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Section 1. Introduction

This short note deals with pseudoconvex domains (and more generally locally hyperconvex domains) on complex spaces with singularities. For strongly pseudoconvex domains the results are well known [13]: any strongly pseudoconvex domain $D \subset X$ is a proper modification of a Stein space at a finite set, in particular it is holomorphically convex.

On the other hand an example of Grauert [12] shows that the pseudoconvexity of $D$ is not sufficient to guarantee its holomorphic convexity. For complex manifolds the most general positive result in this direction seems to be the following theorem of Elencwajg [4]: Let $X$ be a complex manifold and $D \subset X$ a locally Stein open subset. Assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\bar{D}$. Then $D$ is Stein.

The main purpose of this note is to generalize Elencwajg’s theorem for complex spaces with singularities. We are able to prove only the following partial result:

**THEOREM 1:** Let $X$ be a complex space, $D \subset X$ a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\bar{D}$. Then $D$ is Stein.

As a direct consequence we obtain:

**COROLLARY 1:** Let $X$ be a K-complete space and $D \subset X$ a relatively compact open subset which is locally hyperconvex. Then $D$ is Stein. In particular any pseudoconvex domain $D \subset X$ is Stein.

When $X$ is a Stein space the above corollary can be strengthened as follows:

**THEOREM 2.** Let $X$ be a Stein space and $D \subset X$ a locally Stein open subset. Assume that $D$ is locally hyperconvex at $\partial D \cap \text{Sing}(X)$. Then $D$ is a Stein space.

**REMARK 1.**

(a) Corollary 1 for pseudoconvex domains was proved in ([1], Theorem 2) under the additional assumption that $D$ has a globally defined boundary.
(b) A weaker result than Theorem 2 is proved in ([1], Corollary 2). Namely it is assumed that $D$ is strongly pseudoconvex at $\partial D \cap \text{Sing}(X)$.

Section 2. Preliminaries

All complex space are supposed reduced and countable at infinity.

A Stein space is called hyperconvex [18] if there exists a continuous plurisubharmonic exhaustion function $\varphi: X \to (-\infty, 0)$ (the empty set is considered hyperconvex).

Examples of hyperconvex spaces.

Let $D \subset \mathbb{C}^n$ be a Stein open set. Each of the following conditions are sufficient for the hyperconvexity of $D$:

(a) $D$ is bounded and convex [18]
(b) $D$ is bounded and has $C^2$ boundary [2] or $C^1$ boundary [11]
(c) $D$ is a bounded Reinhardt domain containing the origin [5]
(d) $D$ is a tube whose base $\text{Re}(D) \subset \mathbb{R}^n$ is bounded and convex [5].

Other examples can be found in ([5], [8]).

To get examples of hyperconvex spaces in the singular case one may take subspaces or finite morphisms into the nonsingular ones given above. In particular any relatively compact analytic polyhedron in a Stein space is hyperconvex and any Stein space can be exhausted with hyperconvex open sets.

DEFINITION 1. Let $X$ be a complex space, $D \subset X$ an open subset and $A \subset \partial D$ any subset. We say that $D$ is locally hyperconvex at $A$ if for any $x_0 \in A$ there exists an open neighbourhood $U$ of $x_0$ such that $U \cap D$ is hyperconvex. When $A = \partial D$ $D$ is called locally hyperconvex.

DEFINITION 2 ([1], [12], [13]). Let $X$ be a complex space and $D \subset X$ a relatively compact open subset. $D$ is called pseudoconvex if for any $x_0 \in \partial D$ there exists an open neighbourhood $U$ of $x_0$ and a continuous plurisubharmonic function $\varphi: U \to \mathbb{R}$ such that $U \cap D = \{x \in U \mid \varphi(x) < 0\}$.

It is clear from the above definitions that any pseudoconvex domain is locally hyperconvex.

The proof of Theorem 1 relies on a patching technique which allows us to produce a continuous strongly plurisubharmonic exhaustion function $\varphi: D \to \mathbb{R}$. To obtain the Steiness of $D$ we invoke the following result of Narasimham [13]:

THEOREM 3. Let $D$ be a complex space and assume that there exists a continuous strongly plurisubharmonic exhaustion function $\varphi: D \to \mathbb{R}$. Then $D$ is a Stein space.
For the proof of Theorem 2 we shall need the following two results:

**THEOREM 4** ([1], Theorem 4). Let $X$ be a Stein space and $D \subset X$ a locally Stein open subset. Assume that there is an open neighbourhood $U$ of $\partial D \cap \text{Sing}(X)$ such that $D \cap U$ is a Stein space. Then $D$ itself is a Stein space.

**THEOREM 5** ([14], Theorem 2). Let $X$ be a Stein space, $A \subset X$ a closed analytic subset and $V$ an open neighbourhood of $A$. Then there exists a continuous plurisubharmonic function $p : X \to \mathbb{R}$ such that $A \subset \{p < 0\} \subset V$.

Let us recall also the following:

**DEFINITION 3.** A complex space $X$ is called $K$-complete if for any $x_0 \in X$ there is a holomorphic map $f : X \to \mathbb{C}^p$, $p = p(x_0)$ such that $x_0$ is an isolated point of $f^{-1}(f(x_0))$.

It is known [9] that a complex space $X$ of pure dimension $n$ is $K$-complete iff $X$ can be realised as a remified domain over $\mathbb{C}^n$, but we shall not need this result.

In ([1], Lemma 5) it was proved:

**THEOREM 6.** Every relatively compact open subset of a $K$-complete space carries a $C^\infty$ strongly plurisubharmonic function.

### Section 3. Proof of the main results

In the proof of Theorem 1 the existence of some special convex increasing functions on $(-\infty, 0)$ will play an important role. So we state:

**LEMMA 1.** Let $(a_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of negative real numbers such that $a_n \to 0$. Then there exists a function $\tau : (-\infty, 0) \to \mathbb{R}$ with the following properties:

1. $\tau$ is continuous, increasing and convex
2. $\tau \geq 0$
3. $\lim_{x \to 0} \tau(x) = \infty$
4. $\tau(a_{n+1}) - \tau(a_n) < 1$ for every $n \in \mathbb{N}$

**Proof.** We define $\tau$ to be linear on each interval $[a_n, a_{n+1}]$ and to vanish identically near $-\infty$. The precise definition is as follows:

$$
\tau(x) = \begin{cases} 
    n - \frac{a_2}{a_1} \cdots - \frac{a_n}{a_{n-1}} - \frac{x}{a_n} & \text{if } a_n \leq x \leq a_{n+1} \\
    0 & \text{if } x \leq a_1
\end{cases}
$$

Properties (1), (2) and (4) follow easily from the definition of $\tau$ so it remains to
verify (3). Since $\tau$ is increasing it suffices to show that $\tau(a_n) \to \infty$. Now

$$
\tau(a_{n+p}) - \tau(a_n) = \frac{a_n - a_{n+1}}{a_n} + \ldots + \frac{a_{n+p-1} - a_{n+p}}{a_{n+p-1}} \geq \frac{a_n - a_{n+p}}{a_n},
$$

hence for a given $n$ $\tau(a_{n+p}) - \tau(a_n) \geq \frac{1}{2}$ if $p$ is sufficiently large (depending on $n$). It follows that $\tau(a_n) \to \infty$ which proves the lemma.

**Lemma 2.** Let $f_1, \ldots, f_n : (-\infty, 0) \to (-\infty, 0)$ be increasing functions such that for any $i \in \{1, \ldots, n\}$ $\lim_{x \to 0} f_i(x) = 0$. Then there exists a continuous increasing convex function $\tau : (-\infty, 0) \to \mathbb{R}$ such that:

(a) $\lim_{x \to 0} \tau(x) = \infty$

(b) $\tau \circ f_i - \tau \circ f_j$ is bounded for any $i, j \in \{1, \ldots, n\}$

**Proof.** From the assumption “$\lim_{x \to 0} f_i(x) = 0$ for any $i \in \{1, \ldots, n\}$” it follows that there exists an increasing sequence $\{\alpha_v\}_{v \in \mathbb{N}}$ of negative real numbers, $\alpha_v \to 0$ such that:

$$
\max\{f_1(\alpha_v), \ldots, f_n(\alpha_v)\} < \min\{f_1(\alpha_{v+1}), \ldots, f_n(\alpha_{v+1})\} \quad \text{for any} \ v \in \mathbb{N}. \quad (\ast)
$$

If we set $a_v = \min\{f_1(\alpha_v), \ldots, f_n(\alpha_v)\}$ for odd $v$ and $a_v = \max\{f_1(\alpha_v), \ldots, f_n(\alpha_v)\}$ for even $v$ then $a_1 < \ldots < a_v < a_{v+1} < \ldots < 0$ and $a_v \to 0$.

By Lemma 1 there is a continuous convex increasing function

$$
\tau : (-\infty, 0) \to \mathbb{R}, \tau \geq 0, \lim_{x \to 0} \tau(x) = \infty
$$

and

$$
\tau(a_{v+1}) - \tau(a_v) < 1 \quad \text{for any} \ v \in \mathbb{N}.
$$

To prove Lemma 2 it remains to verify that $\tau \circ f_i - \tau \circ f_j$ is bounded. Since $\tau$ is bounded below ($\tau \geq 0$) it suffices to check that $\tau(f_i(x)) - \tau(f_j(x))$ is bounded for $x < 0$ sufficiently close to 0. If $\alpha_{2v} \leq x \leq \alpha_{2v+2}$ then

$$
a_{2v-1} \leq \min\{f_i(x), f_j(x)\} \leq \max\{f_i(x), f_j(x)\} \leq a_{2v+2},
$$

hence $\tau(f_i(x)) - \tau(f_j(x)) < 3$, which proves Lemma 2.

**Lemma 3.** Let $Y$ be a complex space which carries a continuous strongly plurisubharmonic function and let $D \subset Y$ be a relatively compact open subset. Assume that there exists open subsets of $YA_i \subset B_i \subset C_i \in \{1, \ldots, k\}, D \subset \bigcup_{i=1}^k A_i$ and continuous plurisubharmonic exhaustion functions $\varphi_i : C_i \cap D \to \mathbb{R}$ such that
\[ \varphi |_{B_i \cap B_j \cap D} - \varphi |_{B_i \cap B_j \cap D} \text{ is bounded for any } i, j \in \{1, \ldots, k\}. \text{ Then } D \text{ is a Stein space.} \\

\textbf{Proof.} The proof is obtained by a slight modification of the arguments given by M. Peternell in ([16], Lemma 10). For the sake of completeness we shall indicate the modifications to be done.

Take \( p'_i \in C^0_b(Y) \) with \( p'_i \geq 0 \), \( \text{supp } p'_i \subset B_i \) and \( p'_i|_{A_i} = 1 \). We define the functions \( p_i \in C^0_b(Y) \) in the following way: for each \( i \) the functions \( \varphi_j - \varphi_i \) \( j \in \{1, \ldots, k\} \) are bounded on \( \partial B_j \cap A_i \cap D \) so we can choose a sufficiently large constant \( \lambda_i > 0 \) with \( \lambda_i p'_i > \varphi_j - \varphi_i \) on \( \partial B_j \cap A_i \cap D \). We set \( p_i = \lambda_i p'_i \). Since \( p_j = 0 \) on \( \partial B_j \) we have:

\[ p_i + \varphi_i > p_j + \varphi_j \quad \text{on } \partial B_j \cap A_i \cap D \tag{*} \]

Let now \( \varphi \) be a continuous strongly plurisubharmonic function on \( Y \) and let \( A > 0 \) be a sufficiently large constant such that \( A\varphi \) is strongly plurisubharmonic for any \( i \in \{1, \ldots, k\} \). We set \( I = \{1, \ldots, k\} \) and for \( x \in D \) we define \( I(x) \subset I \) by \( I(x) = \{i \in I \mid x \in B_i\} \). If \( x \in D \) we set \( u(x) = \max_{i \in I(x)} \{ p_i(x) + \varphi_i(x) \} \). We show that \( \psi = A\varphi + u \) is a continuous strongly plurisubharmonic exhaustion function on \( D \). It is clear that \( \psi \) is an exhaustion function because \( \varphi_i \) are exhaustion functions on \( C_i \cap D \), hence it remains to verify that \( \psi \) is a continuous strongly plurisubharmonic function on \( D \). Let \( x_0 \in D \) and set \( I(x_0) = \{i \in I \mid x_0 \in \partial B_i\} \). Choose a neighbourhood \( D_{x_0} \subset D \) of \( x_0 \) such that \( D_{x_0} \cap B_i = \emptyset \) if \( i \notin I(x_0) \) and let \( i_0 \in I(x_0) \) with \( x_0 \in A_{i_0} \). For each \( j \in I(x_0) \) it follows from (*) that \( p_{i_0} + \varphi_{i_0} > p_j + \varphi_j \) on \( D_{x_0} \) if \( D_{x_0} \subset A_{i_0} \) is chosen small enough. We get \( u|_{D_{x_0}} = \max_{i \in I(x_0)} \{ p_i(x) + \varphi_i \} \) hence \( \psi|_{D_{x_0}} = \max_{i \in I(x_0)} \{ A\varphi + p_i \} \) which shows that \( \psi \) is a continuous strongly plurisubharmonic function. By Theorem 3 \( D \) is Stein and the proof of Lemma 3 is complete.

\textbf{THEOREM 1.} Let \( X \) be a complex space and \( D \subset X \) a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of \( \overline{D} \). Then \( D \) is Stein.

\textbf{Proof.} Let \( Y \) be a neighbourhood of \( \overline{D} \) and \( \varphi \) a continuous strongly plurisubharmonic function on \( Y \). Choose open subsets \( A_i \subset B_i \subset C_i \subset Y \) \( i \in \{1, \ldots, k\} \) such that:

1. \( D \subset \bigcup_{i=1}^k A_i \)
2. for any \( i \in \{1, \ldots, k\} \) there exists a continuous plurisubharmonic exhaustion function \( v_{ij} : C_i \cap D \to (-\infty, 0) \).

For every \( i, j \in \{1, \ldots, k\} \) such that \( B_i \cap B_j \cap D \neq \emptyset \) we define the function \( E_{ij} : (-\infty, 0) \to (-\infty, 0) \) by \( E_{ij}(x) = \inf \{ v_{ij}(z) \mid z \in B_i \cap B_j \cap D \} \). \( E_{ij} \) are increasing functions and \( \lim_{x \to 0} E_{ij}(x) = 0 \) because \( v_i \) are exhaustion functions. Let \( h : (-\infty, 0) \to (-\infty, 0) \) be the identity map. Now we use Lemma 2 for the finite set of functions \( \{E_{ij}, h\} \) and we get a continuous increasing convex function
\( \tau : (-\infty, 0) \to \mathbb{R} \) such that:

1. \( \lim_{x \to 0} \tau(x) = \infty \)
2. \( \tau - \tau \circ E_{ij} \) is bounded for any \( i, j \in \{1, \ldots, k\} \) with \( B_i \cap B_j \cap D \neq \emptyset \).

Setting \( \varphi_i = \tau \circ v_i \) we get continuous plurisubharmonic exhaustion functions on \( C_i \cap D \). Moreover, if \( z \in B_i \cap B_j \cap D \) \( E_{ij}(v_i(z)) \leq v_j(z) \), therefore \( \varphi_i(z) - \varphi_j(z) \leq (\tau - \tau \circ E_{ij})(v_i(z)) \). From Lemma 3 \( D \) is Stein and the proof of Theorem 1 is complete.

We give now some immediate consequences of Theorem 1. By Theorem 6 we know that any relatively compact open subset of a \( K \)-complete space carries a \( C^\infty \) strongly plurisubharmonic function. Therefore we obtain:

**COROLLARY 1.** Let \( X \) be a \( K \)-complete space and \( D \subset X \) a relatively compact open subset which is locally hyperconvex. Then \( D \) is Stein. In particular any pseudoconvex domain \( D \subset X \) is Stein.

Corollary 1 is a particular case of the following open problem (see [8]):

**LEVI PROBLEM:** Let \( X \) be a \( K \)-complete space and \( D \subset X \) a locally Stein open set. Is \( D \) itself a Stein space?

We show that this is the case at least when \( X \) is a Stein space and \( D \) is locally hyperconvex at \( \partial D \cap \text{Sing}(X) \), namely we prove:

**THEOREM 2.** Let \( X \) be a Stein space and \( D \subset X \) a locally Stein open subset. Assume that \( D \) is locally hyperconvex at \( \partial D \cap \text{Sing}(X) \). Then \( D \) is a Stein space.

**Proof.** For each \( x \in \text{Sing}(X) \) we choose a hyperconvex neighbourhood \( V_x \subset X \) of \( x \) such that \( V_x \cap D \) is hyperconvex. Then \( V = \bigcup_{x \in \text{Sing}(X)} V_x \) is an open neighbourhood of \( \text{Sing}(X) \) and by Theorem 5 there is a continuous plurisubharmonic function \( p \) on \( X \) such that \( B = \{ p < 0 \} \) contains \( \text{Sing}(X) \) and \( \overline{B} \subset V \). We show that \( B \cap D \) is locally hyperconvex. Indeed, for any \( x_0 \in \overline{B} \cap \partial D \subset \overline{B} \subset V \) there exists \( x \in \text{Sing}(X) \) with \( x_0 \in V_x \). On the other hand \( V_x \cap B \cap D = (V_x \cap B) \cap (V_x \cap D) \) which is hyperconvex as an intersection of two hyperconvex open subsets. Therefore \( B \cap D \) is locally hyperconvex and by Corollary 1 and an exhaustion argument it follows that \( B \cap D \) is Stein. In view of Theorem 4 \( D \) itself is a Stein space and the proof is complete.

References


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