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Polynomial structures and generic Torelli for projective hypersurfaces

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An important part of Donagi's proof [2] of generic Torelli for projective hypersurfaces involves recovering the polynomial structure on certain spaces of homogeneous polynomials. In this paper we give a different method for recovering the polynomial structure. Besides being simpler, our method also enables us to prove generic Torelli for some hypersurfaces not covered by Donagi's argument. More precisely, we get the following theorem:

THEOREM. *The period map for smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ is generally injective except possibly for the following cases:*
(i) $d = 3$ *and* $n = 2$ (*cubic surfaces*).
(ii) $d|n + 2$.
(iii) $d = 4$ *and* $4|n$.

REMARKS.
(i) In Donagi's original theorem [2], hypersurfaces with $d = 6$ and $n \equiv 1 \bmod 6$ also had to be excluded.
(ii) The reasons for excluding (i) and (ii) are discussed in [2, §6]. For some recent progress on (ii), see [5]. At the end of the paper we will comment on (iii).

Proof. Let $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$, and let S^r be the graded piece of S in degree r. Then a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d is defined by some $f \in S^d$. J and R will denote the Jacobian ideal and ring of f, and their graded pieces are J^r and R^r respectively. As explained in [2], we may assume that

$$(d - 2)(n - 1) \geqslant 3. \tag{1}$$

To prove generic Torelli, it suffices to show that X can be recovered up to projective equivalence by the algebraic part of its Infinitesimal Variation of Hodge Structure (see [1]).

Let $k = \gcd(d, n + 2)$. Using the symmetrizer lemma, Donagi [2] shows that the algebraic IVHS data determines the map

$$R^k \times R^k \to R^{2k} \tag{2}$$

up to isomorphism. Note that $k < d - 1$ by (1) and our assumption $d \nmid n + 2$. This means that $R^k \simeq S^k$, and as noted in [2], the proof of the theorem is straightforward once we recover the isomorphism $R^k \simeq S^k$ up to $\mathrm{GL}(S^1)$ equivalence (this is what we mean by the phrase "polynomial structure"). Furthermore, when $2k < d - 1$, an easy argument shows that the map (2) determines the polynomial structure (see [2, Lemma 4.2]).

It remains to consider the case $2k \geqslant d - 1$. Since $k \mid d$, this inequality implies either $k = 1$ or $2k = d$. The polynomial structure is automatic when $k = 1$, so that we may assume $2k = d$ and $k > 1$. To recover the polynomial structure in this case, we proceed as follows.

In the process of symmetrizing to obtain (2), one also obtains the map

$$R^k \times R^{2k} \to R^{3k},$$

which together with (2) gives the sequence

$$\wedge^2 R^k \otimes R^k \to R^k \otimes R^{2k} \to R^{3k}. \tag{3}$$

The basic idea is to use this sequence to determine when a codimension 1 subspace of R^k has a base point. The precise result is the following:

PROPOSITION. *Let d and k be as above, and assume that* $2k = d, k > 1$ and $d > 4$. *If $W \subset R^k \simeq S^k$ is a codimension* 1 *subspace, then the sequence*

$$\wedge^2 W \otimes R^k \to W \otimes R^{2k} \to R^{3k} \tag{4}$$

obtained from (3) *is exact at the middle term if and only if W is base point free.*

Before proving the proposition, let's explain how it determines the polynomial structure on R^k. The key point is that the codimension 1 subspaces of $R^k \simeq S^k$ which have a base point (necessarily unique) are the image of the Veronese imbedding $\mathbb{P}^*(S^1) \to \mathbb{P}^*(S^k)$, where for a vector space V, $\mathbb{P}^*(V)$ is the projective space of codimension 1 subspaces of V. It is well known that the image of the Veronese determines the polynomial structure on $R^k \simeq S^k$ [2, Lemma 4.2].

Also, note that the case $d = 4$ not covered by the proposition corresponds exactly to the case $d = 4$, $4|n$ excluded in the statement of the theorem. Thus the theorem follows immediately from the proposition.

Proof of the Proposition. First, assume that W is base point free. Then it follows

from Theorem 2 of [4] that
(i) $\wedge^2 W \otimes S^k \to W \otimes S^{2k} \to S^{3k}$ is exact at the middle term.
(ii) W $\otimes S^1 \to S^{k+1}$ is onto.
From (ii), we see that $W \otimes J^{2k} \to J^{3k}$ is surjective (note that $J^{2k} = J^d = S^1 \cdot J^{d-1}$). Then an easy diagram chase shows that (4) is exact at the middle term.

Now assume that $W \subset R^k$ has a base point a. For $r \geqslant 0$, set $W^r = \{F \in S^r : F(a) = 0\}$. Then W^r is a codimension 1 subspace of S^r, and we may identify W^k with W. Now consider the commutative diagram:

$$\begin{tikzcd}
 & 0 & \\
\wedge^2 W^k \otimes R^k \arrow[r] & W^k \otimes R^{2k} \arrow[r] \arrow[u] & R^{3k} \\
\wedge^2 W^k \otimes S^k \arrow[r] \arrow[u, equal] & W^k \otimes S^{2k} \arrow[r, "\alpha"] \arrow[u] & W^{3k} \arrow[u] \\
 & W^k \otimes J^{2k} \arrow[r, "\beta"] \arrow[u] & J^{3k} \cap W^{3k} \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The map α is surjective, and then a diagram chase shows that β is surjective whenever the top row is exact in the middle. Thus, to prove the proposition, it suffices to show that the map β: $W^k \otimes J^{2k} \to J^{3k} \cap W^{3k}$ is not surjective.

Since the generators $f_{x_0}, \dots, f_{x_{n+1}}$ of the Jacobian ideal J form a regular sequence, we can resolve J using the Koszul sequence of $f_{x_0}, \dots, f_{x_{n+1}}$. The first relations between the generators occur in degree $d-1$, so that the map $S^r \otimes J^{d-1} \to J^{r+d-1}$ is an isomorphism for $r < d-1$. From $2k = d$ and $d > 4$, we see that $k + 1 < d - 1$, and hence

$$\begin{aligned}
\operatorname{Im}(\beta) = W^k \cdot J^{2k} &= W^k \cdot S^1 \cdot J^{d-1} = W^{k+1} \cdot J^{d-1} \simeq W^{k+1} \otimes J^{d-1} \\
J^{3k} &= S^{k+1} \cdot J^{d-1} \simeq S^{k+1} \otimes J^{d-1}.
\end{aligned}$$

Since $J^{3k} \cap W^{3k}$ has codimension 1 in J^{3k}(J^{3k} is base point free), an easy dimension count shows that $\operatorname{Im}(\beta)$ is a proper subspace of $J^{3k} \cap W^{3k}$. Thus β is not surjective, which concludes the proof of both the proposition and the theorem. □

The above argument simplifies the "polynomial structure" part of the proof of generic Torelli for projective hypersurfaces, and we should mention that the proof of the symmetrizer lemma has also been simplified [3].

The last case excluded in our theorem, when $d = 4$ and $4|n$, is frustrating: it is generally believed that this case should be accessilble by IVHS methods, but both Donagi's arguments [2, §5] and the above proposition break down in this situation (one can show that the sequence (4) is exact in the middle for *all* codimension 1 subspaces, base point free or not).

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