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Projective modules and approximation couples

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1. Introduction

In 1976, Quillen and Suslin independently and almost simultaneously showed that every finitely generated projective module over a polynomial ring in n variables with coefficients from a field, is free. Thus they settled the conjecture of Serre affirmatively. In the following year Lindel and Lutkebohmert [3] showed that the above result is valid if the field of coefficients is replaced by a ring of formal power series in a finite number of variables over a field.

The object of this note is to show that if $R \subset \bar{R}$ is an approximation couple (see Definition 1 below) and if a finitely generated, projective $R[X] = R[X_1, \ldots, X_n] = S$ module M becomes free on extension of scalars to $\bar{R}[X]$, then M itself is free as an S-module. In particular, Serre's conjecture is true if the field of coefficients is replaced by an equicharacteristic Henselian ring.

It is a pleasure to acknowledge the helpful discussions we had with Professor Amit Roy while working on this note.

2. Approximation couples

Let $R \subset \bar{R}$ be two commutative rings with the same identity, provided with a topology τ such that R is dense in $\bar{R}$ under the induced topology.

DEFINITION 1. Let $R \subset \bar{R}$ be as above. The pair $R \subset \bar{R}$ is called an approximation couple (or a couple of rings having the approximation property) if the following holds:

For any finite family $\{f_i\}_{i \in I}$ of polynomials in $R[Y_1, \ldots, Y_n]$ and for each common zero $\xi = (\xi_1, \ldots, \xi_n)$ of $\{f_i\}$ in $\bar{R}^n$, we can find a common zero $y = (y_1, \ldots, y_n)$ in R^n which is arbitrarily close to ξ in the product topology in $\bar{R}^n$.

We give below a few examples of approximation couples.

1. Let $(\hat{K}, v)$ be a complete valued field of characteristic 0 and K any algebraically closed field in $\hat{K}$. Then $K \subset \hat{K}$ forms an approximation couple (Lang [2]).

2. Let R be a local ring and $\bar{A} = R[[X]]$. Let A be the Henselization of

$R[X]_{(X)}$ at its maximal ideal. Then $A \subset \bar{A}$ is an approximation couple. (Artin [1]).

3. Let A be the valuation ring of a complete non-archimedean valued field (K, v) of characteristic 0 and $Y = (Y_1, Y_2, \ldots, Y_n)$ be a set of indeterminates over K. In the formal power series ring $A[[Y]]$ introduce a topology with the help of v as follows. For any $f = \Sigma_\nu a_\nu Y^\nu$ where $\nu = \nu_1, \nu_2, \ldots, \nu_n)$, $\nu_i \geqslant 0$, set $v(f) = \mathrm{Sup}_\nu v(a_\nu)$. If A_n denotes the subring of $A[[Y]]$ consisting of elements (which are algebraic over $A[Y]$ and $\bar{A}_n$ is the closure of A_n in $A[[Y]]$ in the above topology, then $A_n \subset \bar{A}_n$ is an approximation couple. (Robba [4]).

In the above examples the rings involved are Noetherian. Schoutens [5] gives an example of a couple of non-Noetherian local rings having the approximation property.

3. Projective modules

In this section we prove the main result of this note.

THEOREM: *Let $R \subset \bar{R}$ be an approximation couple and M be a finitely generated projective S-module where $S = R[X] = R[X_1, \ldots, X_n]$ the X_i being indeterminates over R. If $\bar{M} = \bar{S} \otimes_R M(\bar{S} = \bar{R}[X])$ is free S-module, then M is free as an S-module.*

In particular, the validity of Serre's conjecture for $\bar{R}[X_1, \ldots, X_n]$ modules implies the validity for $R[X_1, \ldots, X_n]$-modules.

Proof. M being a projective module, is a direct summand of a free module over S and as M is also finitely generated we have an N such that $M \oplus N \cong S^m$ for a suitable m. Note that N is also finitely generated and projective. By the hypothesis, the modules $\bar{M} = M \otimes_S \bar{S}$ and $N = N \otimes_S \bar{S}$ are free $\bar{S}$-modules.

Now, consider the exact sequence

$$S^m \xrightarrow{\varepsilon} S^m \xrightarrow{\varphi} M \longrightarrow 0$$

of S-modules, where ε is the projection of S^m on N and φ is the projection on M. Tensoring the above sequence with $\bar{S}$ over S, we get the following exact sequence

$$\bar{S}^m \xrightarrow{\bar{\varepsilon}} \bar{S}^m \xrightarrow{\bar{\varphi}} \bar{M} \longrightarrow 0.$$

Since both $\bar{M}$ and $\bar{N}$ are free over $\bar{S}$ and $\bar{S}^m = \bar{M} \oplus \bar{N}$, for the standard basis $\{e_1, e_2, \ldots, e_m\}$ of $\bar{S}^m$ over $\bar{S}$ where e_i is the m-tuple with 1 at the ith entry and 0 elsewhere, we can find an $\bar{S}$-automorphism $\bar{\sigma}$ of $\bar{S}^m$ namely, $\bar{\sigma}(f_i) = e_i$ (where $f_1, \ldots, f_s$ and $f_{s+1}, \ldots, f_m$ are generators of $\bar{N}$ and $\bar{M}$ respectively, as free

$\bar{S}$-modules) such that the matrix of

$$\bar{\tau} = \bar{\sigma} \cdot \bar{\varepsilon} \cdot \bar{\sigma}^{-1} \tag{1}$$

with respect to the above basis has the form $\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$. In other words we have the following commutative diagram. Here $m = r + s$.

$$\begin{array}{ccccccc}
\bar{S}^m & \xrightarrow{\bar{\varepsilon}} & \bar{S}^m & \xrightarrow{\bar{\varphi}} & \bar{M} & \longrightarrow & 0 \\
{\scriptstyle \bar{\sigma}}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}} & & \downarrow{\scriptstyle \cong} & & \\
\bar{S}^m & \xrightarrow{\bar{\tau}} & \bar{S}^m & \longrightarrow & \bar{S}^r & \longrightarrow & 0
\end{array}$$

Let $A(\bar{\theta})$ (respectively $A(\theta)$) denote the matrix associated with $\bar{\theta} \in \mathrm{End}_{\bar{S}}(\bar{S}^m)$ (respectively $\theta \in \mathrm{End}_S(S^m)$) with respect to the standard basis $\{e_i\}$. In terms of the matrices, (1) can be written as $A(\bar{\tau}) \cdot A(\bar{\sigma}) = A(\bar{\sigma}) \cdot A(\bar{\varepsilon})$ and this yields

$$B(\bar{\sigma}) = A(\bar{\sigma}) \cdot A(\bar{\varepsilon}) - \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} A(\bar{\sigma}) = 0. \tag{2}$$

Note that $A(\varepsilon) = A(\bar{\varepsilon})$. Since $\bar{\sigma}$ is an automorphisms of $\bar{S}^m$ det $A(\bar{\sigma}) = u$ is a unit in $\bar{S}$ and therefore, on replacing f_1 by $u \cdot f_1$ we may assume that

$$\det A(\bar{\sigma}) = 1 \tag{3}$$

Setting $A(\bar{\sigma}) = (\bar{f}_{ij})$ and $B(\bar{\sigma}) = (\bar{g}_{ij})$ we have $\bar{f}_{ij} = \Sigma_\nu r_\nu^{(ij)} X^\nu$ where $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_i \geqslant 0$ and $\bar{r}_\nu^{(ij)} \in \bar{R}$. On replacing $\bar{r}_\nu^{(ij)}$ by indeterminates $T_\nu^{(ij)}$, from equation (2) we get $\bar{g}_{kl} = (\Sigma_\mu P_\mu^{(kl)}(\ldots T_\nu^{(ij)} \ldots) X^\mu)$ is zero on specializing $T_\nu^{(ij)} = \bar{r}_\nu^{(ij)}$. Here $P_\mu^{(kl)}(\ldots \mathrm{T}_\nu^{(ij)} \ldots)$ are polynomials over R(since $A(\varepsilon)$ has entries from $R[X]$). Thus we get a finite set of polynomial equations

$$P_\mu^{(kl)}(\ldots T_\nu^{(ij)} \ldots) = 0 \quad \text{over } R \text{ satisfied by } (\ldots \bar{r}_\nu^{(ij)} \ldots),\ \bar{r}_\nu^{(ij)} \in \bar{R}.$$

Likewise, equation (3) gives another finite system of polynomial equations satisfied by $\{\bar{r}_\nu^{(ij)}\}$. As $R \subset \bar{R}$ is an approximation couple, we can find $r_\nu^{(ij)} \in R$ such that $\{r_\nu^{(ij)}\}$ is a common solution of the polynomial equations arising out of condition (2) and (3). Thus we have an automorphism σ of S^m with

$$A(\sigma) = (f_{ij}), \quad \text{where } f_{ij} = \sum_\nu r_\nu^{(ij)} X^\nu, \quad \nu = (\nu_i, \ldots, \nu_n)$$

such that the following diagram commutes.

$$\begin{array}{ccccccc} S^m & \xrightarrow{\varepsilon} & S^m & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\cong} & & \\ S^m & \xrightarrow[\tau]{} & S^m & \longrightarrow & S^r & \longrightarrow & 0. \end{array}$$

This gives $M \cong S^r$. Thus M is a free S-module.

REMARK 1. In case $n = 0$, conditions (1) and (2) actually lead to m^2 linear equations and one homogeneous polynomial of total degree m in $T_v^{(ij)}$ equated to 1 and the proof gets considerably simplified.

REMARK 2. In view of example 2 and the result of Lindel and Lutkebohmert the theorem above implies that any finitely generated projective module over an equicharacteristic Henselian local domain is free.

References

1. M. Artin, Algebraic approximation of structures over complete local rings. *Inst. Hautes Etudes Sci.* 36 (1969) 23–68.
2. S. Lang, On Quasi algebraic closure. *Ann. Math.* 55 (1952) 379–390.
3. H. Lindel and W. Lutkebohmert, Projective Moduln uber polynomialen Erweiterungen von Potenzreihen algebren. *Archiv. der Math.* 28 (1977) 51–54.
4. P. Robba, Propriété d'approximation pour les éléments algebriques. *Compositio Mathematica* 63 (1987) 3–14.
5. H. Schoutens, Approximation properties for some non-Noetherian local rings. *Pacific Jl. of Math.* 131 (1988) 331–359.