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On the compactification problem for Stein surfaces

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Abstract. A complete classification of compact $\mathbb{C}$-analytic surfaces which occurred as analytic compactifications of Stein surfaces is given. For Stein surfaces X which admit non algebraic compactifications, it is shown that (a) the analytic Kodaira dimension of X is $-\infty$ (b) X also carries some affine structure and (c) all algebraic (resp. non algebraic) compactifications of X are birationally (resp. biholomorphically) equivalent provided $X \neq C^* \times C^*$. Throughout, analytic surfaces will mean 2-dimensional connected $\mathbb{C}$-analytic manifolds. Purely 1-dimensional $\mathbb{C}$-analytic spaces will be referred to simply as analytic curves. Furthermore, all compact analytic surfaces are assumed to be *minimal* [6], i.e. free from exceptional curves of the first kind.

1. Structures of compactifiable Stein surfaces

DEFINITION 1. Let X be a non compact analytic surface. A compact analytic surface M is said to be a *compactification* of X if there exists a $\mathbb{C}$-analytic subvariety $\Gamma \subset M$ such that X is biholomorphic to $M \backslash \Gamma$. Furthermore, M is said to be an *algebraic compactification* (resp. *a non algebraic compactification*) if M is an algebraic surface (resp. a non algebraic surface). X is called *compactifiable* if X admits some compactification M.

PROBLEM 1. Let M be a compactification of some Stein surface X (i.e. 2-dimensional $\mathbb{C}$-analytic subvariety in some $\mathbf{C}^N$). What analytic structures M might be equipped with?

REMARK 1. Since X is Stein, one can easily check that Γ is a connected compact analytic curve.

Now a complete answer for Problem 1 is provided by the following:

THEOREM 1. [3] *Let M be a compactification of some Stein surface X and let $\Gamma := M \backslash X$. Then M is either* (*i*) *algebraic.*

(*ii*) $b_1 = 1, b_2 = 0$, *M admits no non constant meromorphic functions and contains at least one compact analytic curve.*

or (*iii*) $b_1 = 1$, $b_2 = n \geqslant 1$ *and M contains at least one compact analytic curve.* (*Here b_1* (*resp. b_2*) *denoted the first* (*resp. the second*) *Betti number for M*)

REMARK 2. (a) For compact analytic surfaces with $b_1 = 1$ and $b_2 \geqslant 1$, it follows that M admits no non constant meromorphic functions.

(b) For compact analytic surfaces M with $b_1 = 1$ and without non constant meromorphic functions, it is known [6] that M contains only finitely many compact connected analytic curves. Since X is Stein, one can check that Γ is the only compact curve in M. Furthermore it follows from [6] that, for the option (ii), Γ is irreducible; on the other hand it follows from [1a.b], for the option (iii), Γ consists of exactly n irreducible components.

DEFINITION 2 [6]. A compact analytic surface M is called a *Hopf* surface if its universal covering is biholomorphic to $\mathbf{C}^2\backslash\{0\}$. A Hopf surface is called *non elliptic* if it does not contain any non constant meromorphic functions.

Interestingly, the alternatives (ii) and (iii) in Theorem 1 indeed occurred:

EXAMPLE 1. Let M_1 be a *non elliptic Hopf surface* containing exactly one non singular elliptic curve Γ[6]. Then one can check that $X_1 := M_1\backslash\Gamma$ is a Stein surface (see Remark 4 below); in fact, X_1 is biholomorphic to $\mathbf{C}^* \times \mathbf{C}^*$.

EXAMPLE 2. Not until 1978, a first example of compact analytic surface M_2 with $b_1 = b_2 = 1$ containing exactly one compact analytic curve Γ was explicitly exhibited [5]; precisely Γ is a rational curve with an ordinary double point and $\Gamma^2 = 0$. Furthermore, $X_2 := M_2\backslash\Gamma$ is Stein; in fact, X_2 is biholomorphic to an affine $\mathbf{C}$-bundle of degree -1 over some non singular elliptic curve Δ (see Proposition 3 below).

On the other hand, X_1(resp. X_2) also does admit an algebraic compactification, $\mathbf{P}_1 \times \mathbf{P}_1$ (resp. a $\mathbf{P}_1$-bundle over Δ). In view of this strange phenomenon, it is natural to raise the following:

QUESTION 1. Let X be a Stein surface.
If X admits a non algebraic compactification, does X always admit some algebraic compactification?

Notice that the converse to question 1 is false; in fact, the only compactifications of $\mathbf{C}^2$(resp. $\mathbf{C} \times \mathbf{C}^*$) are the algebraic ones! [10][11].

2. Existence of compactifications

Our question 1 is motivated by the following:

PROBLEM 2. Let X be a Stein surface.
If X is compactifiable, does X always admit some algebraic structure?

We are now in a position to exhibit an explicit example of a non elliptic Hopf surface alluded to in Example 1:
Let G be the subgroup of GL(2, $\mathbf{C}$) generated by

$$g := \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

with $0 < |\alpha| < 1$. One can check that [6]

(i) G is a properly discontinuous group with no fixed points on $\mathbf{C}^2\backslash\{0\}$.
(ii) $\mathbf{H}_\alpha := \mathbf{C}^2\backslash\{0\}/G$ is a compact analytic surface without any non constant meromorphic functions, with $b_1 = 1$ and $b_2 = 0$.
(iii) Let $\mathbf{x} := (z, w) \in \mathbf{C}^2$ and let $\Pi: \mathbf{C}^2\backslash\{0\} \to \mathbf{H}_\alpha$ be the natural projection map. Then the punctured line $\mathbf{C}^2\backslash\{0\} \cap \{w = 0\}$ is G-stable and hence is mapped by Π onto a non singular elliptic curve $\Delta_\alpha := \mathbf{C}^*/\langle\alpha\rangle$ which is the only compact analytic curve in $\mathbf{H}_\alpha$.

REMARK 4. [3] Let us use the same notations as in Definition 2 and let us consider the following holomorphic map:

$$\Phi: \mathbf{C}^2\backslash\{w = 0\} \to \mathbf{C}^* \times \mathbf{C}^*$$

$$(z, w) \mapsto [\exp(2\pi\alpha i z/w),\ w^{-1}\exp(\alpha z/w.\log\alpha)]$$

Now one can check that:

(i) Φ is a well defined surjective map.
(ii) $\Phi(\mathbf{x}) = \Phi(\mathbf{x}')$ iff $\mathbf{x}' = g^k(\mathbf{x})$ for some k.

Hence Φ induces a biholomorphism $\mathbf{H}_\alpha\backslash\Delta_\alpha \cong \mathbf{C}^* \times \mathbf{C}^*$.

Also it follows from the construction that

PROPOSITION 1. *$\mathbf{H}_\alpha$ is biholomorphic to $\mathbf{H}_\beta$ iff $\alpha = \beta$.*

In general non elliptic Hopf surfaces are characterized by:

PROPOSITION 2 [6]. *Let M be a compact analytic surface with $b_1 = 1$, $b_2 = 0$ and without any non constant meromorphic functions. Let us assume that M contains at least one compact analytic curve. Then M is biholomorphic to some non elliptic Hopf surface.*

Now for some fixed integer $n > 0$, $\alpha \in \mathbf{C}$ with $0 < |\alpha| < 1$ and $\mathbf{t} := (t_0, \ldots, t_n) \in \mathbf{C}^n$, Enoki [1a] constructed compact analytic surfaces, denoted by $\mathbf{S}_{n,\alpha,t}$ which have the following intrinsic properties:

(i) $b_1 = 1$ and $b_2 = n$.
(ii) $\mathbf{S}_{n,\alpha,t}$ contains a compact analytic curve, denoted by $\mathbf{D}_{n,\alpha,t}$ with $(\mathbf{D}_{n,\alpha,t})^2 = 0$. and
(iii) $\mathbf{S}_{n,\alpha,t}\backslash\mathbf{D}_{n,\alpha,t} =: \mathbf{A}_{n,\alpha,t}$ has a structure of an *affine* $\mathbb{C}$-*bundle* over some non singular elliptic curve $\Delta_\alpha := \mathbf{C}^*/\langle\alpha\rangle$.

DEFINITION 3. $\mathbf{S}_{n,\alpha,t}$ are called *parabolic Inoue surfaces*. Meanwhile $\mathbf{A}_{n,\alpha,t}$ will be referred to as their associated affine $\mathbb{C}$-bundles.

REMARK 5. In the special case where $n = 1$ and $\mathbf{t} \neq \mathbf{0}$, $\mathbf{D}_{1,\alpha,t}$ is a rational curve with an ordinary double point and $\mathbf{S}_{1,\alpha,t}$ is exactly the compact analytic surface alluded to in Example 2.

Now parabolic Inoue surfaces are characterized by the following result:

THEOREM 2. [1a] *Let M be a compact analytic surface with $b_1 = 1$ and $b_2 = n$. Now let assume that M contains a divisor $D \neq 0$ with $D^2 = 0$.*

Then M is biholomorphic to $\mathbf{S}_{n,\alpha,t}$ and $D = r\mathbf{D}_{n,\alpha,t}$ for some $0 < |\alpha| < 1$, $t \in \mathbf{C}^n$ and $r \in Z$.

Our main focus on certain of these $\mathbf{S}_{n,\alpha,t}$ stems from the following

PROPOSITION 3. *If $t \neq 0$ then $\mathbf{S}_{n,\alpha,t}$ contains exactly one compact analytic curve.*

Proof. Let Ξ be an irreducible compact analytic curve in $\mathbf{S}_{n,\alpha,t}$ other than any irreducible components of $\mathbf{D}_{n,\alpha,t}$. Then $\Xi \not\subset \mathbf{D}_{n,\alpha,t}$. Since the first Chern class of $\mathbf{D}_{n,\alpha,t}$ is zero, one has, by considering the intersection number, that $\Xi \cap \mathbf{D}_{n,\alpha,t} = \emptyset$. Since $\mathbf{S}_{n,\alpha,t}$ does not have any non constant meromorphic functions, $\Xi^2 = m \leqslant 0$. Furthermore, by construction, $\mathbf{S}_{n,\alpha,t} \backslash \mathbf{D}_{n,\alpha,t} =: \mathbf{A}_{n,\alpha,t}$ admits an algebraic compactification M which is a $\mathbf{P}_1$-bundle over some non singular elliptic curve $\Delta_\alpha := \mathbf{C}^*/\langle\alpha\rangle$. Now let $\Gamma := M \backslash \mathbf{A}_{n,\alpha,t}$, it follows from [1a] that $\Gamma^2 = n$. Since $\mathbf{H}^2(M, \mathbf{Z})$ is generated by Ξ and F, the fibres of M, one has $\Xi = a\Gamma + bF$. By construction $\Xi.\Gamma = 0$ and $\Xi.F \geqslant 0$, this will imply, by some tedious calculation, that

$$m = -n, a = 1 \quad \text{and} \quad b = -n.$$

Therefore Ξ is a section of the $\mathbf{P}_1$-bundle M and hence of $\mathbf{A}_{n,\alpha,t}$. Consequently $\mathbf{t} = 0$. Q.E.D.

We are now in a position to provide a positive answer for Problem 1 as well as for Question 1.

THEOREM 3. *Let X be a given Stein space. Then X is compactifiable iff X admits some algebraic structure.*

Proof. (a) If X is algebraic it follows readily that X is compactifiable (see e.g. [2])

(b) So let us assume that X is compactifiable and let M be its compactification with $\Gamma := M \backslash X$ and let $\mathbf{A} := [a_{ij}]$ be the intersection matrix determined by the irreducible components Γ_i of Γ. It follows from Theorem 1 that M has three possibilities:

(i) If M is of type (i), certainly X is algebraic.

(ii) If M is of type (ii), one has, in view of Proposition 2 that M is a non elliptic Hopf surface. Following remark 2(ii), one can check that $X = M \backslash \Gamma \cong \mathbf{C}^* \times \mathbf{C}^*$ which certainly carries some affine algebraic structure.

(iii) If M is of type (iii), it follows that $\mathbf{A}$ is negative semidefinite since M does not have any non constant meromorphic functions (see remark 2). Furthermore, since X is Stein, in view of Hartogs theorem, $\mathbf{A}$ cannot be negative definite [3][12a]. Therefore $\mathbf{A}$ is singular, consequently one can find a divisor $D \subset M$ supporting on Γ such that $D^2 = 0$. Hence Theorem 2 and Proposition 3 tell us that, for some $\alpha \in \mathbf{C}$, $\mathbf{t} \in \mathbf{C}^n$ with $\mathbf{t} \neq \mathbf{0}$, $X = M \backslash \Gamma \cong \mathbf{S}_{n,\alpha,t} \backslash \mathbf{D}_{n,\alpha,t} =: \mathbf{A}_{n,\alpha,t}$ which is an affine $\mathbb{C}$-bundle over some non singular elliptic curve. Consequently, again X does admit some affine algebraic structure.

Q.E.D.

REMARK 6. (i) A proof of Theorem 3 given in [12b] was incomplete.
(ii) For $t \neq 0$, $\mathbf{A}_{n,\alpha,t}$ is an affine surface.
(iii) In contrast with the case where $t \neq 0$, $\mathbf{S}_{n,\alpha,0}$ contains exactly two disjoint connected compact analytic curves [1b]:
(a) $\mathbf{D}_{n,\alpha,0}$ which consits of n irreducible components, and (b) E, a non singular elliptic curve with $E^2 = -n$. Furthermore, $X := \mathbf{S}_{n,\alpha,0} \backslash \mathbf{D}_{n,\alpha,0}$ is a strongly pseudoconvex surface; in fact X is holomorphic line bundle over E such that the first Chern class of X is equal to $-n$ (see [12a.b.c])
(iv) $\mathbf{S}_{1,\alpha,0}$ was first constructed by Inoue in 1974 (see e.g. [1a][5])

3. Uniqueness of compactifications

At this stage the natural question is to inquire about the uniqueness of compactification for Stein surfaces. However, in view of the results in section 1, our question can be formulated in a precise manner as follows.

PROBLEM 3. Let M_i ($i = 1$ or 2) be two algebraic (resp. non algebraic) compactifications of some Stein surface X.
Do M_i birationally (resp. bimeromorphically) equivalent?

Unfortunately, the answer is No for both cases as we shall see below.

COUNTEREXAMPLE 1. Let us consider the following construction which is due to Serre [9].

Claim. $\mathbf{C}^* \times \mathbf{C}^*$ is biholomorphic (as algebraic group) to a non trivial extension of an elliptic curve $\mathbf{\Delta}$ by the additive group $\mathbf{C}$.

In fact, let $\mathbf{V} := \mathbf{C}^2$ and let $\mathbf{V} \supset \mathbf{U} := \mathbf{Z}\mathbf{e}_1 + \mathbf{Z}\mathbf{e}_2$ be a discrete subgroup which does not contain in any complex line where $\mathbf{e}_1 := (1, 0)$ and $\mathbf{e}_2 := (0, 1)$. Certainly one can find a complex line $D \cong \mathbf{C}$ in $\mathbf{V}$ such that $D \cap \mathbf{U} = 0$. Now let $\mathbf{I}: \mathbf{V} \to \mathbf{C}$ be a map defined by $\mathbf{I}(\mathbf{e}_1) = 1$ and $\mathbf{I}(\mathbf{e}_2) = i$, then one obtains the following exact sequence of groups:

$$0 \to D \cong \mathbf{C} \to \mathbf{V}/\mathbf{U} \xrightarrow{\mathbf{I}} \mathbf{C}/im\mathbf{I} \to 0$$

It is clear that $\mathbf{V}/\mathbf{U}$ is biholomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ and $\mathbf{C}/im\mathbf{I} \cong \mathbf{C}/\mathbf{Z} + \mathbf{Z}i$ which is some elliptic curve $\mathbf{\Delta}$ and our claim is proved.

Now let M_1 be an algebraic compactification of $X_1 := \mathbf{C}^* \times \mathbf{C}^*$ such that $\Gamma_1 = M_1 \backslash X_1$ is an irreducible compact connected curve (notice that such an M_1 exists, see e.g. [10]). Then one can check that [8] the Albanese map A: $M_1 \to \mathbf{\Delta}$ maps Γ_1 biholomorphically onto $\mathbf{\Delta}$ and M_1 has a structure of a $\mathbf{P}_1$-bundle over $\mathbf{\Delta}$.

Hence one obtains a Stein surface X_1 admitting two algebraic compactifications which are not birationally equivalent: one M_1, a non rational surface and the other one $M_2 := \mathbf{P}_1 \times \mathbf{P}_1$ a rational surface.

COUNTEREXAMPLE 2. Let us consider the following two non elliptic Hopf surfaces $M_1 := \mathbf{H}_\alpha$ and $M_2 := \mathbf{H}_\beta$ with $\alpha \neq \beta$. Let $\Gamma_1 := \Gamma_\alpha$ (resp. $\Gamma_2 := \Gamma_\beta$) be the unique compact analytic curve in M_1 (resp. in M_2). Following Remark 4, one has

$$M_1 \backslash \Gamma_1 \cong \mathbf{C}^* \times \mathbf{C}^* \cong M_2 \backslash \Gamma_2$$

Therefore one obtains a Stein surface $X_2 := \mathbf{C}^* \times \mathbf{C}^*$ admitting two non algebraic compactifications, namely M_1 and M_2 which are not, in view of proposition 1, bimeromorphically equivalent.

Our main purpose here is to investigate further these two counterexamples. First of all few basic ingredients are in order.

Let M be an algebraic compactification of some Stein surface X, let $\Gamma := M \backslash X$ which is assumed to be a divisor with only normal crossings and let $\mathbf{K}_M$ be the canonical bundle of M. Now let us consider a basis $\{\phi_0, \ldots, \phi_N\}$ for the vector space $\mathbf{V} := \mathbf{H}^0(M, \mathbf{O}(m\mathbf{K}_M + (m-1)\Gamma)$ which gives rise to a well defined holomorphic map:

$$\begin{aligned} &\mathbf{\Phi}_m \colon M \to \mathbf{P}_N \\ &w \mapsto \mathbf{\Phi}_m(w) := [\phi_0 : \ldots : \phi_N] \end{aligned}$$

where $\mathrm{N} := \dim. \mathbf{V} - 1$

Following [7] let $N(X) := \{m > 0 \mid \dim \mathbf{V} > 0\}$ and let us define

$$\kappa(X) := \begin{cases} \max._m\{\dim \Phi_m(M)\} & \text{if } N(X) \neq \emptyset \\ -\infty & \text{if } N(X) = \emptyset \end{cases}$$

DEFINITION 4. $\kappa(X)$ is called the *analytic Kodaira dimension* of X.

REMARK 7. (i) Notice that when X is compact, i.e. $\Gamma = \emptyset$, then the analytic Kodaira dimension coincides with the standard notion of *Kodaira dimension* for compact analytic surfaces. [4]

(ii) In the previous definition, if one replaces the vector space $\mathbf{V}$ by $\mathbf{W} := \mathbf{H}^0(M, \mathbf{O}(m(\mathbf{K}_M + \Gamma)))$, then one obtains the notion of *logarithmic Kodaira dimension* $\underline{\boldsymbol{\kappa}}(X)$. Since $\mathbf{V}$ is a subspace of $\mathbf{W}$, hence

$$\boldsymbol{\kappa}(X) \leqslant \underline{\boldsymbol{\kappa}}(X) \tag{$*$}$$

Furthermore, if we replace $\mathbf{V}$ by $\mathbf{U} := \mathbf{H}^0(M, \mathbf{O}(m\Gamma))$ where Γ is some divisor in M, then $\boldsymbol{\kappa}(\Gamma) := \boldsymbol{\kappa}(X)$ is the so called litaka's D-dimension. See [4] [7]

(iii) By definition $\boldsymbol{\kappa}(X) = -\infty, 0, 1$ or 2 and furthermore, one has,

$$\boldsymbol{\kappa}(X) \geqslant \boldsymbol{\kappa}(M) \tag{$**$}$$

Now notice that $\boldsymbol{\kappa}(\mathbf{C}^* \times \mathbf{C}^*) = -\infty$, more generally one has

THEOREM 4. *Let X be a compactifiable Stein surface. Then either $\boldsymbol{\kappa}(X) = -\infty$ or 2.*

Proof. Let $\Gamma = \bigcup \Gamma_i$ where Γ_i are the irreducible components, be the compact analytic curve in M such that $X = M \backslash \Gamma$. Since X is Stein, so Γ is connected and the intersection matrix (Γ_i, Γ_j) is not negative definite [3]

(i) $\boldsymbol{\kappa}(M) = -\infty$. In this case, M is a ruled surface or a $\mathbf{P}_2$. Now if M is a $\mathbf{P}_2$, it follows from [7] that $\boldsymbol{\kappa}(X) = -\infty$ or 2. So let us assume that M is a ruled surface. Now let $\pi: M \to B$ be a surjective morphism onto some compact analytic curve B. It is known that $M \cong \mathbf{P}(\mathbf{E})$ where $\mathbf{E}$ is a locally free sheaf of rank 2 on B. Let ε be a divisor on B corresponding to $\Lambda^2\mathbf{E}$, and let $e := -\deg \varepsilon$. Now let us fix a section Ξ of M with $O_M(\Xi) \cong \mathrm{O}_{\mathbf{P}(\mathbf{E})}(1)$ and let us denote the fibre of M by F. It is shown [2] that

$$\Xi^2 = -e \tag{$*$}$$

From now on let us denote by C an irreducible compact curve in M such that $C \neq \Xi$ and F. Since $H^2(M, \mathbf{Z})$ is generated by Ξ and F, one can write

$$C \equiv a\Xi + bF \tag{$\#$}$$

where $a, b \in \mathbf{Z}$ and $\equiv$ stands for numerical equivalent. Now in view of $(*)$ and $(\#)$

$$C^2 = 2ab - a^2 e. \tag{§}$$

Case 1. For $e > 0$, one has (see [2] Prop. V.2.20) $a > 0$, $b \geqslant ae$.

(a) If for at least one i, one has $\Gamma_i = C$, it follows from (§) that $\Gamma_i^2 > 0$; hence one can find integers $\{a_i\}$ such that $D^2 > 0$ where $D := \Sigma_i a_i \Gamma_i$; but X does not contain any compact curve, so D is ample and X is affine and we are done by Proposition 4 below.

(b) Otherwise, one must have $\Gamma_i = \Xi$ for some i and $\Gamma_j = F$ for all $j \neq i$. Then certainly $\boldsymbol{\kappa}(X) = -\infty$.

In fact, since $\mathbf{K}_M \equiv -2\Xi + nF$ (see e.g. [2]) and since $\Gamma \equiv \Xi + pF$, hence $\Theta := \mathbf{K}_M + \Gamma \equiv -\Xi + kF$ for some integers n, p and k. Hence if $\underline{\boldsymbol{\kappa}}(X) \geqslant 0$ one can find an effective divisor D which is linearly equivalent to $m\Theta$ for some $m > 0$; it will follow that $D \cdot F = -m < 0$ which is a contradiction since $F^2 = 0$. Consequently $\underline{\boldsymbol{\kappa}}(X)$ and hence in view of $(*)$ of Remark 7, $\boldsymbol{\kappa}(X) = -\infty$

Case 2. For $e = 0$, one has ([2], *loc. cit.*) $a > 0$ and $b \geqslant 0$

(a) If $b > 0$, then $\Gamma^2 > 0$; consequently X is affine and we are done by proposition 4 below.

(b) If $b = 0$ and Γ is irreducible then $\Gamma^2 = 0$.

Now if $\Gamma \neq \Xi$ then, since $\Gamma \cdot \Xi = 0$, $X := M \backslash \Gamma$ is not Stein. Hence one must have $\Gamma = \Xi$ and the same argument as in Case 1(b) will tell us that $\boldsymbol{\kappa}(X) = -\infty$.

On the other hand if Γ is reducible, then certainly $\Gamma^2 > 0$; consequently X is affine and we are done again in view of Proposition 4.

Case 2. For $e < 0$, one has (see [2] Prop V. 2.21)
either (i) $a = 1$, $b \geqslant 0$.

Now if $b > 0$, it follows from (§) that $\Gamma^2 > 0$ and we are done by Proposition 4.

On the other hand, if $b = 0$, $(\#)$ tells us that Γ is actual a section of M. Then the argument in Case 1(b) tells us that $\boldsymbol{\kappa}(X) = -\infty$
or (ii) $a \geqslant 2$, $b \geqslant \frac{1}{2}ae$.

Claim. If Γ is irreducible then $b = \frac{1}{2}ae$ cannot occur.

In fact, if it does; then let $Y = a'\Sigma + b'F$ with $b' = \frac{1}{2}a'e$. Consequently $\Gamma \cdot Y = 0$, i.e. there exists a compact analytic curve Y in X which is not possible since X is Stein. Hence our claim is proved.

On the other hand if Γ is reducible then $\Gamma^2 > 0$, consequently X is affine and we are done.

Now for $b > \frac{1}{2}ae$, it follows from (§) that Γ is ample and we are done in view of Proposition 4.

(ii) If $\boldsymbol{\kappa}(M) \geqslant 0$, then it follows from [12c] (Corollary 3) that $\boldsymbol{\kappa}(X) = 2$.

Q.E.D.

On the other hand, in contrast with the Counterexample 2, one has

THEOREM 5. [1b] *Let* $\mathbf{S}_{n,\alpha,t}$*(resp.* $\mathbf{A}_{n,\alpha,t}$*) and* $\mathbf{S}_{m,\beta,s}$ *(resp.* $\mathbf{A}_{m,\beta,s}$*) be two given parabolic Inoue surfaces* (resp. *their associated affine* $\mathbb{C}$*-bundles*). *Then* $\mathbf{S}_{n,\alpha,t}$ *is biholomorphic to* $\mathbf{S}_{m,\beta,s}$ *iff* $\mathbf{A}_{n,\alpha,t}$ *and* $\mathbf{A}_{m,\beta,s}$ *are biholomorphically equivalent.*

In retrospect, in view of these results, it follows that the Counterexamples 1 and 2 are in some extent unique; in fact, one has

THEOREM 6. *Problem* 3 *admits an affirmative answer provided M_i are not ruled surfaces* (resp. *not non elliptic Hopf surfaces*).

Proof. (i) Let M_i ($i = 1$ or 2) be two algebraic compactifications of some Stein surface X. Following Theorem 4, $\kappa(X) = -\infty$ or 2. However, if $\kappa(X) = -\infty$, then in view of $(**)$ in Remark 7, $\kappa(M_i) = -\infty$; therefore, M_i are ruled surfaces [6] which are excluded by our current hypothesis.

Now if $\kappa(X) = 2$, it follows from [7] that M_i are biholomorphic.

(ii) Let M_i be two non algebraic compactifications of some Stein surface X and let Γ_i be the compact analytic curves in M_i such that $X = M_i \backslash \Gamma_i$. In view of Theorem 1 and our hypothesis, M_i are parabolic Inoue surfaces. But $M_i \backslash \Gamma_i$ are biholomorphic as affine $\mathbb{C}$-bundle. Hence it follows, from Theorem 5, that M_i are biholomorphic. Q.E.D.

4. Open questions

In order to complete Theorem 4 it remains for us to establish the following:

PROPOSITION 4. *Let X be an affine surface. Then $\kappa(X) = -\infty$ or* 2.

Proof. Let us use the notations of definition 4. If $N(X) = \emptyset$ for any m, then certainly $\kappa(X) = -\infty$. Otherwise, let us assume that there exists an m_0 such that, for $m \geqslant m_0$

$$N(X) \neq \emptyset \tag{$*$}$$

Since X is affine, one can assume that Γ is ample. Since

$$2m\mathbf{K}_M + (2m-1)\Gamma = D + \Gamma$$

where $D := 2m\mathbf{K}_M + 2(m-1)\Gamma$. Hence in view of the hypothesis $(*)$ with $m \geqslant m_0$, D is linearly equivalent to some effective divisor, i.e. $D + \Gamma \geqslant \Gamma$. Since Γ is ample, therefore $\kappa(2m\mathbf{K}_M + (2m-1)\Gamma) \geqslant \kappa(\Gamma) = 2$. Consequently $\kappa(X) = 2$. Q.E.D.

REMARKS. Notice that Proposition 4 as well as Theorem 4 are false if one replaces the analytic Kodaira dimension by the logarithmic one. Here we would like to exhibit few concrete examples in order to illustrate the main difference between these two concepts.

(a) Let us use the same notations as in counterexample 1 and let $X := \mathbf{C}^* \times \mathbf{C}^*$. Now if M_1 is its compactification, then it follows from the proof of Case 1(b) in Theorem 4 that $\underline{\kappa}(X) = -\infty$. On the other hand, if M_2 was its compactification, then $\underline{\kappa}(X) = 0$ (see (b)(ii) below). Meanwhile, $\kappa(X) = -\infty$ in either cases. This shows that logarithmic Kodaira dimension is not even biholomorphic invariant; meanwhile the analytic Kodaira dimension is indeed bimeromorphically invariant [7] which is quite suitable within the framework of analytic compactifications.

(b) Let $\Gamma \subset \mathbf{P}_2$ be a compact analytic curve and let $X := \mathbf{P}_2 \backslash \Gamma$ be the Stein (and even affine) surface. Let us consider the following cases (see [4] and [7])

(i) if Γ is a union of three lines which meet exactly at one point, then $\underline{\kappa}(X) = \kappa(X) = -\infty$.
(ii) if Γ is a union of three lines in a general position then $\underline{\kappa}(X) = 0$ and $\kappa(X) = -\infty$.
(iii) if Γ is a union of four lines with the configuration as shown by Fig. 1 then $\underline{\kappa}(X) = 1$ and $\kappa(X) = -\infty$.

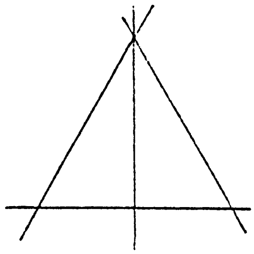

Fig. 1

(iv) if Γ is a union of 4 lines with the configuration as shown by Fig. 2 then $\kappa(X) = \underline{\kappa}(X) = 2$.

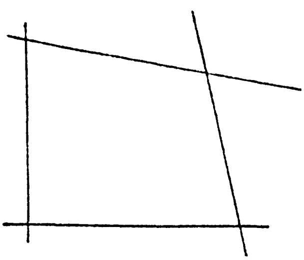

Fig. 2

Also notice that if a Stein surface admits a non elliptic Hopf surface (resp. a parabolic Inoue surface) as its compactification, it follows from its construction (see [6] resp. [1.b]) that X also admits an elliptic rules surface M with invariant $e \geqslant 0$ as its compactification such that $\Gamma = M \backslash X$ is a section of M with $\Gamma^2 = e$. Therefore, from the argument of the Case 1.(b) in the proof of Theorem 4, we obtain, without using Sakai's result [7], the following:

COROLLARY 4. *Let X be a Stein surface which admits a non algebraic compactification. Then $\kappa(X) = -\infty$*

Also with similar arguments, Theorem 4 remains valid if one replaces Stein surface by strongly pseudoconvex surface; this answers affirmatively a question raised in [12c].

As far as the existence of compactifications is concerned, it follows from Theorem 4 and Proposition 4, that Problem 2 can be sharpened as follows:

PROBLEM 2′. Do compactifiable Stein surfaces always admit some affine structure?

Incidentally, in view of the proof of Theorem 3, one has:

PROPOSITION 5. *Problem 2′ admits a positive answer for Stein surfaces which admit non algebraic compactifications.*

On the other hand, as far as the uniqueness of compactifications is concerned, Theorem 6, in effect tells us that $\mathbf{C}^* \times \mathbf{C}^*$, up to biholomorphism, is the only Stein surface admitting non algebraic compactifications which are not bimeromorphically equivalent.

However, regarding the situation for algebraic compactifications, Theorem 6 leaves a bit to be desired, namely

QUESTION 3. Is $\mathbf{C}^* \times \mathbf{C}^*$ the only (up to biholomorphism) Stein surface admitting non birationally equivalent algebraic compactifications?

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Note added in proof: In view of recent developments, we do have now counterexamples for Problem 2 and an affirmative answer to Question 3. Details will appear elsewhere. Also, we received a preprint "Algebraic compactifications of Stein surfaces" by G.K. Sankaran, in which interesting results regarding algebraic compactifications of Stein surfaces were obtained.