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On the dual variety in characteristic 2

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Consider an integral on degenerate variety $X \subset \mathbf{P}^N$. The projective geometry of X is very much reflected in the geometry of its conormal variety $C(X) \subset \mathbf{P}^N \times \mathbf{P}^{N*}$; $C(X)$ is the closure of the pairs (x, H) with $x \in X_{\text{reg}}$ and $T_xX \subseteq H$ (i.e. such that H is tangent to X at x). For instance if the dual variety X^* (i.e. the projection of $C(X)$ into $\mathbf{P}^{n*}$) is not a hypersurface then the general hyperplane tangent to X is tangent to X at infinitely many points. If the algebraically closed base field $\mathbf{F}$ has $\text{char}(\mathbf{F}) = 0$, then it is known that in such a case the general hyperplane tangent to X is tangent to X exactly along a linear space. Without any assumption on X^*, when $\text{char}(\mathbf{F}) = 0$, many nice things are true: biduality (i.e. $(X^*)^* = X$ under the identification of $(\mathbf{P}^{N*})^*$ with $\mathbf{P}^N$), the fact that in the enumerative formulas the solutions comes with multiplicity ones etc. In general such good things are not true when $\text{char}(\mathbf{F}) = p > 0$. Wallace ([15]) was the first to note this fact and to show that there is a class of varieties (the reflexive varieties) for which such facts are true (in particular the biduality of X). A variety X is called reflexive if $C(X) = C(X^*)$ (up to the exchange of the factors in $\mathbf{P}^N \times \mathbf{P}^{N*}$ and the identification of $(\mathbf{P}^{N*})^*$ with $\mathbf{P}^N$). A variety $X \subset \mathbf{P}^N$ is called ordinary if it is reflexive and X^* is a hypersurface. When $\text{char}(\mathbf{F}) = 0$, every variety is reflexive. When $\text{char}(\mathbf{F}) > 0$ this is not true, but the class of reflexive varieties has remarkable stability properties and it is very nice: in the small universe of reflexive varieties everything is nice (see [2], [3], [4]). But there is a disturbing feature: $\text{char}(\mathbf{F}) = 2$. When $\text{char}(\mathbf{F}) = 2$ there is no ordinary variety of odd dimension. In particular the general hyperplane section Y of an ordinary variety cannot be ordinary. But it has many nice features: semi-reflexivity (see [3]) and Y^* is a hypersurface ([3], 5.9 and 5.12), i.e. the general hyperplane tangent to y is tangent only at finitely many points of Y. It was asked in [3], (5.11)(iii), if for such X the following property ($) holds:

($): the general hyperplane tangent to a general hyperplane section Y of an ordinary variety X is tangent exactly at one point of Y.

A proof of ($) for all ordinary X would make authomatically possible to drop the unpleasant assumption "char(**F**) $\neq 2$" in a few statements ([3], 4.10(ii); for the non-existence of bitangencies in the enumeration of the contact formula in [2], §2, part (c) of the main theorem, p. 162, see the discussion in [2], p. 169). Unfortunately, the answer is NO. In Section 1 we construct a very nice family of examples; they are hypersurfaces and we have explicit equations; they are related to the so-called null-correlations (and to the null-correlation bundles) (a very classical topic: see e.g. [1]). But the next best thing happens. These examples are the only ones. This is proved in Section 2. Thus in this paper we prove the following result.

THEOREM 0.1. *Fix an integral non degenerate variety X in $\mathbf{P}^N$, $\dim(X) = n$. Then a general hyperplane tangent to a general hyperplane section Y of X is tangent at more than one point of Y if and only if $N = n + 1$, $n = 2k + 2$ is even, and, up to a change of the homogeneous coordinates $y_1, \dots, y_k, z_1, \dots, z_k$* of $\mathbf{P}^N$, *the equation f of X has the form:*

$$f = \left(\sum_{i=1}^{k} y_i z_i\right) h^2 + b^2 \tag{1}$$

with h and b homogeneous polynomials.

Every integral non degenerate hypersurface X satisfying (1) has two geometrical properties (see §1 for their proof): TX is the restriction to X of a null-correlation bundle of $\mathbf{P}^{n+1}$ and X is canonically isomorphic to its dual variety. Now we can check that such a variety X is not normal (proof: X is singular on the codimension one subset $\{h = b = 0\}$). Thus we have the following corollary.

COROLLARY 0.2 *Fix an integral non degenerate variety $X \subset \mathbf{P}^N$. X has property ($) if it is normal or if it is not a hypersurface.*

While the work on this paper was done, the author was a guest at SFB 170 (Gottingen); he wants to thank very much H. Flenner for some very useful conversations. S. Kleiman detected an error in the first version of this paper and several inaccuracies in the second version; the author wants to thank him both for this reason and for very strong encouragement.

This paper is dedicated to Alessandra.

1. The examples

In this paper we work always over an algebraically closed base field **F** with char(**F**) $= 2$. Fix an odd integer $m > 1$ and set $\mathbf{P} := \mathbf{P}^m$. Let $\mathcal{O}$ and Ω be structural sheaf and the cotangent bundle on $\mathbf{P}$. It is well-known that $\Omega(2)$ is spanned by its global sections. Since m is odd, $c_m(\Omega(2)) = 0$; hence a general section of $\Omega(2)$ never vanishes. Choose a nowhere vanishing section s of $\Omega(2)$. The choice of s is equivalent to the choice of a surjection $t\colon T\mathbf{P} \to \mathcal{O}(2)$. Set $N := \mathrm{Ker}(t)$. N is a rank-$(m-1)$ vector bundle on $\mathbf{P}$. It is called a null-correlation bundle and classically considered (see e.g. [1]).

Thus s induces an exact sequence:

$$0 \to N \to T\mathbf{P} - \mathcal{O}(2) \to 0. \tag{2}$$

Choose homogeneous coordinates $x_0, \ldots, x_m$ on $\mathbf{P}$; we will set $m = 2k - 1$, $y_i := x_{2i-2}$, $1 \leqslant i \leqslant m$, $z_i := x_{2i-1}$, $1 \leqslant i \leqslant m$. Consider the Euler sequence on $\mathbf{P}$:

$$0 \to \mathcal{O} \to (m+1)\mathcal{O}(1) \to T\mathbf{P} \to 0. \tag{3}$$

The map $\mathcal{O} \to (m+1)\mathcal{O}(1)$ in (3) send $c \in \mathcal{O}$ into $(cx_0; \ldots; cx_m)$. By (3) the choice of t is equivalent to the choice of $m+1$ linear forms $l_0, \ldots, l_m$ on $\mathbf{P}$ such that $x_0 l_0 + \cdots + x_m l_m = 0$; t is surjective exactly when the forms $l_0, \ldots, l_m$ have no common zero. Furthermore by the dual of (3) $H^0(\mathbf{P}, \Omega(2))$ can be identified to the set of antisymmetric $(m+1) \times (m+1)$ matrices over $\mathbf{F}$ (even if $\mathrm{char}(\mathbf{F}) = 2$); the matrix given by $l_0, \ldots, l_m$ has as elements in the ith row the coefficients of l_i. The nowhere vanishing of s is equivalent to the fact that the corresponding antisymmetric matrix L is invertible; hence up to a change of coordinates s is uniquely determined. We will choose the coordinates $x_0, \ldots, x_m$ in such a way that $l_{2i-2} = z_i$ and $l_{2i-1} = -y_i$, $1 \leqslant i \leqslant m$.

Let X be a reduced hypersurface of $\mathbf{P}$ with homogeneous equation f, $\deg(f) = d > 2$. Assume that $(\partial/\partial x_i) f = l_i g$ for all i and for some form g of degree $d - 2$. Since X is reduced, $g \neq 0$. Then the map from $T\mathbf{P}|X$ to the normal bundle $\mathcal{O}_X(d)$ to X factors through $\mathcal{O}_X(2)$; since the l_i are nowhere vanishing, we see that the exact sequence

$$0 \to TX \to T\mathbf{P}|X \to \mathcal{O}_X(d)$$

induces the exact sequence

$$0 \to TX \to T\mathbf{P}|X \to \mathcal{O}_X(2) \to 0 \tag{4}$$

which is the restriction to X of (2). In particular TX is locally free and $TX = N|X$.

Claim: X is reflexive.

Proof of the claim. The alternating matrix L corresponding to s gives a null-correlation, i.e. an isomorphism $\mathbf{L}$ between $\mathbf{P}$ and its dual $\mathbf{P}^*$ such that $x \in \mathbf{L}(x)$ for all $x \in \mathbf{P}$. Since the matrix L is alternating, we check easily that if

$y \in \mathbf{L}(x)$ then $x \in \mathbf{L}(y)$. Let $J \subset \mathbf{P} \times \mathbf{P}^*$ be the incidence correspondence, $J = \{(x, H): x \in H\}$. $J = \mathbf{P}(T\mathbf{P}(-1))$ (as space over $\mathbf{P}$) and t gives a section of $J \to \mathbf{P}$ with image the graph graph($\mathbf{L}$) of $\mathbf{L}$. Furthermore, since $TX = N|X, J|(X \times \mathbf{P}^*)$ is the conormal variety $C(X)$ of X. Since $\mathbf{L}$ is an isomorphism, the projection graph($\mathbf{L}$) $\to \mathbf{P}^*$ is an isomorphism. Hence so is its restriction $C(X) \to X^*$. Thus X is reflexive (see [2] or [3] or [4]), proving the claim.

Since (4) is the restriction of (2), for any $x, y \in (X)_{\mathrm{reg}}, y \in T_x X$ if and only if $x \in T_y X$. Furthermore if H is a general hyperplane, say $H = \mathbf{L}(a)$ with $a \in \mathbf{P}$ and a general, for all $x \in H \cap (X_{\mathrm{reg}})$, we have $a \in T_x X$ (i.e. by definition $Y := X \cap H$ is strange with strange point a). In particular every line D in H with $a \in D$ is tangent to X at all points of $Y \cap D$. First assume that for a general line D in H with $a \in D$, card$(Y \cap D) \geqslant 2$. Fix any such D and $c, b \in Y \cap D$ with $c \neq b$. To show that X does not satisfy ($), it is sufficient to check that $(T_c Y) \cap Y = (T_b Y) \cap Y$. Fix a general $e \in (T_c Y) \cap Y$. Then $c \in T_e Y$. Since $a \in T_e Y$, we get $b \in T_e Y$, hence $e \in T_b Y$, hence the thesis. We have to prove the assumption, i.e. that for general $\mathbf{L}(a)$ there is such a line D. First assume $m = 3$, i.e. X a surface. Since X is reflexive, for general $x \in X T_x X$ has order of contact 2 with x at X, hence for general $\mathbf{L}(a)$ a general line D tangent to $X \cap \mathbf{L}(a)$ (i.e. a general line in H through a) has contact order 2 with Y at each point of $Y \cap D$; since $d > 2$, the assumption follows. Now assume $m > 3$. By the case $m = 3$, we see (cutting X with a general $\mathbf{P}^3$) that for a general $x \in X$ and a general tangent line D to X at x, card$(D \cap X) > 1$; $\mathbf{L}(x)$ is the tangent space to X at x, hence contains D. Of course, $\mathbf{L}(x)$ is not a general hyperplane. But we may find a family of points $a(t)$, $t \in T$, $a(t) \in \mathbf{P}$ for all t, with T smooth, irreducible affine curve, and a family of lines $D(t)$, $t \in T$, with $a(t) \in D(t) \subset \mathbf{L}(a(t))$ for every t (hence $D(t)$ tangent to X at each point of $D(t) \cap X$ by the strangeness of $X \cap \mathbf{L}(a(t))$, with $a(0) = x$ for some point $0 \in T$, $a(t)$ general for general t. For general $t \in T$, $\mathbf{L}(a(t))$ and $D(t)$ will work.

Now we have to show that (assuming char($\mathbf{F}$) = 2) such varieties X exist, write their equations and find all possible X. First note that from $(\partial/\partial z_i)f = y_i g$ (resp. $(\partial/\partial y_i)f = z_i g$), $i = 1, \ldots, m$, we get that g contains every variables z_i (resp. y_i) at an even order. Thus $g = h^2$ for some polynomial h of degree $d/2$; in particular d is even. Furthermore, taking derivatives, we see that $f - (\Sigma_i y_i z_i)h^2$ is a square, say b^2. Viceversa, for any even d and any choice of polynomials h, b, with $\deg(h) = -1 + (d/2)$, $\deg(b) = d/2$, define f by the equation (1). The variety $X := \{f = 0\}$ will work if it is reduced and irreducible. Even the reducibility is not a big problem; if it is reducible, any of its reducible components of degree >2 would work (if there are such components). The reader can easily construct examples of f with X integral. Here we show only the existence of at least one example, taking $m = 3$, $h = x_1$, $b = (x_0 + x_2 + x_3)^2$. Note that the restriction of this surface X to the plane $A := \{x_2 = 0\}$ is a curve C with exactly a singular point $(x_0 = x_3 \neq 0, x_1 = 0)$ while the line $\{x_0 = 0\}$ intersects C only at one point with $x_3 = 0$. Hence C cannot be the product of (maybe reducible) conics.

2. End of proof

In this section we will prove 0.1 and 0.2 showing that the examples given in section 1 are the only examples of reflexive varieties without property ($). We will fix an integral non degenerate variety $X \subset \mathbf{P}^N$, $\dim(X) = n$, X not with the property ($). The proof of 0.1 is divided into 2 steps; essentially, in the first step we will handle the case "$N = n + 1$", while in the second step we will handle the case "$N > n + 1$".

Step 1. By [5], Lemma 3 p. 334, the general projection Y of X into $\mathbf{P}^{n+1}$ is reflexive; of course, Y is ordinary and cannot have the property ($). In this step we will work with Y i.e. we will assume $N = n + 1$ and X a hypersurface. By the definition of ordinary variety and [4], 2.4(v), there is a proper closed subset A of X, A containing the singular set $\mathrm{Sing}(X)$ of X, such that for every $x \in X$, $x \notin A$, the tangent space T_xX is tangent to the smooth locus of X only at x. Consider the following assertion (£):

(£): for a general $x \in X$ and a general hyperplane H through x, the proper morphism from the conormal variety of $X \cap H$ in H to the dual of $X \cap H$ has x as its fiber through x (set theoretically).

Note that, fixing H and taking $z \in H$ with z general, it is easy to check that (£) implies ($). Thus if we prove (£) we get a contradiction. For every $y \in X \backslash A$, set $X(y) := X \cap T_yX$, $X(y)' := X(y) \cap (X_{\mathrm{reg}})$. Since $N = n + 1$, by the definition of A for every $y \in X(x)'$, $y \neq x$, $T(x, y) := T_yX \cap T_xX$ is a hyperplane of T_xX. Varying y we get a family of hyperplanes of T_xX of dimension at most $n - 1$.

Assume that for general x and general $y \in T(x)'$, $x \notin T(x, y)$.

Then a general hyperplane H through x does not contain any $T(x, y)$. Since $T_x(X \cap H) = (T_xX) \cap H$, (x, H) will prove (£).

Thus we may assume that for general $x \in X$ and every $y \in T(x)'$, $x \in T(x, y)$. If the family $\{T(x, y)\}$ (x fixed, y variable in $X(x)'$) has not dimension $n - 1$, again the general hyperplane H through x does not contain any $T(x, y)$, again giving (£). Thus till the end of this step we will assume that for general x, a general hyperplane of T_xX containing x is of the form $T(x, y)$. Counting dimensions we see that this implies that for general $x \in X$ and general $y \in (X \cap T_xX)$, we have $x \in T_yX$. This means that for general $x \in X$, the image of $(X_{\mathrm{reg}} \cap T_xX)$ under the Gauss map $\mathbf{g}\colon \mathbf{X}_{\mathrm{reg}} \to \mathbf{P}^*$ is contained in the hyperplane H_x of $\mathbf{P}^*$ corresponding to x. Thus for general $x \in X$, $\mathbf{g}(X_{\mathrm{reg}} \cap T_xX)$ has $X^* \cap H_x$ as closure in $\mathbf{P}^*$. Thus $\mathbf{g}^*(\boldsymbol{O}_{\mathbf{P}^*}(1)) = \boldsymbol{O}_X(1)|X_{\mathrm{reg}}$. We claim that this gives that the induced map in cohomology $\mathbf{g}'^*\colon H^0(\mathbf{P}^*, \boldsymbol{O}_{\mathbf{P}^*}(1)) \to H^0(X_{\mathrm{reg}}, \boldsymbol{O}_X(1))$ has image contained in the image of $H^0(\mathbf{P}, \boldsymbol{O}_{\mathbf{P}}(1))$; indeed since X is non degenerate, $H^0(\mathbf{P}^*, \boldsymbol{O}_{\mathbf{P}^*}(1))$ is generated by the sections corresponding to the hyperplanes H_x, with $x \in X$. The map $\mathbf{g}'^*$ is injective because X^* is not contained in a hyperplane as X is ordinary. Note that (for dimensional reasons and the claim and the injectivity just proven) $\mathbf{g}'^*$ induces an isomorphism j of $\mathbf{P}^*$ onto $\mathbf{P}$. To show that if ($) does not hold X must be of the form considered in Section 1, it is sufficient to check that j^{-1} is a null-correlation, i.e. that $j(c) \in c$ for every $c \in \mathbf{P}^*$. Let M be the matrix associated to j^{-1}. M is antisymmetric if and only if j^{-1} is a null-correlation. Assume that M is not antisymmetric, i.e. that $Q := \{z \in \mathbf{P}: {}^t zMz = 0\}$ is a quadric. Since by construction $x \in j^{-1}(x)$ for all $x \in X$ (note that $x \in T_x X$), this means that X is a quadric (which trivially satisfy ($)). Thus we have described every ordinary hypersurface X without ($).

Step 2. (*case* $N > n + 1$). Now we assume that X is not a hypersurface, and that ($) fails for X; by [5] as at the beginning of step 1 we may assume that $N = n + 2$ and that a general projection of X into $\mathbf{P}^{n+1}$ does not have ($) (hence by step 1 is of the type considered in Section 1). Fix a point $a \in X_{\text{reg}}$. Fix a general point $P \in \mathbf{P}^{n+2}$ and let j (or j_P) be the projection of X into $\mathbf{P}^{n+1}$ from P; set $Z := j(X)$. Let $H := (P, T_a X)$ be the span of P and $T_a X$. For general P we have $H \cap X \neq H \cap X \cap T_a X$. Fix a general $b \in (H \cap X)$. Since $j(b) \in T_{j(a)} Z$, by the classification in the case of hypersurfaces we have $j(a) \in T_{j(b)} Z$, i.e. a is in the span H' of P and $T_b X$. Varying P in H, we see that this is equivalent to $H' = H$. Hence $\dim(T_a X \cap T_b X) = n - 1$. Moving P, we see that this is true for all $a, b \in X_{\text{reg}}$. Fix $a, b, c \in X_{\text{reg}}$. Then either $T_a X \cap T_b X \cap T_c X = T_a X \cap T_b X$ or $T_c X$ is contained in the span H'' of the union of $T_a X$ and $T_b X$. Since $\dim(T_a X \cap T_b X) = n - 1$, we have $\dim(H'') = n + 1$, hence $H'' \neq \mathbf{P}^{n+2}$. Move c. In the latter case we get $X_{\text{reg}} \subset H''$, contradicting the non degeneracy of X. In the former case all $T_c X$, $c \in X_{\text{reg}}$, contain $T_a X \cap T_b X$, hence X cannot be ordinary. This completes the proof of 0.1 and 0.2.

REMARK 2.1. By the classification given it follows that, except for the examples of Section 1, [3], 4.10(ii), holds true even in characteristic 2. For the examples of Section 1 it fails badly since for a general $H \in X^*$, the hyperplane tangent to X^* at H is the dual of $X \cap H$.

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