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On a condition of thinness at infinity

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1. Introduction

Let E be a domain in $\mathbb{R}^m$ $(m \geqslant 3)$. E is said to be thin at a point x_0 if there exists a superharmonic function u, defined in $B(x_0, r) = \{x \in \mathbb{R}^m \,|\, |x - x_0| < r\}$ for some r, such that

$$\liminf_{\substack{y \to x_0, \\ y \in B(x_0, r) \cap E, \\ y \neq x_0}} u(y) > u(x_0)$$

This is the definition given by Brelot [1] to characterize regularity of a point for the Dirichlet problem. Using the Riesz decomposition Theorem it is possible to prove that the "lim inf" can be taken to be infinite. This fact implies that the solid angle subtended by $\partial B(x_0, r) \cap E$ at the point x_0 is very small. In fact, it tends to zero with r. More precisely, if

$$\theta(r) = \frac{\sigma[\partial B(x_0, r) \cap E]}{\sigma[\partial B(x_0, r)]},$$

where σ is the Lebesgue measure on $\partial B(x_0, r)$, then $\theta(r) \to 0$, as $r \to 0$. A proof of this result can be found in [5, p. 211]. Therefore the following natural question arises: how big can $\theta(r)$ be when E is thin at x_0? In this paper we shall give an answer to this question which is best possible in a sense that will be made precise later on.

In the theorem that follows we shall give the condition of thinness, for domains which are thin at infinity, in terms of the characteristic constant of spherical sets. The precise definition of this can be found in Section 2. The corresponding condition for domains which are thin at the origin is the same. This is so since the characteristic constant is invariant under inversion with respect to the unit sphere S^m.

THEOREM 1: *Let E be an unbounded domain in $\mathbb{R}^m$ $(m \geqslant 3)$, containing the origin and thin at infinity. Let E^c be the complement of E; we assume that $(\bar{E})^c$ is regular for the Dirichlet problem. If $\alpha(r)$ is the characteristic constant of $\partial B(0, r) \cap E^c$, then*

$$\int_0^\infty \frac{\alpha(r)}{r}\, dr < \infty.$$

COROLLARY: *Let E be as above and $\theta(r) = [\sigma(\partial B(0, r) \cap E)]/[\sigma(\partial B(0, r)]$.* Then

$$\int_0^\infty \frac{\left\{\log \dfrac{1}{\theta(r)}\right\}^{-1}}{r}\, dr < \infty, \quad \text{if } m = 3$$

and

$$\int_0^\infty \frac{\theta(r)^{m-3}}{r}\, dr < \infty, \quad \text{if } m > 3.$$

This corollary is a consequence of Theorem 1 and the asymptotic expressions (as $\varphi \to \pi$) for the characteristic constant $\alpha(C_\varphi)$ of a spherical cap

$$C_\varphi = \{x = (x_1, x_2, \ldots, x_m) \in S^m |\, \cos\varphi < x_1 \leqslant 1\},$$

where S^m is the unit sphere of $\mathbb{R}^m$. These expressions are

$$\alpha(C_\varphi) \sim \frac{1}{2 \log\left(\dfrac{2}{\pi - \varphi}\right)}, \quad \text{if } m = 3$$

and

$$\alpha(C_\varphi) \sim \frac{\Gamma(m-2)}{\Gamma\left(\dfrac{m-1}{2}\right)\Gamma\left(\dfrac{m-3}{2}\right)} \left\{\frac{\pi - \varphi}{2}\right\}^{m-3}, \quad \text{if } m > 3,$$

as $\varphi \to \pi$. The proofs of these asymptotic results can be found in [2, Th. 4]. We shall prove that Theorem 1 is best possible in the following sense.

THEOREM 2: *Let $\phi(t)$ be a positive decreasing function defined in* $[0, \infty)$ *such that*

$$\int_1^\infty \frac{\phi(t)}{t}\,\mathrm{d}t < \infty. \tag{1.1}$$

Then, there exists an unbounded regular domain $E \subseteq \mathbb{R}^m$ ($m \geqslant 3$), which is thin at infinity and such that $\phi(t) = \mathrm{O}(\alpha(t))$, as $t \to \infty$, where $\alpha(t)$ is the characteristic constant of $\partial B(0, t) \cap E^c$.

2. The characteristic constant

Following [4] let E be a measurable set on the unit sphere S^m of $\mathbb{R}^m$. We shall associate to E a constant $\alpha(E)$ (the characteristic constant of E) in the following way. If E is regular with analytic boundary then we define

$$\lambda(E) = \inf_{f \in \mathfrak{F}} \left\{ \frac{\int_E |\mathrm{grad}\, f|^2\, \mathrm{d}\sigma_m}{\int_E |f|^2\, \mathrm{d}\sigma_m} \right\},$$

where

$$\mathfrak{F} = \{f \mid \text{(i) is continuously differentiable on } S^m, \text{ (ii) } f \text{ vanishes outside } E, \text{ (iii) } f \not\equiv 0 \text{ in } E\}$$

and $\mathrm{d}\sigma_m$ denotes an element of surface area on S^m. Since the class $\mathfrak{F}$ gets larger with increasing E and the infimum smaller, $\lambda(E)$ decreases with increasing domains. If F is a general compact set, we define $\lambda(F)$ as the upper bound of $\lambda(E)$ over the class of all domains E containing F. Finally, if E is a general set then we define $\lambda(E)$ as the lower bound over the class of all compact sets F contained in E. The characteristic constant of E is defined to be the positive root of the equation

$$\alpha(\alpha + (m - 2)) = \lambda(E).$$

If the set E lies on the sphere $S(0, r)$ we define $\alpha(E) = \alpha(\hat{E})$, where $\hat{E}$ is the radial projections of E on S^m (i.e., $\hat{E}$ is the set of all $x/|x|$, where $x \in E$). We remark that the characteristic constant is invariant under inversion with respect to the sphere S^m.

3. Thinness at the origin and thinness at infinity

A set E is thin at the origin if and only if

$$\sum_{n=1}^{\infty} \frac{C(E_n)}{\lambda^{n(m-2)}} < \infty, \tag{3.1}$$

where $0 < \lambda < 1$, $E_n = E \cap \{x \in \mathbb{R}^m | \lambda^{n+1} \leqslant |x| < \lambda^n\}$ and $C(E_n)$ is the Newtonian capacity of E_n. This is the well known Wiener criterion [6. p. 287]. In what follows F^* will denote the image of the set F under inversion with respect to S^m.

LEMMA: *Let E be an open set in $\mathbb{R}^m$ with the origin as a boundary point and E^* the inversion of E with respect to S^m. Then E is thin at* 0 *if and only if*

$$\sum_{n=1}^{\infty} \lambda^{n(m-2)} C((E_n)^*) < \infty, \tag{3.2}$$

where $E_n = E \cap \{x \in \mathbb{R}^m | \lambda^{n+1} \leqslant |x| < \lambda^n\}$ and $0 < \lambda < 1$.

Whenever a set E^* satisfies (3.2) we shall say that E^* is thin at infinity.

Proof. This is a consequence of the Wiener criterion together with the estimates

$$\frac{1}{\lambda^{2(n+1)(m-2)}} C(E_n) \geqslant C((E_n)^*) \geqslant \frac{1}{\lambda^{2n(m-2)}} C(E_n).$$

4. Proof of Theorem 1

Since E is thin at infinity then

$$\sum_{n=1}^{\infty} \lambda^{n(m-2)} C(E_n) < \infty,$$

where $E_n = E \cap \{x \in \mathbb{R}^m | \lambda^{n+1} \leqslant |x| < \lambda^n\}$ and $0 < \lambda < 1$. Therefore there exists a positive Borel measure μ such that the support of μ is contained in $\bar{E}$, $U^{\mu} \leqslant 1$ throughout $\mathbb{R}^m$, where

$$U^{\mu}(x) = \int_{\mathbb{R}^m} \frac{1}{|x - y|^{m-2}} \, d\mu(y)$$

is the Newtonian potential associated to μ, and $U^{\mu}(x) \equiv 1$ quasi-everywhere on $\bar{E}$. A proof of this result can be found in [6, p. 280]. The exceptional set that appears here is empty since $(\bar{E})^c$ is regular. In fact, using the mean value inequality for the superharmonic function U^{μ} it is easy to prove that $U^{\mu}(x) \equiv 1$ in E. On the other hand, if $U^{\mu}(x_0) < 1$ for some x_0 in $\bar{E}$ then x_0 must belong to the boundary of E, and using Brelot's definition of thinness at a point one deduces that E is thin at x_0. This implies that x_0 is an irregular boundary point of $(\bar{E})^c$, which is absurd. Now we set $u(x) = 1 - U^{\mu}(x)$ and notice that u is a non-negative subharmonic function in $\mathbb{R}^m$, bounded above by one and identically zero on $\bar{E}$. If $D(r) = \partial B(0, r) \cap (\bar{E})^c$ and

$$m^2(r, u) = \frac{1}{\sigma_m r^{m-1}} \int_{D(r)} u^2(x) \, d\sigma(x)$$

then we can apply the Carleman–Huber convexity theorem [4, p. 137] to the function u restricted to $(\bar{E})^c$. Therefore there exist two positive constants C and r_0 such that

$$m(r, u) \geqslant C \exp\left[\int_{r_0}^{r/2} \frac{\alpha(t)}{t} \, dt\right],$$

where $\alpha(t)$ is the characteristic constant of $\partial B(0, t) \cap E^c$. Since u is bounded above by one $m(r, u) \leqslant 1$ and thus

$$\int_0^{\infty} \frac{\alpha(t)}{t} \, dt < \infty.$$

5. A certain class of thin sets and proof of Theorem 2

In [3] we have introduced the following domains defined by sequences. Let

$$E = \bigcup_{t \geqslant 0} B((t, 0), r(t)),$$

where $(t, 0) = (t, 0, \ldots, 0) \in \mathbb{R}^m (m > 3)$,

$$r(t) = \begin{cases} a_1 2^2, & 0 \leqslant t < 2^2. \\ a_{n+1} t, & 2^{2^n} < t < 2^{2^{n+1}}, \quad n \geqslant 1, \\ 2a_n 2^{2^n}, & t = 2^{2^n}, \quad n \geqslant 1, \end{cases}$$

and $\{a_n\}$ is a non-increasing sequence of positive real numbers such that $a_1 < 3/10$. This upper bound for a_1 is required in order to prevent two consecutive balls in the sequence $\{B((2^{2n}, 0), r(2^{2n}))\}$ from intersecting. In the case $m = 3$ we define E as above with

$$r(t) = \begin{cases} 2r_1, & t = 0, \\ r_1, & 0 < t < 2, \\ r_n, & 2^{n-1} < t < 2^n, \quad n \geqslant 2, \\ 2r_n, & t = 2^n, \quad n \geqslant 1, \end{cases}$$

where $\{r_n\}_{n \geqslant 1}$ is a non-increasing sequence of positive real numbers with $r_1 < 1/4$. The domain E is thin at infinity provided

$$\sum_{n=0}^{\infty} \frac{1}{\log 2^n r_n^{-1}} < \infty, \quad \text{if } m = 3, \tag{5.1}$$

and

$$\sum_{n=0}^{\infty} 2^n a_{n+1}^{m-3} < \infty, \quad \text{if } m > 3, \tag{5.2}$$

A proof of this result can be found in [3, Ths. 7, 8]. Now we are in a position to prove Theorem 2. Let us first assume that $m \geqslant 4$. We set $\phi_n = \phi(2^{2n})$. Then

$$\int_2^{\infty} \frac{\phi(t)}{t}\, dt = \sum_{n=0}^{\infty} \int_{2^{2n}}^{2^{2n+1}} \frac{\phi(t)}{t}\, dt < \infty.$$

Since $\phi(t)$ is a non-increasing function of t

$$\frac{\log 2}{2} \sum_{n=0}^{\infty} \phi_{n+1} 2^{n+1} \leqslant \sum_{n=0}^{\infty} \int_{2^{2n}}^{2^{2n+1}} \frac{\phi(t)}{t}\, dt \leqslant \log 2 \sum_{n=0}^{\infty} \phi_n 2^n.$$

This says that (1) is equivalent to

$$\sum_{n=0}^{\infty} 2^n \phi_n < \infty \tag{5.3}$$

Using the sequence $\{\phi_n\}$ we define a domain E in $\mathbb{R}^m$ whose associated sequence $\{a_n\}$ is

$$a_n = c\phi_{n-1}^{1/m-3},$$

where c is a positive constant chosen so that $a_1 < 3/10$. As a result of (5.2) and (5.3) E is thin at infinity. On the other hand, since the characteristic constant of a set decreases as the set increases

$$\alpha(t) \geqslant \alpha(C_{\pi-\theta_{n+1}}), \quad 2^{2^n} \leqslant t \leqslant 2^{2^{n+1}} \tag{5.4}$$

where $C_{\pi-\theta_{n+1}}$ is a spherical cap of angle $\pi - \theta_{n+1}$, and $\theta_{n+1} = \sin^{-1} a_{n+1}$. From the asymptotic behaviour of $\alpha(C_\theta)$, as $\theta \to \pi$, in the case $m \geqslant 4$, we obtain

$$\alpha(C_{\pi-\theta_{n+1}}) \geqslant c_1 a_{n+1}^{m-3}, \tag{5.5}$$

for all n sufficiently large, and c_1 a suitable constant. Now combining (5.4) and (5.5) and taking into account that ϕ is a non-increasing function, we get

$$\alpha(t) \geqslant c_1 c^{m-3}\phi_n \geqslant c_1 c^{m-3}\phi(t)$$

where $2^{2n} \leqslant t \leqslant 2^{2n+1}$. This completes the proof in the case $m \geqslant 4$. Let us suppose now that $m = 3$. We set $\phi_n = \phi(2^n)$. Since ϕ is a non-increasing function we can write

$$\log 2 \sum_{n=0}^{\infty} \phi_{n+1} \leqslant \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} \frac{\phi(t)}{t}\, dt \leqslant \log 2 \sum_{n=0}^{\infty} \phi_n,$$

which proves that, for $m = 3$, (1.1) is equivalent to

$$\sum_{n=0}^{\infty} \phi_n < \infty. \tag{5.6}$$

We now proceed as in the case $m \geqslant 4$ defining a domain $E \subseteq \mathbb{R}^3$ associated to a sequence $\{r_n\}_{n \geqslant 1}$, which is defined in the following manner

$$r_{n+1} = c_2 \sum_{k=0}^{\infty} 2^{n+k} \exp\left(\frac{-1}{\phi_{n+k}}\right), \quad n \geqslant 1$$

and (5.7)

$$r_1 = 2r_2,$$

where the constant c_2 is so chosen to make sure that $r_1 < 1/4$. Before carrying on, we must see that the sequence $\{r_n\}_{n \geqslant 1}$ is well defined. In order to do that we ought to prove that the series in (5.7) converges. In fact we are going to prove that

$$\sum_{k=0}^{\infty} 2^k \exp\left(\frac{-1}{\phi_k}\right) < \infty. \tag{5.8}$$

Since $\{\phi_n\}$ is a decreasing sequence then, in view of (5.6), we have that

$$n\phi_n \to 0, \quad \text{as } n \to \infty.$$

Thus

$$\lim_{k \to \infty} \sqrt[k]{2^k \exp\left(\frac{-1}{\phi_k}\right)} = 2 \lim_{k \to \infty} \exp\left(\frac{-1}{k\phi_k}\right) = 0,$$

which proves (5.8). It is obvious that the sequence $\{r_n\}_{n \leqslant 1}$ is decreasing. Therefore we can associate to it a domain $E \subseteq \mathbb{R}^3$. We claim that E is thin at infinity. According to (5.1), all we have to prove is that

$$\sum_{n=0}^{\infty} \left\{\log \frac{2^n}{r_n}\right\}^{-1} < \infty. \tag{5.9}$$

Since $n\phi_n \to 0$, as $n \to \infty$,

$$(n\phi_n)^{-1} \geqslant 2 \log 4, \quad n \geqslant n_0.$$

Then, if $n \geqslant n_0$ and $k \geqslant 0$, we have that

$$(k\phi_{n+k})^{-1} \geqslant (n + k)^{-1} \phi_{n+k}^{-1} \geqslant 2 \log 4.$$

Therefore,

$$2^{-1}\phi_{n+k}^{-1} \geqslant \log 4^k,$$

which allows us to say that

$$\exp\,(2\phi_{n+k})^{-1} \geqslant 4^k.$$

Then,

$$2^k \exp\,(-\phi_{n+k}^{-1}) \leqslant 2^{-k} \exp\,(-2^{-1}\phi_{n+k}^{-1}) \leqslant 2^{-k} \exp\,(-2^{-1}\phi_n^{-1}).$$

From here and using (5.7) we get that

$$\frac{r_{n+1}}{2^n} = c_2 \sum_{k=0}^{\infty} 2^k \exp\left(\frac{-1}{\phi_{n+k}}\right) \leqslant c_2 \exp\left(\frac{-1}{\phi_n}\right) \sum_{k=0}^{\infty} \frac{1}{2^k} = 2c_2 \exp\left(\frac{-1}{2\phi_n}\right).$$

Hence

$$\sum_{n=n_0}^{\infty} \frac{1}{\log\left(\dfrac{2^{n+1}}{r_{n+1}}\right)} \leqslant \sum_{n=n_0}^{\infty} \frac{1}{\dfrac{1}{2\phi_n} - \log c_2} < \infty$$

by (5.6). Therefore (5.9) holds and so E is thin at infinity. In order to complete the proof of Theorem 2 it remains to prove that $\phi(t) = O(\alpha(t))$. Let $t \in [2^n, 2^{n+1}]$. We define

$$\theta_{n+1} = \tan^{-1}\left\{\frac{r_{n+1}}{2^{n+1} - \sqrt{3}r_{n+1}}\right\}$$

and

$$\gamma_{n+1} = 2^{n+1} \sin \theta_{n+1}.$$

Since the characteristic constant of a cap decreases as the cap increases

$$\alpha(t) \geqslant \alpha(C_{\pi-\theta_{n+1}}). \qquad (5.10)$$

On the other hand, the asymptotic expressions for $\alpha(C_\theta)$, as $\theta \to \pi$, allows us to say that there exists a constant c_3 such that

$$\alpha(C_{\pi-\theta_{n+1}}) \geqslant c_3 \frac{1}{2\log\dfrac{2}{\theta_{n+1}}}, \tag{5.11}$$

for all sufficiently large n. Here we are using that $\theta_n \to 0$, as $n \to \infty$, which is a consequence of the fact that $r_n \to 0$. We also have that

$$\theta_{n+1} \geqslant \sin\theta_{n+1} = \frac{\gamma_{n+1}}{2^{n+1}}. \tag{5.12}$$

From (5.10), (5.11) and (5.12) we deduce that

$$\alpha(t) \geqslant c_3 \frac{1}{2\log\dfrac{2^{n+2}}{\gamma_{n+1}}} \tag{5.13}$$

Since $\gamma_{n+1} \geqslant r_{n+1}$ we deduce from (5.13) that

$$\alpha(t) \geqslant c_3 \frac{1}{2\log\dfrac{2^{n+2}}{r_{n+1}}} \tag{5.14}$$

for all sufficiently large n and $t \in [2^n, 2^{n+1}]$. Now if we use the following lower bound for r_{n+1}, which is obtained from (5.7),

$$r_{n+1} \geqslant c_2\, 2^n \exp\left(\frac{-1}{\phi_n}\right), \quad n \geqslant 1,$$

in (5.14) we get

$$\alpha(t) \geqslant \frac{c_3}{2\log\left[2^{n+2}c_2^{-1}2^{-n}\exp\left\{\dfrac{1}{\phi_n}\right\}\right]} = \frac{c_3}{2\log\left[4c_2^{-1}\exp\left\{\dfrac{1}{\phi_n}\right\}\right]}$$

$$= \frac{c_3}{2\log\left[\dfrac{4}{c_2}\right] + \dfrac{1}{\phi_n}} \geqslant c_4\phi_n \geqslant c_4\phi(t),$$

for a suitable constant c_4. In obtaining the last inequality we have taken into account that $t \geqslant 2^n$ and that ϕ is non-increasing. This completes the proof that $\phi(t) = O(\alpha(t))$, as t tends to ∞.

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