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On a result of G. Baumslag

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1. Introduction

Suppose that A and B are residually finite groups and that $A \otimes Z \cong B \otimes Z$, where Z represents an infinite cyclic group and the product is direct. Then it does not follow in general that A and B are isomorphic [3, 4, 5]. However Baumslag [1] has pointed out that A and B must have the same finite images and he has used this result to give simple examples of groups A and B which are not isomorphic but do have the same finite images. These are groups which are extensions of a finite cyclic by an infinite cyclic group. Two such groups may be represented as

$$\begin{aligned} G_{m,s} &= \langle a, b\colon a^m = 1,\ b^{-1}ab = a^s, (m, s) = 1\rangle \\ H_{m,t} &= \langle c, d\colon c^m = 1,\ d^{-1}cd = c^t,\ (m, t) = 1\rangle. \end{aligned} \tag{1}$$

We find necessary and sufficient conditions for the isomorphism of the direct products

$$G_{m,s} \otimes Z \cong H_{m,t} \otimes Z. \tag{2}$$

Using these conditions and a simple property of p-Sylow subgroups we get the converse: if $G_{m,s}$ and $H_{m,t}$ have the same finite images then (2) holds. An example involving just infinite groups shows that this result is not true in general.

Moreover, it is true that $A \otimes Z \cong B \otimes Z$ implies that the automorphism groups Aut (A) and Aut (B) are isomorphic if A and B are the groups in (2).

THEOREM 1. *Let* $G_{m,s}$ and $H_{m,t}$ *be given by* (1). *Then* (2) *holds if and only the system of congruences*

$$\begin{pmatrix} s^x \equiv t \\ t^y \equiv s \end{pmatrix} (\operatorname{mod} m) \tag{3}$$

has a solution $x = u$, $y = v$.

Remark that if $o(t)$ denote the order of t then the greatest common divisors $(o(t), u) = (o(t), v) = 1$. Otherwise, since $t^{uv} \equiv t$, $t^{uv-1} \equiv 1$ and so $o(t) \mid uv - 1$; thus if for any prime p, $p \mid (o(t), u)$ or $p \mid (o(t), v)$ then $p = 1$. Similarly $(o(s), u) = (o(s), v) = 1$. Of course (3) at once implies that $o(s) = o(t)$.

Proof. (a) Assume that (3) holds. Generators for $G_{m,s} \otimes Z$ are $(a, 0)$, $(b, 0)$, and $(1, 1)$ with group multiplication on the first components and addition of integers on the second components. For example $(b, 0)^j(a, 0)^i(1, 1)^k = (b^j a^i, k)$ which is a generic element of $G_{m,s} \otimes Z$. We wish to set up a map $\sigma\colon G_{m,s} \otimes Z \to H_{m,t} \otimes Z$ which will be an isomorphism. For the element $(a, 0)$ of finite order we must have an image of finite order: $(c^r, 0)$, where gcd $(m, r) = 1$. Suppose

$$(a, 0) \to (c^r, 0), (\mathrm{b}, 0) \to (\mathrm{d}^h, \mathrm{f}), (1, 1) \to (\mathrm{d}^k, \mathrm{g}). \tag{4}$$

Since the product is direct, the image (d^k, g) must commute with the other images. Thus $(d^k, g)^{-1}(c^r, 0)\,(d^k, g) = (c^r, 0)$. Performing the calculations we get $(c^{rt^k}, 0) = (c^r, 0)$. This yields $r^{t^k} \equiv r \pmod m$ and so $t^k \equiv 1 \pmod m$. Hence we may put $k = o(t)$.

We want to ensure that σ given by (4) is:

(i) *Injective.* Suppose $(b, 0)^y(\mathrm{a}, 0)^x\ (1, 1)^z \to (1, 0)$. Then $(d^h, f)^y(c^r, 0)^x(d^k, g)^z \equiv (1, 0)$. Carrying out the calculations and using the fact that $t^k \equiv 1 \pmod m$ we get $(d^{hy+kz}c^{rx}, fy + gz) = (1, 0))$. This gives $x \equiv 0 \pmod m$ and the simultaneous integral system $hy + kz = 0, fy + gz = 0$. To have injectivity this system must have only the trivial solution for y and z. To ensure this we need

$$hg - kf \neq 0. \tag{5}$$

(ii) *Surjective.* It suffices to show the existence of p', q' such that $(d^h, f)^{p'}(d^k, g)^{q'} = (d, 0) = (d^{hp'+kq'}, fp' + gq')$. This yields

$$\begin{aligned} hp' + kq' &= 1 \\ fp' + gq' &= 0 \end{aligned} \tag{6}$$

where without loss of generality we can have $\gcd(f, g) = 1$. Choose $h = u$, the solution for x in (3). By the remark preceding the proof, $gcd(u, k) = 1$ so that there are integers p', q' to make $up' + kq' = 1$. Thus the first equation of (6) is satisfied. Taking $f = -q'$, $g = p'$ will now satisfy the second. Since $ug - kf = up' + kq' = 1 \neq 0$ condition (5) also holds. Thus with these choices for h and k (4) establishes the isomorphism (2).

(b) Now assume that (2) holds under the isomorphism $(b, 0) \to (d^y c^x, f)$, $(a, 0) \to (c^r, 0)$. Since $(b, 0)^{-1}(a, 0)\ (b, 0) = (a^s, 0)$ therefore $(c^{-x}d^{-y}, -f)\ (c^r, 0)\ (d^y c^x, f) = (c^{rs}, 0) = (c^{-x}d^{-y}c^r d^y c^x, 0) = (c^{rt^y}, 0)$. It follows that $rt^y \equiv rs \pmod m$

and so $t^y \equiv s \pmod{m}$. By symmetry there exists x such that $s^x \equiv t \pmod{m}$, hence (3) follows and the proof is complete.

THEOREM 2. *Let* $G_{m,s}$ and $H_{m,t}$ *be given by* (1). *If these groups have the same finite images then* (2) *follows.*

Proof. Choose e such that $s^e \equiv 1 \pmod{m}$. Then

$$G_{m,s} \to G = \langle a, b\colon a^m = b^e = 1, b^{-1}ab = a^s \rangle, \quad \text{and } o(G) = me.$$

By assumption $H_{m,t}$ must have a finite factor H and there is an isomorphism $\sigma\colon G \to H$. Let p^k be the highest power of a prime factor p of m, $m = p^k h$, $(h, p) = 1$. Now a^h is an element of order p^k in G. Let S be a p-Sylow subgroup in G which contains a^h. S must have an isomorphic image T in H which is a p-Sylow subgroup of H. Then a^h corresponds under σ to an element w of order p^k in T. Since c^h of order p^k is contained in a p-Sylow subgroup of T, and since all p-Sylow subgroups are conjugate, there is an inner automorphism $\tau_1\colon w \to c^{fh}$, $(f, m) = 1$.

Let $\tau_2\colon c^{fh} \to c^h$, $d \to d$. Suppose $\sigma\colon b \to d^y c^x$. Define $\sigma_P = \sigma\tau_1\tau_2$ (acting on the right). Then (b) $\sigma_P = d^y c^z$. Since any automorphism takes c into a power and since an inner automorphism preserves the first factor d^y, this is the same y as in the image of b under σ, and so remains the same for all p. We now have $\sigma_P\colon a^h \to c^h$, $b \to d^y c^z$. Since $b^{-1}ab = a^s$, $b^{-1}a^h b = a^{hs}$. Under σ_P this gives $(d^y c^z)^{-1} c^h (d^y c^z) = c^{hs} = c^{ht^y}$. Then $t^y \equiv s \pmod{m/h = p^k}$. Since this is true with the same y for all maximal prime power factors of m, we have $t^y \equiv s \pmod{m}$. By symmetry there is a solution $s^x \equiv t \pmod{m}$. By Theorem 1 the proof is complete.

REMARK 1. In the proof of Theorem 2 the full hypothesis was not used. The isomorphism of only a single pair of finite images, G and H in the proof, will ensure that $G_{n,s}$ and $H_{m,t}$ have the same finite images.

REMARK 2. $G_{m,s} \otimes Z \otimes \cdots \otimes Z = H_{m,t} \otimes Z \otimes \cdots \otimes Z$ also imply that conditions (3) hold so that the consequences of this statement are entirely equivalent to those of (2).

3. More generally let A and B be arbitrary groups and let C be an infinite cyclic group. Suppose $A \otimes C \cong B \otimes C$.

If $U = A \otimes C$ then the right hand side of the isomorphism can be viewed as another decomposition $U = B' \otimes C'$ where $B \cong B'$ and $C \cong C'$. In this way we get the equality

$$A \otimes C = B' \otimes C'. \tag{7}$$

Denote by π_1, π_2, respectively π'_1, π'_2 the projections corresponding to these decompositions. Let $\pi_1(B') = A_1$, $\pi'_2(C) = C'_2$. Now in each case the kernel of these restrictive maps is $B' \cap C$. Thus

$$\begin{aligned} &\frac{B'}{B'} \cap C \cong A' < A \\ &\frac{C}{B'} \cap C = C'_2 < C'. \end{aligned} \tag{8}$$

If $B' \cap C \neq 1$ then $C/B' \cap C$ is a finite cyclic group and by the second equation $C'_2 = 1$ so that $B' \cap C = C$, $C < B'$. Since C is a direct factor: $B' = B'' \otimes C$. Then $A \otimes C = B'' \otimes C \otimes C'$ and this modulo C gives $A \cong B'' \otimes C' \cong B'' \otimes C = B' \cong B$. Thus $\mathrm{Aut}(A) = \mathrm{Aut}(B)$. If $B' \cap C = 1$ then the first equation of (8) shows that $B' \cong A'$. Then $\mathrm{Aut}(B) = \mathrm{Aut}(B') = \mathrm{Aut}(A')$. Hence in order to have $\mathrm{Aut}(A) = \mathrm{Aut}(B)$ we must have $\mathrm{Aut}(A) = \mathrm{Aut}(A')$, where A' is a proper normal subgroup of A. For the groups in (2) this is the case.

THEOREM 3. *Let* $G_{m,s}$, $H_{m,t}$ *be given by* (1). *Then the relation* (2) *implies that* $\mathrm{Aut}(G_{m,s}) = \mathrm{Aut}(H_{m,t})$.

Proof. By Theorem 1 there exists $x = u$, $y = v$ satisfying (3). Then the map $\sigma\colon H_{m,t} \to G_{m,s}$ defined by $c \to a$, $d \to b^u$ satisfies the relation $d^{-1}cd = c^t$ and is an isomorphism, so that we have

$$H_{m,t} \cong A' = \langle a, b^u \rangle < G_{m,s} \tag{9}$$

An arbitrary automorphism $\tau \in \mathrm{Aut}(G_{m,s})$ is given by

$$a \to a^r,\ (m, r) = 1;\ b \to a^x b^e, 0 < x < m, e = (+/-)1. \tag{10}$$

If τ is restricted to A' we get an automorphism of A' which we denote by τ'. The map $\tau \to \tau'$ is injective: suppose $\tau \to \tau' = 1$. Then it follows that $a^r = a$, $b^u = (a^x b^e)^u = (b^u)a^{xL}$. This implies that $r = 1$, $e = 1$ and $xL \equiv 0 \pmod m$. Here $L = (1 + s + s^2 + \cdots + s^{u-1}) = (s^u - 1)/(s - 1) \equiv (t - 1)/(s - 1) \pmod m$. Now the relations (3) imply that $(L, m) = 1$, so that $x \equiv 0 \pmod m$ and so $\tau = 1$. We have now $\mathrm{Aut}(G_{m,s}) < \mathrm{Aut}(A') = \mathrm{Aut}(H_{m,t})$. Then the result follows from symmetry.

Recall that a group is called just infinite if it is infinite but all its proper quotient groups are finite. Let A in (7) be just infinite. Since $\pi_2(B') = C_2 < C$, where C_2 is an infinite cyclic group or 1, the definition yields $C_2 = 1$ so that $B' < A$ and B' is just infinite. Symmetrically $A < B'$. Thus $A = B' \cong B$. Now there exists non-isomorphic just infinite groups with the same finite images [2]. For two such groups A and B we cannot have $A \otimes C \cong B \otimes C$.

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