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Linearizing some $\mathbb{Z}/2\mathbb{Z}$ actions on affine space

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Let V be the affine space k^n over an algebraically closed field k, G a linearly reductive group and $A: G \times V \to V$ a group action with a fixed point, say the origin. Then for all $g \in G$ let me denote by $A(g)$ the corresponding automorphism of V. We have

$$A(g) = L(g) + D(g)$$

where $L(g), D(g) \in \operatorname{End} V$, $L(g)$ linear and $D(g)$ the sum of terms of higher degrees. Let me recall the well known linearization problem: is the action A linearizable, i.e. conjugated to the linear action $L: G \times V \to V$ (see e.g. [B] and [K])? Recently counter-examples have been found, see [S] and [K + S], so it is reasonable to study additional assumptions on the action A. One of them is considered in the present paper.

First I want to define some morphism $\sigma_A: V \to V$ which turns out to be a conjugating automorphism for A, provided σ_A is invertible. It will be done using the Reynolds operator i.e. the equivariant projection $\rho: \mathcal{O}(G) \to k$. For a finite dimensional k-space W we have the unique linear map $\int_G: \operatorname{Mor}(G, W) \to W$ such that for all linear maps $f: W \to k$ the induced diagram

$$\begin{array}{ccc} \int_G : \operatorname{Mor}(G, W) & \longrightarrow & W \\ \Big\downarrow f_* & & \Big\downarrow f \\ \rho: \mathcal{O}(G) & \longrightarrow & k \end{array}$$

is commutative. Now let $\phi: G \to \operatorname{End}(V)$ be such a map that the induced map $G \times V \to V$ is an algebraic morphism. Then $W := \text{lin hull}\,(\phi(G))$ is finite dimensional, hence $\int_G \phi$ is a well-defined element of $\operatorname{End}(V)$. Let us apply the above to the map $\phi: G \ni g \mapsto L(g^{-1})A(g) \in \operatorname{End}(V)$ and set $\sigma = \sigma_A = \int \phi$ (compare [J]). We have

$$L(h)\sigma = \int_{g \in G} L(h)L(g^{-1})A(g) = \left(\int_{g \in G} L(hg^{-1})A(gh^{-1})\right)A(h) = \sigma A(h)$$

for all $h \in G$.

So σ invertible implies that $A(h) = \sigma^{-1}L(h)\sigma$. In particular the action A is linearizable. Later we will give an example of an action A which can be linearized but for which σ_A is not invertible.

As mentioned in [J], the morphism σ_A can be interpreted as an average deviation of A from being linear.

CONJECTURE (Kraft, Procesi). Assume for some $d \geqslant 2$

$$A(g) = L(g) + H_d(g) + H_{d+1}(g) + \cdots + H_{2d-2}(g), \quad \text{for all } g,$$

where $H_m(g)$ is a homogeneous endomorphism of V of degree m. Then σ_A is invertible. In particular the action A is linearizable.

THEOREM. *The above conjecture is true in the following cases*

1. *G linearly reductive, $d = 2$ and* char $k \neq 2$,
2. *G diagonalizable, $d = 2$ and* char k *arbitrary*,
3. $G = \mathbb{Z}/2\mathbb{Z}$, *$d$ arbitrary and* char $k = 0$.

Cases 1 *and* 2 *are the objects of* [J].

Proof for the case 3. Let I denote the identity map of V. We can write: $G = \{I, L + D\}$, where L and D are endomorphisms of V, L linear and $D = H_d + \cdots + H_{2d-2}$. We have $L^2 = (L + D)^2 = I$. It follows that

$$LD + D(L + D) = 0. \tag{1}$$

Let me denote by $\tilde{H}_d$ the d-linear symmetric map from V^d to V corresponding to H_d. Then we have

$$D(L + D) = DL + d\tilde{H}_d(L, \ldots, L, H_d) + \cdots$$

where the first summand consists of terms of degrees $d, \ldots, 2d - 2$, the second is of degree $2d - 1$ and all further summands have higher degrees. Considering the possible cancellations in (1) we obtain:

$$-LD = DL = D(L + D). \tag{2}$$

By defnition $\sigma = \frac{1}{2}(I + (I + LD)) = I - \frac{1}{2}DL$. We will prove that $I + \frac{1}{2}DL$ is the inverse of σ.

LEMMA. $D(I + mDL) = D$ *for* $m = 0, 1, 2, \ldots$

Proof. Suppose the above holds for some $m - 1, m > 0$. By (2), $D = D(I + DL)$. Therefore

$$D = D(I + (m - 1)DL)(I + DL) = D(I + DL + (m - 1)DL(I + DL)).$$

On the other hand $DL(I + DL) = -LD(I + DL) = -LD = DL$, and we are done.

Since char(k) = 0 the Lemma implies that $D(I + rDL) = D$ for all $r \in k$. Then taking $r = \frac{1}{2}$ we have

$$(I - \tfrac{1}{2}DL)(I + \tfrac{1}{2}DL) = I + \tfrac{1}{2}DL + \tfrac{1}{2}LD(I + \tfrac{1}{2}DL) = I,$$

and the same applies if we interchange the order of factors at the left hand side. Q.E.D.

EXAMPLE OF A NON INVERTIBLE σ. Let the linear endomorphism L of k^2 be given by $L(x, y) = (x, -y)$ and an automorphism τ by $\tau(x, y) = (x - (x + y)^2, y + (x + y)^2)$ so that $\tau^{-1}(x, y) = (x + (x + y)^2, y - (x + y)^2)$. The automorphism $\tau^{-1}L\tau$ has order two, so it defines an action of the group of order two on k^2. The corresponding endomorphism $\sigma = \frac{1}{2}(I + L\tau^{-1}L\tau)$ takes (x, y) to $(x - u + v, y + u + v)$, where $u = \frac{1}{2}(x + y)^2$, $v = \frac{1}{2}(x - y - 2(x + y)^2)^2$. Direct computation shows that the Jacobian determinant of σ is

$$J(\sigma) = 1 - 4(x^2 + y^2) + 8(x^3 + y^3) + 24(x^2y + xy^2).$$

Therefore the endomorphism σ is not invertible, while the considered group action can obviously be linearized.

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