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On the moduli of curves with theta-characteristics

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0. Introduction

Let C be a smooth complete connected curve of genus g (i.e. C is a compact connected Riemann surface of genus g) over the field of complex numbers \mathbf{C} . We denote by K_C the canonical line bundle of C .

DEFINITION. A line bundle \mathcal{L} is called a *theta-characteristic* on C if $\mathcal{L}^2 := \mathcal{L} \otimes \mathcal{L} \simeq K_C$.

If \mathcal{L} is a line bundle on C then ‘deg \mathcal{L} ’ denotes its degree. Since $\deg K_C = 2g - 2$, we have $\deg \mathcal{L} = g - 1$ for any theta-characteristic \mathcal{L} on C . If $g \geq 1$, then $\text{Pic}(C) \simeq \text{Pic}^0(C) \times \mathbf{Z}$ (where $\text{Pic}(C)$ is the Picard group of C ; $\text{Pic}^0(C)$ is the identity component of $\text{Pic}(C)$) and $\text{Pic}^0(C)$ is a complex torus of dimension g . Thus if $g \geq 1$, then there are 2^{2g} theta-characteristics on C .

DEFINITION. Let \mathcal{L} be a line bundle on C . Then \mathcal{L} is said to be even (resp. odd) theta-characteristic if $h^0(\mathcal{L}) := \dim_{\mathbf{C}} H^0(C, \mathcal{L})$ is even (resp. odd).

Among the 2^{2g} theta-characteristics on C , $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$) of them are even (resp. odd) [see [M] p. 190].

It follows from a theorem of Mumford [see [M] p. 184] and the fact about the monodromy action [see [ACGH] p. 294] that the moduli of curves with theta-characteristics (i.e., the variety parametrizing isomorphism classes of (C, \mathcal{L}) , C a curve of genus g as above and \mathcal{L} a theta-characteristic on C) has exactly two connected components \mathcal{M}_g^+ and \mathcal{M}_g^- corresponding to even and odd theta-characteristics respectively. If \mathcal{M}_g denotes the moduli of genus g curves, then we have covering projections $\mathcal{M}_g^+ \rightarrow \mathcal{M}_g$ and $\mathcal{M}_g^- \rightarrow \mathcal{M}_g$ of degree $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ respectively.

Let $\mathcal{M}_g^r \subset \mathcal{M}_g$ be the closure of the locus of all curves C , such that on C there is a theta-characteristic \mathcal{L} with $h^0(\mathcal{L}) = r$, with its natural subscheme structure (see §1). It follows that \mathcal{M}_g^r is the locus of the curves C possessing a theta-characteristic \mathcal{L} with $h^0(\mathcal{L}) \geq r$ and $h^0(\mathcal{L}) \equiv r(2)$. Note if r is 0 or 1 $\mathcal{M}_g^r = \mathcal{M}_g$. In this note, following a suggestion of M. V. Nori, we give a method to compute Zariski tangent spaces to \mathcal{M}_g^r in the moduli-stack. Using the above method we find the dimension of the Zariski tangent spaces to \mathcal{M}_g^r at a hyperelliptic curve.

We give an example of a \mathcal{M}_g^r which is not reduced as a scheme. Also we give example of \mathcal{M}_g^r which is not irreducible.

We refer to Teixidor I Bigas. M. [T], for a detailed study of the above moduli for small r .

It is a great pleasure for me to thank Prof. Madhav. V. Nori, for his help and constant encouragement.

1. Method to compute Zariski tangent space

For the standard facts about moduli [see [T] p. 100]. Let C be a curve of genus g as in the introduction. Let U be a neighbourhood of $[C]$ in a suitable cover of the moduli space \mathcal{M}_g of genus g curves. Note that the tangent space to U at $[C]$ can be identified with $H^1(C, T_C)$, where T_C is the tangent bundle of C . Let \mathbf{C} be the corresponding universal curve over U , i.e., we have a proper smooth morphism

$$\pi: \mathbf{C} \rightarrow U,$$

such that for each point $x \in U$, $C_x := \pi^{-1}(x)$ is a smooth curve of genus g with suitable universal properties. Let L be a line bundle on \mathbf{C} such that for $x \in U$, $L_x := L|_{\pi^{-1}(x)}$ is a theta-characteristic on C_x . Since $\text{deg } L_x = g - 1$, by Riemann-Roch theorem [see [H] p. 295] we see that $h^0(L_x) = h^1(L_x)$, where $h^0(L_x) := \dim_{\mathbf{C}} H^0(C_x, L_x)$ and $h^1(L_x) := \dim_{\mathbf{C}} H^1(C_x, L_x)$. Set $\mathcal{L} = L|_{[C]}$, where $[C]$ is the point of U corresponding C . If $h^0(\mathcal{L}) = r$, then by using semi-continuity theorem [see [H] p. 281–291] we get a morphism (changing U by a suitable neighbourhood of $[C]$, if necessary)

$$\theta: U \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})) \simeq M_r(\mathbf{C}),$$

such that the scheme-theoretic inverse image of the origin is the locus of curves in U corresponding to \mathcal{M}_g^r defined in the introduction. We are interested in the tangent space mapping

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})),$$

of θ at $[C] \in U$.

First note that by Serre’s duality theorem [See [H] p. 295] $H^0(C, \mathcal{L})$ (resp. $H^0(C, K_C^2)$), where $K_C^2 := K_C \otimes K_C$ is naturally dual to $H^1(C, \mathcal{L})$ (resp. $H^1(C, T_C)$).

THEOREM 1. *The mapping*

$$\Theta^\vee: H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) \rightarrow H^0(C, K_C^2),$$

defined by

$$(fe_1, ge_1) \mapsto (fdg - gdf)e_1^2,$$

(where $fe_1, ge_1 \in H^0(C, \mathcal{L})$), is dual to Θ (up to a scalar multiple).

Proof. Let $t \in H^1(C, T_C)$. First we describe the homomorphism

$$\Theta(t): H^0(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L}).$$

Choose an affine covering $\{U_1, U_2\}$ of C such that

$$\mathcal{L}(U_1) \simeq \mathcal{O}_C(U_1)e_1 \quad \text{and} \quad \mathcal{L}(U_2) \simeq \mathcal{O}_C(U_2)e_2$$

and also we have

$$K_C(U_1) \simeq \mathcal{O}_C(U_1)e_1^2 = \mathcal{O}_C(U_1)h \, da$$

$$K_C(U_2) \simeq \mathcal{O}_C(U_2)e_2^2 = \mathcal{O}_C(U_2)h_1 \, db,$$

where \mathcal{O}_C is the structure sheaf of C and a, b, h, h_1 are rational functions on C . If $\alpha_{12} \in \mathcal{O}_C(U_1 \cap U_2)^*$ is the transition function of \mathcal{L} , then by our assumption on \mathcal{L} , α_{12}^2 is the transition function for K_C , where $\mathcal{O}_C(U_1 \cap U_2)^*$ is the group of invertible elements of $\mathcal{O}_C(U_1 \cap U_2)$. Now $t \in H^1(C, T_C)$ gives an infinitesimal deformation $C_t[\varepsilon]$ as follows: let $D: \mathcal{O}_C(U_1 \cap U_2) \rightarrow \mathcal{O}_C(U_1 \cap U_2)$ be the derivation corresponding to 't', then $C_t[\varepsilon]$ is defined by gluing

$$\text{Spec} \left(\mathcal{O}_C(U_1) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right) \quad \text{and} \quad \text{Spec} \left(\mathcal{O}_C(U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right)$$

along $\text{Spec}(\mathcal{O}_C(U_1 \cap U_2) \otimes \mathbf{C}[\varepsilon]/(\varepsilon^2))$ by the function

$$f \mapsto f + \varepsilon D(f).$$

If $K_{C_t[\varepsilon]}$ is the relative cononical bundle of $C_t[\varepsilon] \rightarrow \text{Spec}(\mathbf{C}[\varepsilon]/(\varepsilon^2))$, then it is easy to verify that $K_{C_t[\varepsilon]}$ is given by the transition function

$$\alpha_{12}^2 \left(1 + \varepsilon \left(\frac{d(D(a))}{da} + \frac{D(h)}{h} \right) \right) \in \left(\mathcal{O}_C(U_1 \cap U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right)^*.$$

Then

$$\alpha_{12} \left(1 + \frac{1}{2} \varepsilon \left(\frac{d(D(a))}{da} + \frac{D(h)}{h} \right) \right)$$

gives transition function for a line bundle \mathcal{L}_1 on $C_t[\varepsilon]$ such that $\mathcal{L}_1^2 \simeq K_{C_t[\varepsilon]}$ and $\mathcal{L}_1|_C \simeq \mathcal{L}$. Also on $C_t[\varepsilon]$ we have an exact sequence

$$0 \rightarrow \varepsilon\mathcal{L} \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L} \rightarrow 0.$$

From this exact sequence we get a coboundary homomorphism

$$\psi: H^0(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L}).$$

Using Čech-cohomology with respect to the covering $\{U_1, U_2\}$, we get

$$\psi(fe_1) = \left(\frac{f}{2} \left(\frac{d(D(a))}{da} + \frac{D(h)}{h} \right) + D(f) \right) e_1,$$

where $fe_1 \in H^0(C, \mathcal{L})$. But $\Theta(t)$ is nothing but ψ . Note that if $fe_1, ge_1 \in H^0(C, \mathcal{L})$ then cup product of $\psi(fe_1)$ and fe_1 gives an element

$$\psi(fe_1)ge_1 \in H^1(C, K_C).$$

But

$$\begin{aligned} \psi(fe_1)ge_1 &= \left(\frac{gf}{2} \left(\frac{d(D(a))}{da} + \frac{D(h)}{h} \right) + D(f)g \right) e_1^2 \\ &= \frac{fgh}{2} d(D(a)) + \frac{gfD(h)}{2} da + (ghD(f)) da \\ &= \frac{fgh}{2} d(D(a)) + \frac{fgD(a)}{2} dh + ghD(a) df \\ &= \frac{D(a)h}{2} (f dg - g df) + \frac{1}{2}d(fghD(a)) \end{aligned}$$

(In the above equation we have used the fact $D(h)da = D(a)dh$ and $D(a)df = D(f)da$). So if $p \in C - U_1$, then

$$\text{res}_p(\psi(fe_1)ge_1) = \text{res}_p \left(\frac{D(a)h}{2} (f df - g dg) \right).$$

On the other hand the derivation D corresponding to $t \in H^1(C, T_C)$ induces (by cup product) a homomorphism

$$D: H^0(C, K_C^2) \rightarrow H^1(C, K_C).$$

Composing the homomorphism with

$$\text{res} := \sum \text{res}_p: H^1(C, K_C) \rightarrow \mathbf{C},$$

where summation is over all $p \in C - U_1$, gives that the homomorphism

$$\begin{aligned} H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) &\rightarrow H^0(C, K_C^2) \\ (fe_1, ge_1) &\mapsto (fdg - gdf)h \, da, \end{aligned}$$

is dual to the homomorphism (up to a scalar multiple)

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})).$$

This proves the theorem.

COROLLARY 1. *Image of Θ is contained in the set of alternating matrices.*

Proof. The corollary follows immediately from the theorem because Θ^\vee is clearly zero on symmetric tensors.

COROLLARY 2. [See [Ha] p. 616]. *If $\mathcal{M}_g^r \neq \emptyset$, then every irreducible component of \mathcal{M}_g^r has codimension at most $(r(r - 1))/2$ in \mathcal{M}_g .*

Proof. From the corollary (1) above and the definition of \mathcal{M}_g^r , corollary (2) follows immediately.

2. Examples

First we compute the tangent space map described above, at hyperelliptic curve. If C is a hyperelliptic curve of genus g , then C is the normalization of the plane curve

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i),$$

($a_i \in \mathbf{C}$ and $a_i \neq a_j$ for $1 \leq i, j \leq 2g + 2$). Then

$$H^0(C, K_C) = \mathbf{C} \frac{dx}{y} \oplus \mathbf{C}x \frac{dx}{y} \oplus \cdots \oplus \mathbf{C}x^{g-1} \frac{dx}{y},$$

and

$$\begin{aligned}
 H^0(C, K_C^2) &= \mathbf{C} \left(\frac{dx}{y} \right)^2 \oplus \mathbf{C}x \left(\frac{dx}{y} \right)^2 \oplus \dots \\
 &\quad \oplus \mathbf{C}x^{2g-2} \left(\frac{dx}{y} \right)^2 \oplus \mathbf{C}y \left(\frac{dx}{y} \right)^2 \oplus \dots \\
 &\quad \oplus \mathbf{C}x^{g-3}y \left(\frac{dx}{y} \right)^2.
 \end{aligned}$$

Given integer r ($0 \leq r \leq [(g + 1)/2]$), then set $s = (g - 1) - 2(r - 1)$. If

$$\mathcal{L} = \pi^* \mathcal{O}_{\mathbf{P}^1}(r - 1) \otimes \mathcal{O}_C \left(\sum_{k=1}^s t_{i_k} \right),$$

where $\pi: C \rightarrow \mathbf{P}^1$ is the covering ramified precisely over a_i ($1 \leq i \leq 2g + 2$) and $t_{i_k} \in \pi^{-1}(\{a_1, \dots, a_{2g+2}\})$, then \mathcal{L} is a theta-characteristic on C with $h^0(\mathcal{L}) = r$. Conversely every theta-characteristic \mathcal{L} on C with $h^0(\mathcal{L}) = r$, is of the above form. Fix a theta-characteristic \mathcal{L} on C with $h^0(\mathcal{L}) = r$, then

$$\Theta^\vee: H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) \rightarrow H^0(C, K_C)$$

is induced by

$$(x^a e_1, x^b e_1) \rightarrow (x^a dx^b - x^b dx^a) e_1^2 = (b - a)x^{a+b-1}y \left(\frac{dx}{y} \right)^2,$$

(where $x^a e_1, x^b e_1 \in H^0(C, \mathcal{L})$). Now if $r \geq 2$ it is easy to see that image of Θ^\vee is a $2r - 3$ dimensional subspace of $H^0(C, K_C^2)$. So by the above theorem if $r \geq 2$ it follows that at (C, \mathcal{L}) the tangent mapping

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L}))$$

has rank $(2r - 3)$, hence the $\ker(\Theta)$ is of codimension $2r - 3$ in $H^1(C, T_C)$. Thus we have proved the following:

THEOREM 2. *In a suitable covering space of \mathcal{M}_g , the Zariski tangent space to \mathcal{M}_g^r ($r \geq 2$) at (C, \mathcal{L}) has dimension $3g - 2r$, where C is an hyperelliptic curve and \mathcal{L} is a theta-characteristic on C with $h^0(\mathcal{L}) = r$.*

THEOREM 3. *\mathcal{M}_8^4 is non-reduced scheme of dimension 15 and $(\mathcal{M}_8^4)_{\text{red}}$ is the locus of hyperelliptic curves.*

Proof. By Theorem (2), if $(C, \mathcal{L}) \in \mathcal{M}_8^4$ is such that C is hyperelliptic curve of

genus 8 and \mathcal{L} a theta-characteristic on C with $h^0(\mathcal{L}) = 4$, then the Zariski tangent space at (C, \mathcal{L}) is of dimension 16. On the other hand we show that if C is a curve of genus 8 with a theta-characteristic \mathcal{L} such that $h^0(\mathcal{L}) = 4$, then C is hyperelliptic, this will prove the theorem.

CLAIM. *If $(C, \mathcal{L}) \in \mathcal{M}_8^4$ then C hyperelliptic.*

Proof (of the claim). Suppose \mathcal{L} has a base point p , then $\mathcal{L}(-p)$ is a degree 6 line bundle on a genus 8 curve with 4 linearly independent sections, hence by Clifford's theorem [see [H] p. 343] C is hyperelliptic, so claim is proved if \mathcal{L} has a base point. Hence we can assume that \mathcal{L} has no base point. Let

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^3$$

be the corresponding morphism. Since $h^0(\mathcal{L}^2) = 8$ and $h^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 10$, there are at least two linearly independent quadrics vanishing on $\phi_{\mathcal{L}}(C)$. This implies, since $\phi_{\mathcal{L}}(C)$ is not contained in any hyperplane, degree of $\phi_{\mathcal{L}}(C)$ in \mathbf{P}^3 is ≤ 4 . But $\deg \mathcal{L} = 7$, so \mathcal{L} must have a base point which contradicts our assumption on \mathcal{L} . This proves the claim.

Since locus of hyperelliptic curves is a 15 dimensional subvariety of \mathcal{M}_8 theorem follows.

Next we will describe moduli \mathcal{M}_g^3 for small g . Note that $\mathcal{M}_g^3 = \emptyset$ for $1 \leq g \leq 4$ by Clifford's theorem. For $g \geq 5$ we have the following:

THEOREM [see [T] p. 113]. *The locus \mathcal{M}_g^3 has pure codimension in \mathcal{M}_g if $g \geq 5$, and a generic point of any of its components is a curve C which has only one \mathcal{L} with $\mathcal{L}^2 \simeq \mathcal{K}_C$ such $h^0(\mathcal{L}) = 3$ if $g \geq 6$. Moreover this theta-characteristic gives a birational morphism of C into \mathbf{P}^2 if $g \geq 6$.*

- (1) *When $g = 5$, it follows by Clifford's theorem that \mathcal{M}_5^3 is precisely the locus of hyperelliptic curves.*
- (2) *Next $g = 6$. Let C be a curve of genus 6 and \mathcal{L} be a theta-characteristic on C with $h^0(\mathcal{L}) = 3$. If \mathcal{L} has no base point then clearly \mathcal{L} gives embedding of C in \mathbf{P}^2 . Locus $(\mathcal{M}_6^3)^0$ smooth plane curves of degree 6 is locally closed in moduli \mathcal{M}_6 of genus 6 curves and again by Clifford's theorem it follows that $\mathcal{M}_6^3 = (\mathcal{M}_6^3)^0 \cup \mathcal{H}_6$, where \mathcal{H}_6 is the locus of hyperelliptic curves.*

THEOREM 4. *\mathcal{M}_7^3 is an irreducible subvariety of dimension 15 in the moduli-space \mathcal{M}_7 .*

Proof. By tangent space computations it follows that \mathcal{M}_7^3 has dimension ≥ 15 . It follows by Clifford's theorem that if $C \in \mathcal{M}_7^3$ and if the corresponding theta-characteristic \mathcal{L} on C has a base point then C must be a hyperelliptic curve. But moduli of hyperelliptic curves is of dimension 13, hence C cannot be a general

member of \mathcal{M}_7^3 . So on a general member $C \in \mathcal{M}_7^3$ there exists theta-characteristic \mathcal{L} with $h^0(\mathcal{L}) = 3$ and \mathcal{L} does not have base points. Let

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2$$

be the corresponding morphism. Then the image curve can have degree 2, 3 or 6. But again by dimension count we get that if C is general then image C is a degree 6 in \mathbf{P}^2 , hence $\phi_{\mathcal{L}}$ is birational onto its image. Since $\mathcal{L}^2 \simeq K_C$, we see that image of C under $\phi_{\mathcal{L}}$ is a degree 6 curve having exactly three ordinary double points lying on a line and no other singularities. Now fix a line $l \subset \mathbf{P}^2$ and three distinct points p_1, p_2, p_3 on l then the exact sequence

$$0 \rightarrow \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \bigoplus_{i=1}^3 \frac{\mathcal{O}_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2} \rightarrow 0,$$

where $m_{\mathbf{P}^2, p_i}$ is the ideal sheaf of the point $p_i \in \mathbf{P}^2$, after tensoring with $\mathcal{O}_{\mathbf{P}^2}(6)$ gives the following cohomology exact sequence

$$\begin{aligned} 0 \rightarrow H^0\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) &\rightarrow H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6)) \\ \rightarrow \bigoplus_{i=1}^3 H^0\left(\mathbf{P}^2, \frac{\mathcal{O}_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}(6)\right) &\rightarrow H^1\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) \rightarrow 0. \end{aligned}$$

But it is easy to see that

$$H^1\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) = 0.$$

Now using the fact that $p_i (1 \leq i \leq 3)$ lie on a line and Bertini's theorem [See [H], p. 274] we get an open set

$$U \subset \mathbf{P}\left(H^0\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right)\right),$$

such that if $C \in U$ then C is irreducible plane curve of degree 6 and has double points at $p_i (1 \leq i \leq 3)$ and no other singularities. Note that $\dim U = 18$ and general member is a nodal curve. If we vary $l \subset \mathbf{P}^2$ and $p_i \in l (1 \leq i \leq 3)$, we get a 23 dimensional irreducible locally closed subvariety W of $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6)))$ such that if $C \in W$ then C has exactly three ordinary double points all of them lie on a line and has no other singularities. On W , $\text{PGL}(3)$ acts with finite stabilizer at each of its points. Now the quotient V of W by $\text{PGL}(3)$ gives a dense open subset of \mathcal{M}_7^3 . Since dimension of V is 15 and V is irreducible theorem follows immediately.

THEOREM 5. \mathcal{M}_8^3 is an irreducible subvariety of dimension 18 in the moduli space \mathcal{M}_8 .

Proof. By the theorem quoted above of Teixidor I Bigas, each irreducible component of \mathcal{M}_8^3 is 18 dimensional and whose general member C has a theta-characteristic \mathcal{L} such that \mathcal{L} gives a birational morphism

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2.$$

As above using the fact that $\mathcal{L}^2 \simeq K_C$, we get $\phi_{\mathcal{L}}(C)$ is a curve of degree 7 and has exactly 7 ordinary double points all of them lie on smooth conic and has no other singularities. Fix a smooth conic $E \subset \mathbf{P}^2$ and 7 distinct points p_1, \dots, p_7 on it. Consider the exact sequence

$$0 \rightarrow \prod_{i=1}^7 m_{p_i}^2 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \bigoplus_{i=1}^7 \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \rightarrow 0,$$

where m_{p_i} is the ideal sheaf of p_i in \mathbf{P}^2 . It is easy to see that

$$H^1 \left(\mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) = 0.$$

Hence from the above exact sequence, after tensoring with $\mathcal{O}_{\mathbf{P}^2}(7)$ we get a cohomology exact sequence

$$0 \rightarrow H^0 \left(\mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) \rightarrow H^0 \left(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7) \right) \rightarrow \bigoplus_{i=1}^7 \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \rightarrow 0.$$

Again it is easy to see that there exists an open set

$$U \subset \mathbf{P} \left(H^0 \left(\mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) \right)$$

such that every curve parametrized by U is irreducible and has ordinary double points exactly at the points p_1, \dots, p_7 and no other singularities. Now varying the conic and the 7 points on it we get a 26 dimensional irreducible locally closed sub variety W of $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7)))$, on which $\text{PGL}(3)$ acts with finite stabilizer at each of its points. The quotient V , of W by $\text{PGL}(3)$ is a dense open subset of \mathcal{M}_8^3 . Thus \mathcal{M}_8^3 is a irreducible codimension 3 subvariety of \mathcal{M}_8 .

THEOREM 6. \mathcal{M}_9^3 has exactly two irreducible components each of dimension 21 in \mathcal{M}_9 .

Proof. Again by the theorem of Teixidor I Bigas, each irreducible component

of \mathcal{M}_3^3 is 21 dimensional and whose general members is a curve C with a theta-characteristic \mathcal{L} which give rise to a birational morphism

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2.$$

As above the fact that \mathcal{L} is theta-characteristic gives that $\phi_{\mathcal{L}}(C)$ is a curve of degree 8 and has exactly 12 ordinary double points all of which lie on degree 3 curve and has no other singularities. Also note that the above 12 points on the degree 3 curve has the property twice the sum of these 12 points is the zeros of a section of $\mathcal{O}_{\mathbf{P}^2}(8)$ restricted to the degree 3 curve. Since we are interested in an open subset of \mathcal{M}_3^3 , we look at curves C as above with corresponding singularities of $\phi_{\mathcal{L}}(C)$ lie on a smooth degree 3 curve. We fix a smooth degree 3 curve $E \subset \mathbf{P}^2$ and 12 distinct points p_1, \dots, p_{12} on it such that $2 \sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2}(8)|_E))$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(5)) & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8)) & \rightarrow & H^0(E, \mathcal{O}_E(8)) \rightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(5)) & \rightarrow & V & \rightarrow & \mathbf{C}\sigma \rightarrow 0 \end{array}$$

where σ is a section of $\mathcal{O}_E(8)$ corresponding to $2 \sum_{i=1}^{12} p_i$, V is the inverse image of $\mathbf{C}\sigma$ in $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8))$, $\dim_{\mathbf{C}} V = 22$. From V we have the following mapping

$$V \rightarrow \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}$$

whose image is the 12 dimensional subspace

$$\bigoplus_{i=1}^{12} \frac{m_{E, p_i}}{m_{E, p_i}^2}.$$

So

$$W = \ker \left(V \rightarrow \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2} \right)$$

is a 10 dimensional subspace of $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8))$. Again by Bertini's theorem $\mathbf{P}(W)$ contains an open set $U_{(E, (p_i))}$ such that if C is a curve corresponding to a point of $U_{(E, (p_i))}$ then C irreducible degree 8 curve which has ordinary double points at p_i ($1 \leq i \leq 12$) and has no other singularities. Now the variety H parametrizing $(E, \sum_{i=1}^{12} p_i)$, $E \subset \mathbf{P}^2$ degree 3 smooth curve, p_1, \dots, p_{12} distinct points on E such that $2 \sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2}(8)|_E))$ is 20 dimensional. Note that H has two

connected (irreducible) components (see, Introduction) of dimension 20 corresponding to two types of points p_1, \dots, p_{12} namely whether $\sum_{i=1}^{12} p_i$ is in $\mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2(4)}|_E))$ or not. The above construction gives a variety X fibred over H with fibres the 9 dimensional variety $U_{(E,(p_i))}$. On $X = X_1 \cup X_2$, $\mathbf{PGL}(3)$ acts with finite stabilizer at each of its point and the quotient W is a dense open subset of \mathcal{M}_9^3 . This proves that \mathcal{M}_9^3 has two irreducible components.

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