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Concordance of compacta

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1. Introduction and main results

The purpose of this paper is to prove a new complement theorem for compact subsets of a manifold. Consider two compacta embedded in the interior of a piecewise linear manifold M of dimension greater than four. We show that the compacta have homeomorphic complements in M provided that they satisfy the inessential loops condition, have fundamental codimension three, and are shape concordant in $M \times [0, 1]$.

Let us begin with some definitions. Throughout this note M denotes a closed manifold of dimension $n \geqslant 5$, and I the unit interval $[0, 1]$. If X is a subset of $M \times I$ and $t \in I$, then X_t denotes the t-level $X \cap (M \times \{t\})$ and X_∂ denotes $X_0 \cup X_1$. We will often identify M with $M \times \{0\}$ or $M \times \{1\}$ and thus consider X_0 and X_1 to be subsets of M. A compact subset X of $M \times I$ is said to be a *shape concordance* from X_0 to X_1 if the inclusion $X_i \hookrightarrow X$ is a shape equivalence for each $i = 0, 1$. We say that a compact subset Y of a space Z satisfies the *inessential loops condition* (abbreviated ILC) if for every neighborhood U of Y in Z there exists a neighborhood V of Y in U such that each loop in $V - Y$ which is null-homotopic in V is also null-homotopic in $U - Y$. A subset X of $M \times I$ satisfies the *proper inessential loops condition* if X_0, X_1 and X satisfy the ILC in M_0, M_1 and $M \times I$ respectively. We use '$\simeq$' to denote 'is homotopic to' and '$\cong$' to denote 'is homeomorphic to'. The statement $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ means that X and Y have the same shape. The *fundamental dimension* of X is defined by $\mathrm{Fd}(X) = \min\{\dim Y \mid \mathrm{Sh}(X) = \mathrm{Sh}(Y)\}$. We refer to [10] for other definitions related to shape theory.

The first *complement theorem* was proved by Chapman [1]. Let X_0 and X_1 be two compacta with $\mathrm{Sh}(X_0) = \mathrm{Sh}(X_1)$. If X_0 and X_1 are nicely embedded (as Z-sets) in the Hilbert cube, then their complements are homeomorphic [1]. In [15], a similar complement theorem is shown to hold if X_0 and X_1 are ILC embedded in n-dimensional Euclidean space, $n \geqslant 5$ and $2\,\mathrm{Fd}(X_0) + 2 \leqslant n$. There are also complement theorems for compacta in manifolds; in a manifold it is necessary to add the additional assumption that the embeddings are homotopic.

Beyond the trivial range, complements of shape equivalent ILC compacta in Euclidean space need not be homeomorphic. For example, if X_0 is the disjoint union of two copies of the 3-sphere and $X_1 \cong X_0$, it is possible to embed X_0 and X_1 into the 7-sphere in such a way that their complements do not even have the same homotopy type. Therefore additional conditions on the embeddings of X_0 and X_1 are needed in order to guarantee the existence of a homeomorphism of their complements. The main result of this paper supplies such conditions. It can be viewed as a version of Stallings' Neighborhood Isotopy Theorem for shape theory.

THEOREM. *Let X_0 and X_1 be two compact* ILC *subsets of the interior of the* PL *n-manifold M, $n \geqslant 5$. If X_0 and X_1 are shape concordant in $M \times I$ and* $\mathrm{Fd}(X_0) \leqslant n-3$, *then* $M - X_0 \cong M - X_1$.

REMARK 1. Partial results for $n = 4$ have been given in [17] and [9], but the general theorem is not known to hold in dimension 4.

REMARK 2. The theorem is a generalization of the complement theorems in [11] and [8]. In those theorems additional hypotheses are imposed on the compacta: in [11] it is assumed that the compacta have the shape of a finite complex and in [8] restrictions are placed on the homotopy pro-groups. We emphasize that neither 1-movability of X_0 nor the Mittag-Leffler condition on pro-$\pi_*(X_0)$ is required here. Our theorem supplies a positive answer to [12, Question 2(a)].

REMARK 3. The theorem is also (indirectly) a generalization of the complement theorems of [6] and [7]. That is so because [5] can be used to supply the shape concordance needed in the hypothesis of our theorem. In order to see this, note that if pro-$\pi_i(X_0)$ is stable for $i < r$ and Mittag-Leffler for $i = r$, then X_0 is pointed r-movable (for example, refer to [2, Theorem 4] and [10, Theorem 6, p. 203]). Thus two compacta X_0 and X_1 which satisfy the hypotheses of [6, Theorem 3] are actually shape concordant by [5, Theorem 1].

One way in which the proof given here differs from that in [8] is that we do not appeal to the theorem in [6], but give a new, independent proof of the complement theorem which uses only the Concordance Implies Isotopy Theorem of Hudson. The main Theorem is a direct consequence of the following Propositions. They spell out two technical facts about shape concordances which may be of independent interest: If ILC compacta are concordant at all then they are concordant via a proper ILC concordance and if the concordance satisfies the proper ILC, then the complement of the concordance is a product.

PROPOSITION 1. *Suppose $X_0, X_1 \subset M^n$, $n \geqslant 5$, are* ILC *compacta which are shape concordant via a compactum X in $M \times I$. If* $\mathrm{Fd}(X_0) \leqslant n-3$, *then there is a concordance Y from X_0 to X_1 such that Y satisfies the proper inessential loops condition.*

PROPOSITION 2. *Suppose* $X_0, X_1 \subset M^n$, $n \geqslant 5$, *are compacta which are shape concordant via a compactum* X *in* $M \times I$. *If* $\mathrm{Fd}(X_0) \leqslant n-3$ *and* X *satisfies the proper inessential loops condition, then* $(M \times I) - X \cong (M - X_0) \times I$.

REMARK 4. Proposition 2 is reminiscent of Hudson's Concordance Implies Isotopy Theorem [3].

2. Construction of proper ILC concordances

In this section we will show how to replace a concordance whose ends are ILC embedded in M with a proper ILC one. Observe that without loss of generality we may assume in all the aforementioned statements that

(a) $X \cap (M \times [0, \frac{1}{3}]) = X_0 \times [0, \frac{1}{3}]$, and
(b) $X \cap (M \times [\frac{2}{3}, 1]) = X_1 \times [\frac{2}{3}, 1]$.

In the following, if A is a subset of B, ι_A will denote the inclusion map $\iota\colon A \to B$.

LEMMA 1. *Let* $X \subset M \times I$ *be a shape concordance between* X_0 *and* X_1 *and let* U *be a given neighborhood of* X. *Then there is a smaller neighborhood* V *of* X *and a homotopy* $h_t\colon V \to U$ *such that*

(1) $h_0 = \iota_V$,
(2) $h_1(V) \subset U_0$, *and*
(3) $h_t \mid V_0 = \iota_{V_0}$ *for every* $t \in I$.

Proof. Since $\iota_{X_0}\colon X_0 \to X$ is a shape equivalence, there is a neighborhood W of X in U, a neighborhood W'_0 of X_0 in W_0 and a homotopy $f_t\colon W \to U$ such that

(1.1) $f_0 = \iota_W$,
(1.2) $f_1(W) \subset U_0$, and
(1.3) $f_1 \mid W'_0 \simeq \iota_{W'_0}$ in U_0.

We must now improve the homotopy to achieve condition (3). Choose V to be a PL manifold neighborhood of X such that $V \subset (W - W_0) \cup W'_0$. By use of (1.3) and the homotopy extension property we can assume that

(1.3)′ $f_1 \mid V_0 = \iota_{V_0}$.

We then use the technique in the proof of [14, Theorem 1.4.11] to find the homotopy h_t that we need.

LEMMA 2. *Let* $X \subset M \times I$ *be a shape concordance from* X_0 *to* X_1 *and let* U *be a neighborhood of* X. *If* X_∂ *satisfies* ILC *in* $(M \times I)_\partial$, *then there exists a polyhedron* Q *in* U_1, *a* PL *map* $g\colon Q \times I \to U$ *and a homotopy* $\beta_t\colon X \to U$ *such that*

(1) Q *is a spine of a neighborhood of* X_1 *in* $U_1 \subset M_1$ *with* $\dim Q \leqslant \mathrm{Fd}(X_1)$,
(2) $g(x, 1) = x$ *for all* $x \in Q$,

(3) $g(x, 0) \in U_0$ *for all* $x \in Q$,
(4) $\beta_0 = \iota_X: X \to U$,
(5) $\beta_1(X) \subset g(Q \times I) \cup P$, *where P is a spine of a neighborhood of* X_0 *in* U_0, *and*
(6) $\beta(X_\partial \times I) \subset U_\partial$.

Proof. Let k denote $\mathrm{Fd}(X_0) = \mathrm{Fd}(X_1) \leqslant n - 3$. Let N_0 be a neighborhood of X_0 in U_0 such that $N_0 \searrow P$ where P is a polyhedron of dimension k from [16, Theorem 4.1]. Let V be a neighborhood of X in $U' = (U - U_0) \cup (\mathrm{Int}\, N_0)$ given by Lemma 1 such that $\iota_V: V \to U'$ is homotopic (rel V_0) to a map $f: V \to U'_0 = \mathrm{Int}\, N_0$. Following this homotopy by the collapse $N_0 \searrow P$ gives us a homotopy $\alpha_t: V \to U' \subset U$ with $\alpha_0 = \iota_V$ and $\alpha_1(V) \subset P$.

From [16, Theorem 4.1], there is a neighborhood N_1 of X_1 such that $N_1 \times [\frac{2}{3}, 1] \subset V$ and $N_1 \searrow Q$, where Q is a subpolyhedron of dimension k. Let r_t denote a strong deformation retraction of N_1 to Q. Let $\varphi: [\frac{2}{3}, 1] \to [0, 1]$ be a PL map with $\varphi([\frac{5}{6}, 1]) = 1$ and $\varphi(\frac{2}{3}) = 0$. Define a homotopy $\mu_t: X \to V$ by $\mu_t = \iota$ on $X \cap (M \times [0, \frac{2}{3}])$, and $\mu_t(x, s) = (r_{t \cdot \varphi(s)}(x), s)$ if $x \in X_1$ and $\frac{2}{3} \leqslant s \leqslant 1$. Notice that $\mu_1(X) \cap (M \times [\frac{5}{6}, 1]) \subset Q \times [\frac{5}{6}, 1]$.

Set $X^1 = \mu_1(X) \cap (M \times [0, \frac{5}{6}])$. Define a homotopy $\lambda_t: \mu_1(X) \to U$ by $\lambda_t \mid X^1 = \alpha_t \mid X^1$, and $\lambda_t \mid Q \times [\frac{5}{6}, 1]$ stretches out the fibers so that λ_t maps $Q \times [\frac{5}{6}, 1]$ onto $(Q \times [\frac{5}{6}, 1]) \cup \alpha((Q \times \{\frac{5}{6}\}) \times [0, t])$. Define g in such a way that $g(Q \times [0, 1]) = Q \times [\frac{5}{6}, 1] \cup \alpha((Q \times \{\frac{5}{6}\}) \times [0, 1])$ and, finally, define β to be the homotopy μ followed by λ.

DEFINITION. If $A \subset K \times I$, we define the *shadow* of A by

$$\Sigma(A) = \{(x, t) \mid (x, s) \in A \text{ for some } s \geqslant t\}.$$

LEMMA 3. *Let* $X \subset M^n \times I$, $n \geqslant 5$, *be a shape concordance from* X_0 *to* X_1 *and* let U be a neighborhood of X. If X_∂ *is* ILC *in* $(M \times I)_\partial$ *and* $\mathrm{Fd}(X_0) = k \leqslant n - 3$, *then there is a subpolyhedron L of U such that*

(1) $\dim L_\partial = k$ *and* $\dim L = k + 1$,
(2) L_1 *is a spine of a neighborhood of* X_1 *in* U_1,
(3) $L \searrow L_0$, *and*
(4) *there is a homotopy* $\beta_t: X \to U$ *such that* $\beta_0 = \iota_X$, $\beta_1(X) \subset L$ *and* $\beta_t(X_\partial) \subset U_\partial$ *for all* $t \in I$.

Proof. The proof is essentially the same as that of [8, Lemma 3], but for the sake of completeness we include an outline here. Let $r = 2k + 2 - n$ and let $V^{(0)} = U$. Define inductively a sequence of neighborhoods of X: $V^{(0)} \supset V^{(1)} \supset \cdots \supset V^{(r)}$ such that each inclusion satisfies the conclusions of Lemma 1, above.

Let $g_1: K \times I \to V^{(r)}$ be a map given by Lemma 2, where K is a spine of a neighborhood of X_1 in $V_1^{(r)}$. Assume that g_1 is in general position with singular set $S^{(1)}$ of dimension $\leqslant r - 1$. We propose to construct L by attaching 'fins' to

$g_1(K\times I)$ along the shadow of the singular set. Let $\Sigma^{(1)}\subset K\times I$ be the shadow of $S^{(1)}$; then $S^{(1)}\subset\Sigma^{(1)}$, $\dim\Sigma^{(1)}\leqslant r$ and $g_1(K\times I)\searrow g_1(\Sigma^{(1)}\cup(K\times\{0\}))$. Let $L^{(1)}=g_1(K\times I)$ and $L^{(1),*}=L^{(1)}\cup(g_1(\Sigma^{(1)})\times I)/\sim$ where $\sim$ is defined by $x\sim(x,0)$ for every $x\in g_1(\Sigma^{(1)})$. By the choice of $V^{(r)}$, we can extend $\iota\colon L^{(1)}\to V^{(r)}$ to a map $g_2\colon L^{(1),*}\to V^{(r-1)}$ such that $g_2(L^{(1),*})\cap M_1=K\times\{1\}$ and $g_2(L^{(1),*})\cap M_0 = g_2[g_1(K\times\{0\})\cup(g_1(\Sigma^{(1)}\cap K\times\{0\})\times I)\cup(g_1(\Sigma^{(1)})\times\{1\})]$. Assume that g_2 is in general position, with singular set $S^{(2)}$ of dimension $\leqslant r-2$ and shadow $\Sigma^{(2)}$. Define $L^{(2)}=g_2(L^{(1),*})\subset V^{(r-1)}$ and $L^{(2),*}=L^{(2)}\cup(g_2(\Sigma^{(2)})\times I)/\sim$. Continue to define $L^{(3)}=g_3(L^{(2),*})\subset V^{(r-2)}$, and so on. Finally, $L=g_{r+1}(L^{(r),*})\subset V^{(0)}$ and $L\searrow L_0$.

From now on, whenever i is an integer, we will define $\bar{i}$ to be 0 if i is even and 1 otherwise. If P is a subpolyhedron of a PL manifold, $N(P)$ denotes a regular neighborhood of P (refer to [4]).

Proof of Proposition 1. Since $M\times I$ is an ANR, the homotopy of X given by Lemma 3 may be extended to a neighborhood of X in $M\times I$. Applying Lemma 3 repeatedly gives a sequence of neighborhoods $U^{(j)}$ of X and polyhedra $L^{(j)}\subset U^{(j)}$ with $X=\bigcap_{j=0}^{\infty}U^{(j)}$ such that for each j there is a homotopy $\beta^{(j+1)}$: $U^{(j+1)}\times I\to U^{(j)}$ such that $\beta_0^{(j+1)}=\iota_{U^{(j+1)}}$, $\beta_1^{(j+1)}\colon U^{(j+1)}\to L^{(j)}$, $\beta^{(j+1)}(U_\partial^{(j+1)}\times I)\subset U_\partial^{(j)}$, and $L^{(j)}\searrow L_{\bar{j}}^{(j)}$. Let $r=2\operatorname{Fd}(X_0)+2-n$. For each $j>r$ and $k>j$, we claim that there is an isotopy h_t of $M\times I$ rel$((M\times I)-U^{(j-r)})\cup(M\times I)_\partial$ such that $h_0=id_{M\times I}$, and $h_1(L^{(k)})\subset N(L^{(j)}\cup U_{\bar{j}}^{(j-r)})$.

Proof of the claim. Make $\beta^{(j+1)}$ a PL map and put the track $\beta^{(j+1)}(L^{(k)}\times I)\subset V^{(j)}$ of the homotopy $\beta^{(j+1)}$ in general position with respect to $L^{(k)}\cup L^{(j)}$. By piping, we can find a polyhedron $\Sigma\subset\beta^{(j+1)}(L^{(k)}\times I)$ such that Σ contains the image of the singular set of $\beta^{(j+1)}$, $\dim\Sigma\leqslant\operatorname{Fd}(X_0)$, and $\beta^{(j+1)}(L^{(k)}\times I)\searrow\Sigma\cup\beta_1^{(j+1)}(L^{(k)})$. The collapse $L^{(j)}\searrow L_{\bar{j}}^{(j)}$ gives a homotopy of $\Sigma\cap L^{(j)}$ down to $U_{\bar{j}}^{(j)}$. Use Lemma 1 to attach cells to $L^{(j)}\cup\Sigma\subset L^{(j)}\cup\beta^{(j+1)}(L^{(k)}\times I)$ as in the proof of Lemma 3 to obtain a polyhedron $P\subset U^{(j-r)}$ with the following properties:

(1) $P\searrow L^{(j)}\cup P_{\bar{j}}$, and
(2) there is an isotopy h_t which is rel$(M\times I)-U^{(j-r)}$ and pushes $L^{(k)}$ into a regular neighborhood of P.

By using thin collars on $(M\times I)_\partial$ in $M\times I$, we may assume that $h_t\,|\,(M\times I)_\partial=id$. On the other hand, observe that $P\cup U_{\bar{j}}^{(j-r)})\searrow L^{(j)}\cup U_{\bar{j}}^{(j-r)}$. Therefore $N(P\cup U_{\bar{j}}^{(j-r)})\cong N(L^{(j)}\cup U_{\bar{j}}^{(j-r)})$ up to PL isotopy. This completes the proof of the claim.

We now return to the proof of Proposition 1. By the claim (taking subsequences, if necessary, and replacing $N(L^{(j)}\cup U_{\bar{j}}^{(j-r)})$ with $h^{-1}(N(L^{(j)}\cup U_{\bar{j}}^{(j-r)}))$), we may assume further that $N(L^{(j+1)}\cup U_{\overline{j+1}}^{(j+1)})\subset$

$N(L^{(j)} \cup U_{\bar{j}}^{(j)})$ for each j. Let $Y = \bigcap_{j=0}^{\infty} N(L^{(j)} \cup U_{\bar{j}}^{(j)})$. Then

$$\bigcap_{\bar{j}=0} N(L^{(j)} \cup U_{\bar{j}}^{(j)}) = Y = \bigcap_{\bar{j}=1} N(L^{(j)} \cup U_{\bar{j}}^{(j)}).$$

Note that Y satisfies the proper ILC since any 2-dimensional polyhedron can be pushed off Y and that $X_0 = Y_0 \hookrightarrow Y$ and $X_1 = Y_1 \hookrightarrow Y$ are shape equivalences since $N(L^{(j)} \cup U_{\bar{j}}^{(j)}) \searrow U_{\bar{j}}^{(j)}$.

3. Complements of proper ILC concordances

In this section we will construct a sequence of neighborhoods of a proper ILC concordance. We will apply Hudson's Concordance Implies Isotopy Theorem to gradually straighten out the neighborhoods and eventually prove that the complement is a product.

DEFINITION. Let $P \subset L$ be two polyhedra in $M \times I$. A collapse $L \searrow P$ is *proper* if the collapse induces a strong deformation retraction which keeps L_0 in $M \times \{0\}$ and L_1 in $M \times \{1\}$.

LEMMA 4. *Suppose $X \subset M^n \times I$, $n \geqslant 5$, is a shape concordance from X_0 to X_1 which satisfies the proper* ILC *and* $\mathrm{Fd}(X_0) = k \leqslant n - 3$. *For every neighborhood U of X there exists a compact* PL *manifold neighborhood V of X in U and a proper collapse $V \searrow P$, where P is a compact polyhedron of dimension $\leqslant n - 2$. Furthermore,* $\dim P_i = k$ *and P_i is a spine of V_i for $i = 0, 1$.*

Proof. Since X_∂ satisfies ILC in $(M \times I)_\partial$, we can use [16, Theorem 4.1] to find a PL manifold neighborhood V of X in U such that

(a) $V \cap (M \times [0, \frac{1}{6}]) = V_0 \times [0, \frac{1}{6}]$,
(b) $V \cap (M \times [\frac{5}{6}, 1]) = V_1 \times [\frac{5}{6}, 1]$, and
(c) for each $i = 0, 1$, $V_i \searrow P_i$, where P_i is a polyhedron of dimension k.

It is then clear that there is a proper collapse $V \searrow \tilde{P}$ for some n-dimensional polyhedron $\tilde{P}$ with $\tilde{P} \cap V_i = P_i$ ($i = 0, 1$). A triangulation of $\tilde{P}$ determines, in a natural way, a handle decomposition of V modulo $V_0 \cup V_1$. This handle decomposition has handles of index $\leqslant n$. We wish to eliminate the handles of index n and $n - 1$; a spine of the remaining manifold is the polyhedron P we seek.

First, consider an n-handle. Its cocore is an arc whose endpoints miss X. Since $\mathrm{Fd}(X) \leqslant n - 3$, we can push the arc off X with an isotopy and thus eliminate the n-handle. Next, consider an $(n - 1)$-handle. Its cocore is a 2-disk whose boundary misses X. We claim that it is possible to push this cocore off X with an ambient isotopy as well and thus eliminate the $(n - 1)$-handle. In order to demonstrate this, we must produce a homotopy which pushes the disk off X, keeping its boundary fixed. Let $X' = X \cap (M \times [\frac{1}{6}, \frac{5}{6}])$. Since X' is in the interior of $M \times I$, it is possible to push the cocore off X' in $M \times I$ [15, Lemma 1]. Projecting $M \times I$ onto $M \times [\frac{1}{6}, \frac{5}{6}]$ gives a homotopy which pushes the cocore into $((U - X) \cap (M \times [\frac{1}{6}, \frac{5}{6}]) \cup U_{1/6} \cup U_{5/6}$. Now apply [15, Lemma 1] again to push the cocore off X_t in U_t for $t = \frac{1}{6}$ and $\frac{5}{6}$.

For convenience we make the following technical definition.

DEFINITION. A *collapsible defining sequence* for a concordance X in $M \times I$ is a sequence of PL manifold neighborhoods $N^{(i)}$ of X in $M \times I$ such that

(1) each $N^{(i+1)} \subset \operatorname{Int} N^{(i)}$,
(2) $X = \bigcap_{i=0}^{\infty} N^{(i)}$, and
(3) $N^{(i)} \searrow N_{\bar{i}}^{(i)}$ for each i.

LEMMA 5. *Let $X \subset M \times I$ be a proper* ILC *concordance from* X_0 to X_1. *If* $\dim M = n \geqslant 5$ *and* $\operatorname{Fd}(X_0) = k \leqslant n - 3$, *then there is a collapsible defining sequence for X in $M \times I$.*

Proof. Choose a sequence of compact PL manifold neighborhoods $\{U^i \mid i = 0, 1, \ldots\}$ of X such that $X = \bigcap_{i=0}^{\infty} U^i$. We will inductively construct $N^{(i)}$ so that $N^{(0)} \subset U^0$ and $N^{(i)} \subset U^i \cap \operatorname{Int} N^{(i-1)}$ for $i > 0$. We will describe the construction for i even (say $i = 0$). The case i odd is handled similarly by just interchanging the subscripts 0 and 1.

Let $r = 2\operatorname{Fd}(X_0) + 2 - n$. Use Lemma 3 to choose a subpolyhedron L of U^r such that $L \searrow L_0 \subset U_0^r$. By Lemma 4 there is a small neighborhood V of X which has a codimension-3 spine P. Lemma 3 gives a homotopy $\beta_t\colon P \to U^r$ satisfying the following properties:

(1) $\beta_0 = \iota_P$,
(2) $\beta_1(P) \subset L$, and
(3) $\beta_t(P_\partial) \subset U_\partial^r$ for all $t \in I$.

We can therefore engulf P (and thus also V) with a regular neighborhood of $L \cup U_0^{r-r} = L \cup U_0^0$ (see the proof of Proposition 1). Let $N^{(0)}$ be a small collar of U_0^0 together with the inverse image of a regular neighborhood of L under the engulfing isotopy. Then $X \subset V \subset N^{(0)} \subset U^0$ and $N^{(0)} \searrow N_0^{(0)} \subset U_0^0$.

To continue, define $U^{(1)}$ to be the neighborhood $V \cap U^r$ of X. From Lemma 3 there is a polyhedron $K \subset U^{(1)}$ such that $K \searrow K_1$. As above, we can produce a PL isotopy $\operatorname{rel}(M \times I - U^{(1)}) \cup (M \times \partial I)$ which takes K to K^* such that $K^* \cup U_1^{(1)}$ has a regular neighborhood $N^{(1)}$ with $X \subset N^{(1)} \subset U^{(1)}$ and $N^{(1)} \searrow N_1^{(1)} \subset U_1^{(1)}$. Continuing this procedure will complete the proof of the Lemma.

Proof of Proposition 2. Let $N^{(0)}$ be a neighborhood of X (from Lemma 5) such that $N^{(0)} \searrow N_0^{(0)}$ and let $K^{(0)}$ be a spine of $N_1^{(0)}$. Regular neighborhood theory shows that $N^{(0)} \cong N_0^{(0)} \times I$. Thus the projection map $K^{(0)} \times \{1\} \to K^{(0)} \subset N_1^{(0)}$ extends to a PL embedding $e: (K^{(0)} \times I, K^{(0)} \times \{0\}) \to (N^{(0)}, N_0^{(0)})$. By [3] there is an isotopy h^1 of $M \times I$ which pushes $e(K^{(0)} \times I)$ to the straight vertical copy $K^{(0)} \times I$ and has compact support. Now use Lemma 5 again to find a smaller neighborhood $N^{(1)}$ of $h_1^1(X)$ such that $N^{(1)} \searrow N_1^{(1)}$ and let $K^{(1)}$ be a spine of $N_0^{(1)}$. Again, $N^{(1)} \cong N_1^{(1)} \times I$, so there exists a PL embedding of $K^{(1)} \times I$ into $N^{(1)}$. Apply [3] again to get an isotopy which straightens out this embedding of $K^{(1)} \times I$. The composition of the two isotopies will push X into $N(K^{(0)}) \times I$, where $N(K^{(0)})$ is a small regular neighborhood of $K^{(0)}$ in M_0.

Now replace M with $N(K_0^{(0)})$. Continuing in the fashion described in the preceding paragraph gives an infinite sequence of isotopies whose composition converges, on $(M \times I) - X$, to a homeomorphism between $(M \times I) - X$ and $(M - X_0) \times I$.

REMARK 5. Proposition 2 can also be proved by applying the proper h-cobordism theorem of Siebenmann [13] to the cobordism $(M \times I - X; M_0 - X_0, M_1 - X_1)$. First use the fact that X is a proper ILC concordance to verify conditions $(\pi_1)_\infty$ and $(H_*)_\infty$ in [13, Proposition IV] and thus conclude that the cobordism is a proper h-cobordism. Then use a collapsible defining sequence for X to prove that both obstructions σ_∞ and τ' vanish.

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