

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 76, n° 1-2 (1990), p. 1-5

http://www.numdam.org/item?id=CM_1990__76_1-2_1_0

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2-Cocycles and Azumaya algebras under birational transformations of algebraic schemes

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Received 17 November 1988; accepted in revised form 23 November 1989

The basic question whether the injection $\text{Br}(X) \rightarrow H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ is an isomorphism arose at the very definition of the Brauer group of an algebraic scheme X . Positive answers are known in the following cases:

1. the topological Brauer group $\text{Br}(X_{\text{top}}) \cong H^2(X, \mathcal{O}_{\text{top}}^*)_{\text{tors}} \cong H^3(X, \mathbb{Z})_{\text{tors}}$ (J.-P. Serre); in the étale (algebraic) case the isomorphism is proved for
2. smooth projective surfaces (A. Grothendieck);
3. abelian varieties;
4. the union of two affine schemes (R. Hoobler, O. Gabber).

The first author has formulated a birational variant of the basic question, while considering the unramified Brauer group in [1]. The group $\text{Br}_V(K(X)) = \bigcap \text{Br}(A_v) \subseteq \text{Br}(K(X))$ (intersection taken over all discrete valuation subrings A_v of the rational function field $K(X)$) is isomorphic to $H^2(\tilde{X}, \mathcal{O}^*)$, where \tilde{X} is a nonsingular projective model of X , i.e. a nonsingular projective variety birationally equivalent to X .

QUESTION. Given a cocycle class $\gamma \in H^2(\tilde{X}, \mathcal{O}^*)$, is it possible to find a nonsingular projective model \tilde{X} such that γ is represented by a \mathbb{P}^n -bundle (i.e. by an Azumaya algebra) on \tilde{X} ?

The case where X is a nonsingular projective model of V/G , with G a γ -minimal group and V a faithful representation of G , was considered in [2]. O. Gabber in his letter to Bogomolov (12.1.1988) has given an affirmative answer to the question in the case of general algebraic spaces. In this paper we give a simple version of his proof for algebraic schemes.

Let X be a scheme, $\gamma \in H^2(X, \mathcal{O}^*)$, $\{U_i\}$ an affine cover of X . Then the restriction of γ to each U_i is represented by an Azumaya algebra A_i . If we would have isomorphisms $A_i|_{U_i \cap U_j} \cong A_j|_{U_i \cap U_j}$, we could glue the sheaves $\{A_j\}$ and get an

Azumaya algebra on X , representing γ . But we have isomorphisms $A_{ij|U_i \cap U_j} \otimes \text{End}(E_{ij}) \cong A_{j|U_i \cap U_j} \otimes \text{End}(E_{ji})$ for certain vector bundles E_{ij}, E_{ji} on $U_i \cap U_j$.

THEOREM. *Let X be a noetherian scheme, $\gamma \in H^2(X, \mathcal{O}_X^*)$. There exists a proper birational morphism $\alpha: \tilde{X} \rightarrow X$ such that $\alpha^*(\gamma)$ is represented by an Azumaya algebra on \tilde{X} .*

Proof. It is enough to consider X which are connected. Suppose that $\{U_i\}$ is an affine open cover of X and that γ is non-trivial on at least one U_i . We will construct an Azumaya algebra on a birational model of X by an inductive process which involves adjoining one by one proper preimages of the subsets U_i and, by an appropriate birational change of the scheme and Azumaya algebra obtained, extending the new algebra to the union. We start with some affine open subset U_0 and an Azumaya algebra A_0 on it.

Now suppose by induction that we already have an Azumaya algebra \bar{A}_k on the scheme X_k , a Zariski-open subset of the scheme \bar{X}_k , equipped with a proper birational map $\bar{\alpha}_k: \bar{X}_k \rightarrow X$ such that $X_k = \bar{\alpha}_k^{-1}(U_0 \cup \dots \cup U_k)$. Let U_{k+1} intersect $U_0 \cup \dots \cup U_k$ and $\bar{U}_{k+1} = \bar{\alpha}_k^{-1}(U_{k+1})$. Suppose that on U_{k+1} , γ is represented by the Azumaya algebra A_{k+1} . In the same vein as above we have an isomorphism

$$\bar{A}_k|_{\bar{X}_k \cap \bar{U}_{k+1}} \otimes \text{End}(E_{k,k+1}) \cong \bar{\alpha}_k^*(A_{k+1})|_{\bar{X}_k \cap \bar{U}_{k+1}} \otimes \text{End}(E_{k+1,k})$$

and we need to extend $E_{k,k+1}$ to X_k and $E_{k+1,k}$ to \bar{U}_{k+1} from their intersection. After this we will change \bar{A}_k and $\bar{\alpha}_k^*(A_{k+1})$ by the other representatives $\bar{A}_k \otimes \text{End}(E_{k,k+1})$, $\bar{\alpha}_k^*(A_{k+1}) \otimes \text{End}(E_{k+1,k})$ of the same Brauer classes and glue these Azumaya algebras, hence the proof.

First, extend both sheaves E as coherent sheaves. This can be done by the following

LEMMA. *Let X be a noetherian scheme, $U \subseteq X$ a Zariski-open subset, E a coherent sheaf on U . Then there exists a coherent sheaf E' on X such that $E'|_U \cong E$. This is Ex. II.5.15 in [4].*

Note that we can assume that in our inductive process we add neighborhoods U_{k+1} of no more than one irreducible component (or an intersection of irreducible components) of X , different from those contained in X_k . Thus we assume $X_k \cap U_{k+1}$ to be connected and the rank of E to be constant on $X_k \cap U_{k+1}$, hence E' will be locally generated by n elements, where n is the rank of E .

LEMMA (see [3], Lemma 3.5). *Let X be a noetherian scheme, E a coherent sheaf on X , locally free outside a Zariski closed subset Z on X . Then there exists a coherent sheaf I of ideals on X such that the support of \mathcal{O}_X/I is Z with the following*

property. Let $\alpha: \tilde{X} \rightarrow X$ be the blowing up of X with center I , then the sheaf $\bar{\alpha}(E) :=$ the quotient of $\alpha^*(E)$ by the subsheaf of sections with support in $\alpha^{-1}(Z)$, is locally free on \tilde{X} .

Proof. The proof consists of two parts. First: to reduce the number of local generators to get this number constant on the connected components of X (the minima are the values of the (local) rank function of E). Second, to force the kernel of the (local) presentations $\mathcal{O}_V^m \rightarrow E|_V \rightarrow 0$ to vanish for all neighborhoods from some cover $\{V\}$. Both parts are proved by indicating the suitable coherent sheaves of ideals and blowing up X with respect to these sheaves. Let $\mathcal{O}_V^m \xrightarrow{f} E|_V \rightarrow 0$ be a local presentation of E . Then $\text{Ker}(f)$ is generated by all relations $\sum_{i=1}^m c_i a_i = 0$ where $\{a_i\}$ stand for the free basis of \mathcal{O}_V^m . The coherent sheaf of ideals in the first case is the sheaf defined locally as the ideal I_V in \mathcal{O}_V generated by all c_i such that $\sum_{i=1}^m c_i a_i \in \text{Ker}(f)$ and in the second case as $J_X = \text{Ann}(\text{Ker}(f))$. As the number of generators is constant in the case we are interested in, we give the details only for the second part of the proof and refer to [3] for the first.

Let $\alpha: X' \rightarrow X$ be the blowing up of X with respect to J_X and let $\bar{\alpha}(E)$ be as in the statement of the Lemma. Let

$$0 \rightarrow (\text{Ker}(\bar{f}))|_{V'} \rightarrow \mathcal{O}_{V'}^m \xrightarrow{\bar{f}} \bar{\alpha}(E)|_{V'} \rightarrow 0$$

be the local presentation of $\bar{\alpha}(E)$. We have $\alpha^{-1}(\text{Ann}(f)) \subseteq \text{Ann}(\text{Ker}(\bar{f}))$. Let $\mathfrak{p} \in Z'$, $V' = \text{Spec}(A')$ an affine neighborhood of \mathfrak{p} in X' and let $\sum_{i=1}^m c_i a_i \in \text{Ker}(\bar{f})|_{V'}$ map to a nonzero element in $\text{Ker}(\bar{f})_{\mathfrak{p}}$. Denote by γ a generator of the invertible sheaf $\alpha^{-1}(\text{Ann}(f))$ on $V'' = \text{Spec}(A'') \subseteq V'$ for suitable A'' . It is clear that there exists for given \mathfrak{p} and V'' a finite sequence of open affine neighborhoods V''_1, \dots, V''_s such that $X' \setminus Z' = V''_1, V'' = V''_s$ and $V''_j \cap V''_{j+1} \neq \emptyset$ for $j = 1, \dots, s-1$. So suppose $V' \cap (X' \setminus Z') \neq \emptyset$ and $\mathfrak{q} \in V'' \cap (X' \setminus Z')$. Then $(c_i)_{\mathfrak{q}} = 0$ for $i = 1, \dots, m$ and $\mathfrak{q} \in \text{Spec}(A''_i)$ hence $\gamma^k c_i = 0$ for $i = 1, \dots, m$ for some k . Since γ is not a zero divisor, we conclude that $c_i = 0$ for $i = 1, \dots, m$. Thus (maybe after considering a finite sequence of points $\mathfrak{q}_1, \dots, \mathfrak{q}_s$) we prove that $(\text{Ker}(f))_{\mathfrak{p}}$ is trivial for every $\mathfrak{p} \in X'$. \square

In this way we glue the two sheaves \bar{A}_k and A_{k+1} and get an Azumaya algebra on $\tilde{X}_k \cup \tilde{U}_{k+1}$. As the scheme X is quasi-compact, we obtain an Azumaya algebra on \tilde{X} after a finite number of such steps.

Now we have to show that this process can be done in such a way that the class $[A]$ of the Azumaya algebra A constructed in this way is equal to $\bar{\alpha}^*(\gamma)$. Again this goes by induction on k . We have $X_{k+1} = U \cup V$ with $U = \bar{\alpha}_{k+1}^{-1}(U_0 \cup \dots \cup U_k)$ and $V = \bar{\alpha}_{k+1}^{-1}(U_{k+1})$. We have the exact sequence

$$H^1(U \cap V, \mathcal{O}^*) \rightarrow H^2(X_{k+1}, \mathcal{O}^*) \rightarrow H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$$

and by induction hypothesis, $\bar{\alpha}_{k+1}^*(\gamma) - [\bar{A}_{k+1}]$ maps to zero in $H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$ so it comes from $\beta \in H^1(U \cap V, \mathcal{O}^*)$. By blowing up X_{k+1} we may assume that β is represented by a line bundle which extends to U . Then β maps to zero in $H^2(X_{k+1}, \mathcal{O}^*)$, hence $\bar{\alpha}_{k+1}^*(\gamma) - [\bar{A}_{k+1}] = 0$. \square

Note that we need not bother about the compatibility of isomorphisms, because at each step we choose a new isomorphism between the Azumaya algebra A on $U_1 \cup \dots \cup U_j$ from the preceding step and A_k on U_k , modulo $\text{End}(E)$, $\text{End}(E_k)$.

COROLLARY 1. *Let G be a finite group, V a faithful complex representation of G . Then there exists a nonsingular projective model X of V/G such that $\text{Br}(X) = H^2(X, \mathcal{O}^*)$.*

Proof. The group $H^2(X, \mathcal{O}^*)$ is a birational invariant of nonsingular projective varieties and is isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$ if X is a model of V/G (see [1]). It remains to recall that the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is finite. \square

COROLLARY 2. *Let X be a noetherian scheme over \mathbb{C} , Z a closed subscheme of X and $\gamma \in H_Z^2(X, \mathcal{O}^*)$. Then there exists a proper morphism $\alpha: X' \rightarrow X$ which is an isomorphism above $X \setminus Z$ and maps γ to zero in $H_{\alpha^{-1}(Z)}^2(X', \mathcal{O}^*)$.*

Proof. First, let's have $\alpha(\gamma)$ map to zero in $H^2(X, \mathcal{O}^*)$. To do this, desingularize X by $X' \rightarrow X$. Then in the following exact sequence (in étale cohomology), β will be injective:

$$\begin{array}{ccccccc} H^1(X' \setminus Z', \mathcal{O}^*) & \rightarrow & H_Z^2(X', \mathcal{O}^*) & \rightarrow & H^2(X', \mathcal{O}^*) & \xrightarrow{\beta} & H^2(X' \setminus Z', \mathcal{O}^*) \\ & & & & \uparrow & & \\ & & & & H_Z^2(X, \mathcal{O}^*) & & \end{array}$$

The injectivity is due to the injectivity of $H^2(X', \mathcal{O}^*) \rightarrow H^2(K(X'), \mathcal{O}^*)$ for a nonsingular irreducible scheme X' .

Now γ comes from $\gamma' \in H^1(X' \setminus Z', \mathcal{O}^*) = \text{Pic}(X' \setminus Z')$. It is obvious that Picard elements lift to Picard elements by the blowing ups from the theorem. Thus from the diagram

$$\begin{array}{ccccc} H^1(X'', \mathcal{O}^*) & \rightarrow & H^1(X'' \setminus Z'', \mathcal{O}^*) & \rightarrow & H_Z^2(X'', \mathcal{O}^*) \\ & & \uparrow & & \uparrow \\ & & H^1(X' \setminus Z', \mathcal{O}^*) & \rightarrow & H_Z^2(X', \mathcal{O}^*) \end{array}$$

we conclude that γ becomes trivial on Z'' by $X'' \rightarrow X'$ which extends γ' to X'' . \square

Now let us return to the problem of an isomorphism $\text{Br}(X) \rightarrow H^2(X, \mathcal{O}^*)$ for

nonsingular quasi-projective varieties. The theorem reduces the general problem to the following

QUESTION. Let X' be a blowing up of a nonsingular variety X along a smooth subvariety S and let A' be an Azumaya algebra on X' . Does there exist an Azumaya algebra A on X such that its inverse image on X' is equivalent to A' ?

In case the restriction of A' to the pre-image of S is trivial, the question reduces to the one, whether a vector bundle on this preimage can be extended to X as a vector bundle. For example, if $\dim(X) = 2$ then S is a point and its proper preimage is a \mathbb{P}^1 with self-intersection -1 . Since the map $\text{Pic}(X') \rightarrow \text{Pic}(\mathbb{P}^1)$ is surjective, any vector bundle on \mathbb{P}^1 can be extended to X' .

Therefore we obtain a simple proof of the basic theorem in the case $\dim(X) = 2$ using the birational theorem.

In the case of $\dim(X) = 3$ the same procedure reduces the basic problem to the analogous problem of extending vector bundles from \mathbb{P}^2 and ruled surfaces to a variety of dimension three.

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