > \frac{1}{2}\dim M + 2k$ and $0 < \mu < \lambda$. Here u_0 is an element in $H^m(M, H)$. A fixed point of T is a solution of (1.1). The initial data of the fixed point is

$$u(0) = \pi_- u_0 - \int_0^\infty \pi_0 N(u)(s)\,\mathrm{d}s - \int_0^\infty \pi_+ N(u)(s)\,\mathrm{d}s.$$

We will study T in Banach space $\mathscr{B}_{\mu,m}$ for $m > \frac{1}{2}\dim M + 2k$ and $0 < \mu < \lambda$.

In this section we will prove:

THEOREM 3.1. *For $m > \frac{1}{2}\dim M + 2k$, $0 < \mu < \lambda$ and $|u|_{\mu,m} < 1$, there exist constants C_1 and C_2 such that*

(a) $|Tu|^2_{\mu,m} \leqslant C_1\{\|\pi_- u_0\|^2_{H^m} + |u|^4_{\mu,m}\}$,

(b) $|Tu - \mathrm{e}^{-Jt}\pi_- u_0|^2_{\mu,m} \leqslant C_2|u|^4_{\mu,m}$.

Proof. We will construct $\mathscr{B}_{\mu,m}$-estimate by separating (3.1) freely.

Step 1. The H^m-estimate.

(i) To estimate π_--part we apply Lemma 2.1 to

$$f_-(t) = \mathrm{e}^{-Jt}\pi_- u_0 + \int_0^t \mathrm{e}^{-J(t-s)}\pi_- N(u)(s)\,\mathrm{d}s.$$

This function $f_-(t)$ satisfies the equation:

$$\begin{cases} \dfrac{\partial f_-}{\partial t} = -J(f_-) + \pi_- N(u) \\ u(0) = \pi_- u_0. \end{cases}$$

Lemma 2.3 implies that

$$\begin{aligned} \mathrm{e}^{2\mu t}\|f_-(t)\|^2_{H^m} &\leqslant \|\pi_- u_0\|^2_{H^m} + C\int_0^t \|\pi_- N(u)(s)\|^2_{H^{m-k}}\,\mathrm{d}s \\ &\leqslant \|\pi_- u_0\|^2_{H^m} + C\int_0^t \|u(s)\|^2_{H^m}\|u(s)\|^2_{H^{m+k}}\,\mathrm{d}s \\ &\leqslant \|\pi_- u_0\|^2_{H^m} + C\sup_{t>0}\mathrm{e}^{2\mu t}\|u(s)\|^2_{H^m}\int_0^t \mathrm{e}^{-2\mu s}\|u(s)\|^2_{H^{m+k}}\,\mathrm{d}s \\ &\leqslant \|\pi_- u_0\|^2_{H^m} + C|u|^4_{\mu,m}. \end{aligned}$$

Therefore we obtain

$$\mathrm{e}^{2\mu t}\|\pi_- Tu(t)\|^2_{H^m} \leqslant \|\pi_- u_0\|^2_{H^m} + C|u|^4_{\mu,m}. \tag{3.2}$$

(ii) We will estimate

$$\pi_+ Tu(t) = -\int_t^\infty e^{-J(t-s)}\pi_+ N(u)(s)\,ds \tag{3.3}$$

in H^m-norm. To estimate π_+-part we apply Lemma 2.3 to

$$f_+(t) = -\int_t^\infty e^{-J(t-s)}\pi_+ N(u)(s)\,ds.$$

This function $f_+(t)$ satisfies the equation:

$$\begin{cases} \dfrac{\partial f_+}{\partial t} = -J(f_+) + \pi_+ N(u) \\ u(0) = 0. \end{cases}$$

From Lemma 2.3, it is easily shown that

$$e^{2\mu t}\|\pi_+ Tu(t)\|_{H^m}^2 \leqslant C|u|_{\mu,m}^4. \tag{3.4}$$

(iii) Finally we will estimate in H^m-norm

$$\pi_0 Tu(t) = -\int_t^\infty \pi_0 N(u)(s)\,ds. \tag{3.5}$$

This is essentially the same as (ii), we can show that

$$e^{2\mu t}\|\pi_0 Tu(t)\|_{H^m}^2 \leqslant C|u|_{\mu,m}^4. \tag{3.6}$$

Combining (3.2), (3.4) and (3.6), we conclude that

$$e^{2\mu t}\|Tu(t)\|_{H^m}^2 \leqslant \|\pi_- u_0\|_{H^m}^2 + C|u|_{\mu,m}^4, \tag{3.7}$$

and also

$$e^{2\mu t}\|Tu(t) - e^{-Jt}\pi_- u_0\|_{H^m}^2 \leqslant C|u|_{\mu,m}^4. \tag{3.8}$$

Step 2. The $|\!|\!|\cdot|\!|\!|_{m+k}$-estimate.

As remarked in the previous section, there exists a constant C such that

$$\|\pi_+ Tu(t)\|_{H^{m+k}}^2 + \|\pi_0 Tu(t)\|_{H^{m+k}}^2 \leqslant C(\|\pi_+ Tu(t)\|_{H^m}^2 + \|\pi_0 Tu(t)\|_{H^m}^2). \tag{3.9}$$

Using estimate (3.5), (3.6) and (3.9), and integration in t we, therefore, obtain

$$|||\pi_+ Tu|||^2_{m+k} + |||\pi_0 Tu|||^2_{m+k} \leqslant C|u|^4_{\mu,m}. \tag{3.10}$$

To estimate π_--part we apply Lemma 2.1 to

$$f_-(t) = e^{-Jt}\pi_- u_0 + \int_0^t e^{-J(t-s)}\pi_- N(u)(s)\, ds.$$

From Lemma 2.1, it follows that

$$|||f_-|||^2_{m+k} \leqslant \|\pi_- u_0\|^2_{H^m} + \int_0^\infty \|\pi_- N(u)(s)\|^2_{H^{m-k}}\, dt.$$

Hence we obtain

$$\begin{aligned} |||\pi_- Tu|||^2_{m+k} &\leqslant \|\pi_- u_0\|^2_{H^m} + C|||u|||^2_{m+k} \sup_{t>0} \|u(t)\|^2_{H^m} \\ &\leqslant \|\pi_- u_0\|^2_{H^m} + C|u|^4_{\mu,m}. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we show that

$$|||Tu|||^2_{m+k} \leqslant \|\pi_- u_0\|^2_{H^m} + C|u|^4_{\mu,m}, \tag{3.12}$$

and also

$$|||Tu - e^{-Jt}\pi_- u_0|||^2_{m+k} \leqslant C|u|^4_{\mu,m}. \tag{3.13}$$

Therefore we obtain (a) by (3.7) and (3.12), and (b) by (3.8) and (3.13). ∎

Note that we have shown:

PROPOSITION 3.2. *Under the conditions in Theorem 3.1, the following estimate holds*:

$$\|Tu(t) - e^{-Jt}\pi_- u_0\|_{H^m} \leqslant C|u|^4_{\mu,m}\, e^{-2\mu t}. \tag{3.14}$$

Theorem 3.1 implies the following corollary which is realized an apriori estimate for the integral equation (3.1).

COROLLARY 3.3. *For $m > \frac{1}{2} \dim M + 2k$, and $0 < \mu < \lambda$, there is an $\varepsilon > 0$ such that if $\|\pi_- u_0\|_{H^m}^2 < \varepsilon$ then the ε-ball whose center is $e^{-Jt}\pi_- u_0$ in $\mathscr{B}_{\mu,m}$ is mapped into itself by T.*

4. The proof of Theorem A

In this section, we show that if ε, in Corollary 3.3, is chosen small enough then T is contraction on $\mathscr{B}_{\mu,m}$. As in conclusion of this argument, the existence of stable and unstable manifolds will be shown.

THEOREM 4.1. *For $m > \frac{1}{2} \dim M + 2k$ and $0 < \mu < \lambda$, if $|u|_{\mu,m}$, $|v|_{\mu,m} < 1$ then there exists a constant C such that*

$$|T(u) - T(v)|_{\mu,m}^2 \leqslant C(|u|_{\mu,m}^2 + |v|_{\mu,m}^2)|u - v|_{\mu,m}^2. \tag{4.1}$$

Proof. As before we break up the proof into two steps:

Step 1. The H^m-estimate.

The H^m-norm of the difference of $Tu(t)$ and $Tv(t)$ is separated into three parts. Each part is estimated by essentially same argument of the proof of Theorem 3.1:

$$\|Tu(t) - Tv(t)\|_{H^m} \leqslant \|\pi_-(Tu(t) - Tv(t))\|_{H^m}$$
$$+ \|\pi_0(Tu(t) - Tv(t))\|_{H^m} + \|\pi_+(Tu(t) - Tv(t))\|_{H^m}.$$

To estimate the π_--part, we apply Lemma 3.3 to

$$f_-(t) = \int_0^t e^{-J(t-s)}\pi_-(N(u)(s) - N(v)(s))\, ds.$$

This function $f_-(t)$ satisfies the equation:

$$\begin{cases} \dfrac{\partial f_-}{\partial t} = -J(f_-) + \pi_-(N(u) - N(v)) \\ f_-(0) = 0. \end{cases}$$

From Lemma 2.3, the following estimate holds:

$$e^{2\mu t}|||f_-|||_{m+k}^2 \leqslant C\int_0^t \|\pi_-(N(u)(s) - N(v)(s))\|_{H^{m-k}}^2\, ds. \tag{4.2}$$

On the other hand, from the condition (C3) we get

$$C \int_0^t \|\pi_-(N(u)(s) - N(v)(s))\|^2_{H^{m-k}}\, \mathrm{d}s$$

$$\leqslant C \int_0^t [\|u(s)\|^2_{H^m}\|u(s) - v(s)\|^2_{H^{m+k}} + \|v(s)\|^2_{H^{m+k}}\|u(s) - v(s)\|^2_{H^m}]\, \mathrm{d}s. \tag{4.3}$$

Hence we get

$$\mathrm{e}^{2\mu t}\|\pi_-(Tu(t) - Tv(t))\|^2_{H^m} \leqslant C(|u|^2_{\mu,m} + |v|^2_{\mu,m})|u - v|^2_{\mu,m}. \tag{4.4}$$

Estimates of the same type hold for the π_+ and π_0-parts:

$$\mathrm{e}^{2\mu t}\|\pi_+(Tu(t) - Tv(t))\|^2_{H^m} \leqslant C(|u|^2_{\mu,m} + |v|^2_{\mu,m})|u - v|^2_{\mu,m}, \tag{4.5}$$

and

$$\mathrm{e}^{2\mu t}\|\pi_0(Tu(t) - Tv(t))\|^2_{H^m} \leqslant C(|u|^2_{\mu,m} + |v|^2_{\mu,m})|u - v|^2_{\mu,m}. \tag{4.6}$$

Step 2. The $|||\cdot|||_{m+k}$-estimate.

Since $\mathrm{Im}(\pi_+ + \pi_0)$ is finite dimensional, there exists a constant C such that

$$\|(\pi_+ + \pi_0)(Tu(t) - Tv(t))\|^2_{H^{m+k}} \leqslant C\|(\pi_+ + \pi_0)(Tu(t) - Tv(t))\|^2_{H^m}. \tag{4.7}$$

Therefore from (4.5–6) we obtain

$$\|(\pi_+ + \pi_0)(Tu(t) - Tv(t))\|^2_{H^{m+k}} \leqslant C(|u|^2_{\mu,m} + |v|^2_{\mu,m})|u - v|^2_{\mu,m}. \tag{4.8}$$

To estimate π_--part we apply Lemma 2.1 to

$$f_-(t) = \int_0^t \mathrm{e}^{-J(t-s)}\pi_-(N(u)(s) - N(v)(s))\, \mathrm{d}s.$$

From Lemma 2.1, f_- satisfies that

$$|||f_-|||^2_{m+k} \leqslant \int_0^\infty \|\pi_-(N(u)(s) - N(v)(s))\|^2_{H^{m-k}}\, \mathrm{d}s.$$

Hence we obtain

$$|||\pi_-(Tu - Tv)|||^2_{m+k}$$

$$\leqslant |||u|||^2_{m+k} \sup_{t>0} \|u(t) - v(t)\|^2_{H^m} + |||u - v|||^2_{m+k} \sup_{t>0} \|v(t)\|^2_{H^m}$$

$$\leqslant (|u|^2_{\mu,m} + |v|^2_{\mu,m})|u - v|^2_{\mu,m}. \tag{4.9}$$

Therefore, combining (4.4–6) and (4.8–9), we obtain (4.1). ■

One of the main result of this paper will follow:

COROLLARY 4.2. *For* $m > \frac{1}{2}\dim M + 2k$ and $0 < \mu < \lambda$, *there exists an* $\varepsilon > 0$ *such that for every* $u_0 \in \operatorname{Im} \pi_-$ *with*

$$|e^{-Jt}u_0|_{\mu,m} < \varepsilon,$$

then there is a unique solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = -J(u) + N(u) & \text{on } M \times [0, \infty) \\ \pi_- u(0) = u_0 & \text{on } M \times (0) \end{cases}$$

in $\mathscr{B}_{\mu,m}$ *with* $|u|_{\mu,m} < \varepsilon$. *Moreover the solution tends to zero in* H^m*-norm as* $t \to \infty$ *with exponential order.*

Proof. From Corollary 3.3, there exists an $\varepsilon > 0$ such that if $u_0 \in \operatorname{Im}\pi_-$ and $|e^{-Jt}u_0|_{\mu,m} < \varepsilon$, then the map T is contraction. The existence and the uniqueness immediately follow from this fact. Proposition 3.2 implies that the H^m-norm of the solution $u(t)$ tends to zero as $t \to \infty$. ■

By using (3.1) one can show that the solution $u(t)$ depends smoothly on u_0 in the H^m-topology. The stable manifold as a submanifold of $H^m(M, H)$ has the codimension dim (Im $(\pi_+ + \pi_0)$).

To obtain an unstable manifold we use the integral equation:

$$\begin{aligned} T^- u(t) &= e^{-Jt}\pi_+ u_0 + \int_0^t \pi_+ e^{-J(t-s)} N(u)(s)\, ds - \int_t^{-\infty} \pi_0 N(u)(s)\, ds \\ &\quad - \int_t^{-\infty} \pi_- e^{-J(t-s)} N(u)(s)\, ds, \quad \text{for } t < 0. \end{aligned} \tag{4.10}$$

As similar Corollary 4.2 one can show that T^- is contraction on

$$\mathscr{B}^-_{\mu,m} \subset L^2(\mathbf{R}_-; H^{m+k}(M, H)) \cap L^\infty(\mathbf{R}_-; H^m(M, H)).$$

Therefore we obtain:

COROLLARY 4.3. *For* $m > \frac{1}{2}\dim M + 2k$ and $0 < \mu < \lambda$, *there exists an* $\varepsilon > 0$ *such that for every* $u_0 \in \operatorname{Im} \pi_+$ *with*

$$|e^{-Jt}u_0|^-_{\mu,m} < \varepsilon,$$

then there is a unique solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = -J(u) + N(u) & \text{on } M \times (-\omega, 0] \\ \pi_+ u(0) = u_0 & \text{on } M \times (0) \end{cases}$$

in $\mathscr{B}^-_{\mu,m}$ *with* $|u|^-_{\mu,m} < \varepsilon$. *Moreover the solution tends to zero in* H^m*-norm as* $t \to -\infty$ *with exponential order.*

Corollary 4.3 implies that there exists a space of initial data with which the solutions are asymptotically stable for the backwards evolution equation. This space is called by the unstable manifold. The unstable manifolds as a submanifolds of $H^m(M, H)$ has the dimension $\dim(\operatorname{Im} \pi_+)$.

5. The gradient flow equation for geometric variational problems

In this section, we present a quasi-linear parabolic equation originated in differential geometry.

Let E be a smooth fiber bundle over (M, g) with the fiber a Riemannian manifold (N, h). We denote by $C^\infty(E)$ the set of smooth sections of E, and $\mathbf{R}_M$ a trivial line bundle over M: $M \times \mathbf{R}$. Let $J^r(E)$ be the r-th-ordered jet bundle of E and j_r: $C^\infty(E) \to C^\infty(J^r(E))$ be the jet extension map. For a fiber bundle morphism: F: $J^r(E) \to \mathbf{R}_M$, $L = F_* \circ j_r$ is realized as an r-th ordered nonlinear differential operator from E to $\mathbf{R}_M$, where F_*: $C^\infty(E) \to C^\infty(\mathbf{R}_M)$ is the canonical map induced by F. Now, we define a functional $\mathscr{L}$: $C^\infty(E) \to \mathbf{R}$ by

$$\mathscr{L}(s) = \int_M L(s)(x)\, \mathrm{d}\mu_M \quad \text{for } s \in C^\infty(E). \tag{5.1}$$

Since $L = F_* \circ j_k$, this can be written as

$$\mathscr{L}(s) = \int_M F(x, s(x), Ds(x), \cdots, D^k s(x))\, \mathrm{d}\mu_M \quad \text{for } s \in C^\infty(E). \tag{5.2}$$

In this section, we consider a heat equaton of $\mathscr{L}$:

$$\begin{cases} \dfrac{\partial s}{\partial t} = -\mathscr{E}\mathscr{L}(s) \\ s = s_1. \end{cases} \tag{5.3}$$

This equation (5.3) is realized as the equation governing the gradient flow of the functional.

The aim of this section is to prove asymptotic behavior of (5.3) near a critical point. To do so, we rewrite (5.3) to a "linearized" equation as below.

We define a vector bundle neighborhood (abbreviated VBN) of a fiber bundle [11, p44]. For $s \in C^0(E)$, a VBN ξ in E of s is a vector bundle over M such that ξ is an open subbundle of E and $s \in C^0(\xi)$. For $s \in C^0(E)$, we define $T_s(E)$ as a vector bundle over M which is $s^*(\ker(d\pi)|M)$, where π: $E \to M$ is the projection of the bundle. Canonically, we can identify $T_s(E)$ as a VBN in E of s. Let $\mathcal{O}_E$ be a neighborhood in E which contains $s_0(M)$. It is obvious that $\mathcal{O}_E$ is diffeomorphic to a neighborhood $\mathcal{O}$ in $T_s(E)$ by the exponential map $\exp_{s_0(x)}\sigma(x)$. Using this diffeomorphism, we define a functional $\mathscr{L}'$: $C^\infty(\mathcal{O}) \to \mathbf{R}$ by

$$\mathscr{L}'_{s_0}(\sigma) = \int_M L'_{s_0}(\sigma)(x)\, d\mu_M \quad \sigma \in C^\infty(\mathcal{O}). \tag{5.4}$$

Since we can denote that $L'_{s_0} = F'_{s_0*} \circ j'_k$ where j'_k is the jet extension map and F'_{s_0}: $C^\infty(J^k(\mathcal{O})) \to C^\infty(\mathbf{R}_M)$,

$$\mathscr{L}'_{s_0}(\sigma) = \int_M F'_{s_0}(x, \sigma(x), D^1\sigma(x), \cdots, D^k\sigma(x))\, d\mu_M.$$

We see that s_0 is a critical point of $\mathscr{L}$ if and only if the zero section of $\mathcal{O}$ is a critical point of $\mathscr{L}'$. In local coordinates, the Euler-Lagrange operator of $\mathscr{L}$ and $\mathscr{L}_{s_0}$ are denoted by

$$(\mathscr{E}\mathscr{L}(s))^\alpha = \frac{(-1)^{|A|}}{\sqrt{g}} h^{\alpha\beta} D_A\left(\sqrt{g}\,\frac{\partial F}{\partial p_A^\beta}(x, s(x))\right) \tag{5.6}$$

and

$$(\mathscr{E}\mathscr{L}(\sigma)'_{s_0})^\alpha = \frac{(-1)^{|A|}}{\sqrt{g}} h^{\alpha\beta} D_A\left(\sqrt{g}\,\frac{\partial F'_{s_0}}{\partial p_A^\beta}(x, \sigma(x))\right), \tag{5.7}$$

respectively. Furthermore $d/dt|_{t=0}\, \mathscr{E}\mathscr{L}'_{s_0}(\sigma_t)$ for $\sigma_t = t\sigma$ is called the Jacobi operator of $\mathscr{E}\mathscr{L}'_{s_0}$ which can be written as follows:

$$\begin{aligned} J(\sigma)^\alpha_{s_0} = \frac{(-1)^{|A|}}{\sqrt{g}} \Bigg\{ & \frac{\partial h^{\alpha\beta}}{\partial p^\gamma} D_A\left(\sqrt{g}\,\frac{\partial F'_{s_0}(x, 0)}{\partial p_A^\beta}\right)\sigma^\gamma \\ & + h^{\alpha\beta} D_A\left(\sqrt{g}\,\frac{\partial^2 F'_{s_0}(x, 0)}{\partial p_A^\beta \partial p_B^\gamma} D_B\sigma^\gamma\right)\Bigg\}, \end{aligned} \tag{5.8}$$

where $A = (i_1, \cdots, i_l)$, $|A| = l$, p^α_A are local coordinates in $J^k(\mathcal{O})$, $D_A = \partial^l/\partial x_{i_1} \ldots \partial x_{i_l}$, $1 \leqslant i_1, \cdots, i_l \leqslant \dim M$, $1 \leqslant \alpha, \beta, \gamma \leqslant \dim N$, and the summention convention are used.

Here we define the ellipticity of $\mathscr{E}\mathscr{L}$. The Euler–Lagrange operator $\mathscr{E}\mathscr{L}$ is called elliptic whenever J is an elliptic operator. In this section, we assume that $\mathscr{E}\mathscr{L}$ is elliptic.

PROPOSITION 5.1. *For the functional $\mathscr{L}'_{s_0}$, the Euler–Lagrange equation of $\mathscr{L}'_{s_0}$ is*

$$(\mathscr{E}\mathscr{L}'_{s_0}(0))^\alpha = \frac{(-1)^{|A|}}{\sqrt{g}} h^{\alpha\beta} D_A \left(\sqrt{g} \frac{\partial F'_{s_0}}{\partial p^\alpha_A}(x, 0) \right) = 0 \quad \textit{for } \alpha = 1, \cdots, \dim N.$$

Let $s_0 \in C^\infty(E)$ be a critical point of $\mathscr{L}$. The *index* of s_0, denoted by Index (s_0), is the maximal dimension of negative definite subspace of $C^\infty(E)$ by J_{s_0}. The *nullity* of s_0, denoted by Null(s_0), is the dimension of the kernel of J_{s_0}. We call a critical point s_0 *stable* and *weakly stable* whenever Index(s_0) = Null$(s_0) = 0$ and Index$(s_0) = 0$, respectively.

Under the above situation, we reduce (5.3) to an evolution equation. If the initial value s_1 of (5.3) satisfies the property:

$$|s_1, s_0| := \sup_{x \in M} \mathrm{d}_N(s_1(x), s_0(x)) < i_N,$$

where i_N denotes the injectivity radius of N, then by the above argument we consider, instead of (5.3), the following equation:

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} = -\mathscr{E}\mathscr{L}'_{s_0}(\sigma) & \text{on } M \times [0, \omega) \\ \sigma = \sigma_1 & \text{on } M \times (0). \end{cases} \tag{5.9}$$

In the similar manner of [13], the Taylor expansion of (5.9) yields a new evolution equation. We can calculate as follows:

$$\begin{aligned} \mathscr{E}\mathscr{L}'_{s_0}(\sigma_t) &= \mathscr{E}\mathscr{L}'_{s_0}(\sigma_t)|_{t=0} + t \cdot \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathscr{E}\mathscr{L}'_{s_0}(\sigma_t) \\ &\quad + \int_0^t (t - \xi) \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=\xi} \mathscr{E}\mathscr{L}'_{s_0}(\sigma_t)\, \mathrm{d}\xi, \end{aligned} \tag{5.10}$$

and

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathscr{E}\mathscr{L}'_{s_0}(\sigma_t)^\alpha\right|_{t=\xi} = \frac{(-1)^{|A|}}{\sqrt{g}}\left\{\frac{\partial^2 h^{\alpha\beta}}{\partial p^\delta \partial p^\gamma}\,D_A\left(\sqrt{g}\,\frac{\partial F'_{s_0}(x,\,\xi\sigma(x))}{\partial p^\beta_A}\right)\sigma^\gamma\sigma^\delta\right.$$

$$+\,2\,\frac{\partial h^{\alpha\beta}}{\partial p^\gamma}\,D_A\left(\sqrt{g}\,\frac{\partial^2 F'_{s_0}(x,\,\xi\sigma(x))}{\partial p^\beta_A\partial p^\delta_B}\,D_B\sigma^\delta\right)\sigma^\gamma$$

$$\left.+\,h^{\alpha\beta}D_A\left(\sqrt{g}\,\frac{\partial^3 F'_{s_0}(x,\,\xi\sigma(x))}{\partial p^\beta_A\partial p^\gamma_B\partial p^\delta_C}\,D_B\sigma^\gamma D_C\sigma^\delta\right)\right\}. \quad (5.11)$$

Hence we get the evolution equation, if s_0 is a critical point of $\mathscr{L}$ and s satisfies the above property, instead of (5.3),

$$\begin{cases}\dfrac{\partial\sigma}{\partial t} = -J(\sigma) + N(\sigma) & \text{on } M\times[0,\,\omega)\\[2mm] \sigma = \sigma_1 & \text{on } M\times(0).\end{cases} \quad (5.12)$$

The non-linear term of (5.12) is a differential operator of the $2k$-th order and its $2k$-th-ordered terms have coefficients at most k-th-ordered terms. Hence $N(\sigma)$ satisfies the condition (1.2). Clearly, from the definition $\mathscr{L}'_{s_0}$ the zero is a stationary solution of (5.12) and $J(0) = N(0) = 0$. From the assumption of ellipticity of $\mathscr{E}\mathscr{L}$, J satisfies the condition (C2). Therefore the equation (5.12) is the quasi-linear parabolic equation of type (1.1).

Applying Theorem A to (5.9), we conclude that:

PROPOSITION 5.2. *For* $m > \frac{1}{2}\dim M + 2k$ and $0 < \mu < \lambda$, *there exists an* $\varepsilon > 0$ *such that for every* $\sigma_0 \in \operatorname{Im}\pi_-$ *with*

$$|e^{-Jt}\sigma_0|_{\mu,m} < \varepsilon,$$

then there is a unique solution to

$$\begin{cases}\dfrac{\partial\sigma}{\partial t} = -\mathscr{E}\mathscr{L}'_{s_0}(\sigma) & \text{on } M\times[0,\,\infty)\\[2mm] \pi_-\sigma(0) = \sigma_0 & \text{on } M\times(0)\end{cases}$$

in $\mathscr{B}_{\mu,m}$ and the solution satisfies the estimate

$$\|\sigma(t)\|_{H^m} \leqslant Ce^{-\mu t}$$

for some constant C.

By the transformation $s(x, t) = \exp_{s_0(x)}\sigma(x, t)$, $\sigma(x, t)$ is the solution of (5.9) if and only if $s(x, t)$ is the solution of (5.3).

Therefore we have:

THEOREM 5.3. *For $m > \frac{1}{2}\dim M + 2k$, if the initial value s_1 satisfies some condition, then the equation of the gradient flow of the functional $\mathscr{L}$:*

$$\begin{cases} \dfrac{\partial s}{\partial t} = -\mathscr{E}\mathscr{L}(s) & \text{on } M \times [0, \infty) \\ s(0) = s_1 & \text{on } M \times (0) \end{cases}$$

has a unique global solution in some Banach space and the solution exponentially tends to a critical point in $H^m(E)$.

In the statement of Theorem 5.3, what is some condition for s_1 is that $\sigma_1(x) := (\exp_{s_0})^{-1}s_1(x)$ satisfies the conditions for the initial value of Proposition 5.2. In particular in the case that the critical point s_0 is weakly stable, we can show more precise result.

We define that the critical set S of vector field X on an infinite-dimensional manifold $\mathscr{B}$ is non-degenerate if and only if S satisfies following conditions.

(D1) The set S is a smooth manifold as a submanifold in $\mathscr{B}$.

(D2) For each point $p \in S$, $T_pS = \operatorname{Ker} d_pX$.

We call that $\sigma \in T_{s_0}(E)$ is a *Jacobi field* whenever $J(\sigma) = 0$. Furthermore Jacobi field $\sigma \in T_{s_0}(E)$ is called *integrable* if there exists a 1-parameter family $\{u_t\}$ of critical points with $u_0 = 0$ such that $\partial u_s/\partial t|_{t=0} = \sigma$ [16]. In our case, $S =$ the set of critical points of $\mathscr{L}'_{s_0}$. The connected component of S which contains the zero is denoted by S^0. It is easily shown that if S is non-degenerate then all Jacobi fields are integrable.

We assume that S is non-degenerate. This assumption is equivalent to that the critical set of $\mathscr{L}$ is non-degenerate.

If the zero section of $T_{s_0}(E)$ is a weakly stable critical point and S^0 is non-degenerate, then, for all σ which is not contained in S^0, $\pi_-\sigma$ is satisfies the conditions of Proposition 5.2.

Therefore we have:

THEOREM 5.4. *We assume that s_0 is a weakly stable critical point of $\mathscr{L}$ and the connected component of the set of critical points which contains s_0 is non-degenerate. For $m > \frac{1}{2}\dim M + 2k$, the initial value s_1 is close to a critical point then there exists a unique global solution of*

$$\begin{cases} \dfrac{\partial s}{\partial t} = -\mathscr{E}\mathscr{L}(s) & \text{on } M \times [0, \infty) \\ s(0) = s_1 & \text{on } M \times (0) \end{cases}$$

in some Banach space and the solution tends to a critical point with exponential order.

Here we remark that in the above statement what is some condition is that $\sigma(x) := (\exp_{s_0(x)})^{-1} s_1(x)$ is close to zero.

EXAMPLE. *Harmonic maps and Eells–Sampson equation*
In the above situation, let $E = M \times N$ and

$$F(x, f, p) := \tfrac{1}{2} g^{ij} h_{\alpha\beta}(f) p_i^\alpha p_j^\beta .$$

A critical point of the functional, which is defined by F, is called a harmonic map. The Euler–Lagrange operator of this functional is:

$$(\mathscr{E}\mathscr{L}(f))^\alpha = -\left(\Delta_M f^\alpha + g^{ij} \Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}\right).$$

We denote this operator $-\Delta f$ which is called by the tension field of f. The gradient flow equation is called by Eels–Sampson equation. In this case, Jacobi operator J is:

$$J_f(\sigma) = \Delta^f \sigma - \mathrm{Trace} R^N(\mathrm{d}f, \sigma)\, \mathrm{d}f \quad \text{for } \sigma \in C^\infty(f^{-1}TN),$$

where Δ^f is called the rough Laplacian on the vector bundle $f^{-1}TN$ over M. Therefore if N has non-positive curvature, J is a positive operator: i.e., all harmonic maps are weakly stable.

In 1964, J. Eells and J.H. Sampson showed that if N had non-positive curvature there exists at least one harmonic map in each homotopy class of $C^\infty(M, N)$. In their paper, they prove that, under the assumption that N has non-positive curvature, Eells–Sampson equation has unique global solution for arbitrary smooth initial values and the solution tends to a harmonic map [5, 9].

Applying Theorems 5.3 and 5.4 to this example, we obtain the existence theorem for Eells–Sampson equation, there exists a set of initial values such