

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 74, n° 2 (1990), p. 165-168

http://www.numdam.org/item?id=CM_1990__74_2_165_0

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Projective modules and approximation couples

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1. Introduction

In 1976, Quillen and Suslin independently and almost simultaneously showed that every finitely generated projective module over a polynomial ring in n variables with coefficients from a field, is free. Thus they settled the conjecture of Serre affirmatively. In the following year Lindel and Lutkebohmert [3] showed that the above result is valid if the field of coefficients is replaced by a ring of formal power series in a finite number of variables over a field.

The object of this note is to show that if $R \subset \bar{R}$ is an approximation couple (see Definition 1 below) and if a finitely generated, projective $R[X] = R[X_1, \dots, X_n] = S$ module M becomes free on extension of scalars to $\bar{R}[X]$, then M itself is free as an S -module. In particular, Serre's conjecture is true if the field of coefficients is replaced by an equicharacteristic Henselian ring.

It is a pleasure to acknowledge the helpful discussions we had with Professor Amit Roy while working on this note.

2. Approximation couples

Let $R \subset \bar{R}$ be two commutative rings with the same identity, provided with a topology τ such that R is dense in \bar{R} under the induced topology.

DEFINITION 1. Let $R \subset \bar{R}$ be as above. The pair $R \subset \bar{R}$ is called an approximation couple (or a couple of rings having the approximation property) if the following holds:

For any finite family $\{f_i\}_{i \in I}$ of polynomials in $R[Y_1, \dots, Y_n]$ and for each common zero $\xi = (\xi_1, \dots, \xi_n)$ of $\{f_i\}$ in \bar{R}^n , we can find a common zero $y = (y_1, \dots, y_n)$ in R^n which is arbitrarily close to ξ in the product topology in \bar{R}^n .

We give below a few examples of approximation couples.

1. Let (\bar{K}, v) be a complete valued field of characteristic 0 and K any algebraically closed field in \bar{K} . Then $K \subset \bar{K}$ forms an approximation couple (Lang [2]).

2. Let R be a local ring and $\bar{A} = R[[X]]$. Let A be the Henselization of

$R[X]_{(X)}$ at its maximal ideal. Then $A \subset \bar{A}$ is an approximation couple. (Artin [1]).

3. Let A be the valuation ring of a complete non-archimedean valued field (K, v) of characteristic 0 and $Y = (Y_1, Y_2, \dots, Y_n)$ be a set of indeterminates over K . In the formal power series ring $A[[Y]]$ introduce a topology with the help of v as follows. For any $f = \sum_v a_v Y^v$ where $v = (v_1, v_2, \dots, v_n), v_i \geq 0$, set $v(f) = \sup_v v(a_v)$. If A_n denotes the subring of $A[[Y]]$ consisting of elements (which are algebraic over $A[Y]$) and \bar{A}_n is the closure of A_n in $A[[Y]]$ in the above topology, then $A_n \subset \bar{A}_n$ is an approximation couple. (Robba [4]).

In the above examples the rings involved are Noetherian. Schoutens [5] gives an example of a couple of non-Noetherian local rings having the approximation property.

3. Projective modules

In this section we prove the main result of this note.

THEOREM: *Let $R \subset \bar{R}$ be an approximation couple and M be a finitely generated projective S -module where $S = R[X] = R[X_1, \dots, X_n]$ the X_i being indeterminates over R . If $\bar{M} = \bar{S} \otimes_R M (\bar{S} = \bar{R}[X])$ is free S -module, then M is free as an S -module.*

In particular, the validity of Serre's conjecture for $\bar{R}[X_1, \dots, X_n]$ modules implies the validity for $R[X_1, \dots, X_n]$ -modules.

Proof. M being a projective module, is a direct summand of a free module over S and as M is also finitely generated we have an N such that $M \oplus N \cong S^m$ for a suitable m . Note that N is also finitely generated and projective. By the hypothesis, the modules $\bar{M} = M \otimes_S \bar{S}$ and $\bar{N} = N \otimes_S \bar{S}$ are free \bar{S} -modules.

Now, consider the exact sequence

$$S^m \xrightarrow{\varepsilon} S^m \xrightarrow{\varphi} M \longrightarrow 0$$

of S -modules, where ε is the projection of S^m on N and φ is the projection on M . Tensoring the above sequence with \bar{S} over S , we get the following exact sequence

$$\bar{S}^m \xrightarrow{\bar{\varepsilon}} \bar{S}^m \xrightarrow{\bar{\varphi}} \bar{M} \longrightarrow 0.$$

Since both \bar{M} and \bar{N} are free over \bar{S} and $\bar{S}^m = \bar{M} \oplus \bar{N}$, for the standard basis $\{e_1, e_2, \dots, e_m\}$ of \bar{S}^m over \bar{S} where e_i is the m -tuple with 1 at the i th entry and 0 elsewhere, we can find an \bar{S} -automorphism $\bar{\sigma}$ of \bar{S}^m namely, $\bar{\sigma}(f_i) = e_i$ (where f_1, \dots, f_s and f_{s+1}, \dots, f_m are generators of \bar{N} and \bar{M} respectively, as free

\bar{S} -modules) such that the matrix of

$$\bar{\tau} = \bar{\sigma} \cdot \bar{\varepsilon} \cdot \bar{\sigma}^{-1} \tag{1}$$

with respect to the above basis has the form $\begin{pmatrix} I_s & 0 \\ 0 & \end{pmatrix}$. In other words we have the following commutative diagram. Here $m = r + s$.

$$\begin{array}{ccccccc} \bar{S}^m & \xrightarrow{\bar{\varepsilon}} & \bar{S}^m & \xrightarrow{\bar{\phi}} & \bar{M} & \longrightarrow & 0 \\ \bar{\sigma} \downarrow & & \downarrow \bar{\sigma} & & \downarrow \cong & & \\ \bar{S}^m & \xrightarrow{\bar{\tau}} & \bar{S}^m & \longrightarrow & \bar{S}^r & \longrightarrow & 0 \end{array}$$

Let $A(\bar{\theta})$ (respectively $A(\theta)$) denote the matrix associated with $\bar{\theta} \in \text{End}_{\bar{S}}(\bar{S}^m)$ (respectively $\theta \in \text{End}_S(S^m)$) with respect to the standard basis $\{e_i\}$. In terms of the matrices, (1) can be written as $A(\bar{\tau}) \cdot A(\bar{\sigma}) = A(\bar{\sigma}) \cdot A(\bar{\varepsilon})$ and this yields

$$B(\bar{\sigma}) = A(\bar{\sigma}) \cdot A(\bar{\varepsilon}) - \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} A(\bar{\sigma}) = 0. \tag{2}$$

Note that $A(\varepsilon) = A(\bar{\varepsilon})$. Since $\bar{\sigma}$ is an automorphisms of \bar{S}^m $\det A(\bar{\sigma}) = u$ is a unit in \bar{S} and therefore, on replacing f_1 by $u \cdot f_1$ we may assume that

$$\det A(\bar{\sigma}) = 1 \tag{3}$$

Setting $A(\bar{\sigma}) = (\bar{f}_{ij})$ and $B(\bar{\sigma}) = (\bar{g}_{ij})$ we have $\bar{f}_{ij} = \sum_{\nu} r_{\nu}^{(ij)} X^{\nu}$ where $\nu = (v_1, \dots, v_n)$, $v_i \geq 0$ and $\bar{r}_{\nu}^{(ij)} \in \bar{R}$. On replacing $\bar{r}_{\nu}^{(ij)}$ by indeterminates $T_{\nu}^{(ij)}$, from equation (2) we get $\bar{g}_{kl} = (\sum_{\mu} P_{\mu}^{(kl)} (\dots T_{\nu}^{(ij)} \dots) X^{\mu})$ is zero on specializing $T_{\nu}^{(ij)} = \bar{r}_{\nu}^{(ij)}$. Here $P_{\mu}^{(kl)} (\dots T_{\nu}^{(ij)} \dots)$ are polynomials over R (since $A(\varepsilon)$ has entries from $R[X]$). Thus we get a finite set of polynomial equations

$$P_{\mu}^{(kl)} (\dots T_{\nu}^{(ij)} \dots) = 0 \quad \text{over } R \text{ satisfied by } (\dots \bar{r}_{\nu}^{(ij)} \dots), \bar{r}_{\nu}^{(ij)} \in \bar{R}.$$

Likewise, equation (3) gives another finite system of polynomial equations satisfied by $\{\bar{r}_{\nu}^{(ij)}\}$. As $R \subset \bar{R}$ is an approximation couple, we can find $r_{\nu}^{(ij)} \in R$ such that $\{r_{\nu}^{(ij)}\}$ is a common solution of the polynomial equations arising out of condition (2) and (3). Thus we have an automorphism σ of S^m with

$$A(\sigma) = (f_{ij}), \quad \text{where } f_{ij} = \sum_{\nu} r_{\nu}^{(ij)} X^{\nu}, \quad \nu = (v_1, \dots, v_n)$$

such that the following diagram commutes.

$$\begin{array}{ccccccc}
 S^m & \xrightarrow{\varepsilon} & S^m & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 \sigma \downarrow & & \downarrow \sigma & & \downarrow \cong & & \\
 S^m & \xrightarrow{\tau} & S^m & \longrightarrow & S^r & \longrightarrow & 0.
 \end{array}$$

This gives $M \cong S^r$. Thus M is a free S -module.

REMARK 1. In case $n = 0$, conditions (1) and (2) actually lead to m^2 linear equations and one homogeneous polynomial of total degree m in $T_v^{(i)}$ equated to 1 and the proof gets considerably simplified.

REMARK 2. In view of example 2 and the result of Lindel and Lutkebohmert the theorem above implies that any finitely generated projective module over an equicharacteristic Henselian local domain is free.

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