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Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d\rangle$

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The aim of this paper is to classify map germs $(\mathbb{C}^2, 0) \to \mathbb{C}$ and germs of curve singularities in $\mathbb{C}^2$ given by an equation of the type $f = (x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij}x^iy^j = 0$ with a fixed Milnor number $\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$. Here we always suppose $p < q$ and $\gcd(p, q) = 1$.

The moduli space $M_{p,q,\mu}$ of the map germs described above is an affine Zariski-open subset of $\mathbb{C}^{2(p-1)(q-1)-p-q+2+[q/p]}$ devided by a suitable action of μ_{2pq} (the group of $2pq$-roots of unity) depending on $\mu(f)$.

The moduli space $T_{p,q,\mu}$ of all plane curve singularities described above (which is the moduli space of all plane curve singularities with the semigroup $\langle 2p, 2q, \mu - 2(p-1)(q-1)+1\rangle$ if μ is even) is $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}$ devided by a suitable action of μ_d, $d = \mu - (2p-1)(2q-1)$.

In both cases we also get an algebraic universal family. It turns out that the Tjurina-number $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, \partial f/\partial x, \partial f/\partial y) = \mu(f) - (p-1)(q-1)$ depends only on $\mu(f)$ and p and q.

Constructing the moduli spaces we use the graduation of $\mathbb{C}[[x, y]]$ defined by p, q: $\deg x^iy^j = iq + jp$.

We use the following idea to construct the moduli spaces: Let $\mu = (2p-1)(2q-1) + d$. We prove that for all f of the above type we can choose the same monomial base of $\mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$ (Lemma 2). We choose α, β and that $\alpha < p, \alpha q + \beta p = 2pq + d$. Hence $\mu(f_0) = \mu$ with $f_0 = (x^p + y^q)^2 + x^\alpha y^\beta$. Then we consider a universal μ-constant unfolding of f_0 as a "global" family (Lemma 3). The parameter space U of that unfolding is an affine open subset of $\mathbb{C}^{2(p-1)(q-1)-p-q+2+[q/p]}$. The group μ_{2pq} acts on U and $M_{p,q,\mu} = U/\mu_{2pq}$.

To construct $T_{p,q,\mu}$ we consider the Kodaira-Spencer map of the universal μ-constant unfolding. The Kernel of the Kodaira-Spencer map is a Lie-algebra acting on U. The integral manifolds of that Lie-algebra are the analytically trivial subfamilies of the unfolding.

We choose a suitable section transversal to those integral manifolds, which turns out to be isomorphic to $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}$. The group μ_d acts on the corresponding family and we prove that $T_{p,q,\mu} = \mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$.

1. A normal form for map germs $(\mathbb{C}^2, 0) \to \mathbb{C}$ with initial term $(x^p + y^q)^2$

LEMMA 1. *Let*

$$f = \left(x^p + y^q + \sum_{iq+jp>pq} h_{ij}x^iy^j\right)^2 + \sum_{iq+jp\geqslant 2pq+d} w_{ij}x^iy^j$$

then $\mu(f) \geqslant (2p-1)(2q-1) + d$, *and* $\mu(f) = (2p-1)(2q-1) + d$ *iff*

$$f_d := \sum_{iq+jp=2pq+d} (-1)^{[i/p]} w_{ij} \neq 0.$$

Proof. Either f is irreducible or the components of f have the same tangent direction. This implies that

$$\mu(\tilde{f}) = \mu(f) - 2p(2p-1),$$

where $\tilde{f}$ is the blowing up

$$\tilde{f} = \frac{f(xy, y)}{y^{2p}} = \left(x^p + y^{q-p} + \sum h_{ij}x^iy^{i+j-p}\right)^2 + \sum w_{ij}x^iy^{i+j-2p}$$

$$= \left(x^p + y^{q-p} + \sum_{i(q-p)+jp>(q-p)p} h_{i,j-i+p}x^iy^j\right)^2 +$$

$$+ \sum_{i(q-p)+jp\geqslant 2(q-p)p+d} w_{i,j-i+2p}x^iy^j.$$

Using induction we may assume that

$$\mu(\tilde{f}) \geqslant (2p-1)(2(q-p)-1) + d$$

and

$$\mu(\bar{f}) = (2p-1)(2(q-p)-1) + \dot{d} \text{ iff } 0 \neq \sum_{iq+jp=2pq+d} (-1)^{[i/p]} w_{ij}$$

$$= \sum_{i(q-p)+jp=2(p-q)p+d} (-1)^{[i/p]} w_{i,j-i+2p}.$$

This yields the if part of the result. Now if f is as above and $\mu(f) > (2p-1)(2q-1)+d$, the condition $f_d = 0$ says that $(x^p + y^q)$ divides $\sum_{iq+jp=2pq+d} w_{ij}x^iy^j$, and adding $-\frac{1}{2}\sum_{iq+jp=2pq+d} w_{ij}x^iy^j$ to the first part of f one gets $f = (x^p + y^q + \cdots)^2 +$ terms of degree greater than $2pq + d$. Continuing this way, we get the result.

LEMMA 2. *Let $f = (x^p + y^q)^2 + \sum_{iq+jp>2pq} h_{ij}x^iy^j$ and $\mu(f) = (2p-1)(2q-1)+d$. Let γ, δ such that $\gamma q + \delta p = 3pq - q - p + d, \gamma < p$. Let $B = \{(i,j) \in N^2/i < 2p-1, j < q-1\} \cup \{(i,j) \in N^2/i < p, j < q\} \cup \{(i,j), (i,j) \in N^2/i < \gamma, j < \delta + q\}$. Then $\{x^iy^j\}_{(i,j)\in B}$ is a base of $\mathbb{C}[[x,y]]/(\partial f/\partial x, \partial f/\partial y)$.*

Proof. We use the algorithm of Mora (cf. [3]) to compute a Groebner base of the ideal $(\partial f/\partial x, \partial f/\partial y)$. We consider $\mathbb{C}[[x,y]]$ as a graded ring with $\deg x = q$, $\deg y = p$. Let $f_1 = 1/2p(\partial f/\partial x)$ and $f_2 = 1/2q(\partial f/\partial y)$.

Consider $s(f_1, f_2) = y^{q-1}f_1 - x^{p-1}f_2$ and let f_3 be the reduction of $s(f_1, f_2) = y^{q-1}f_1 - x^{p-1}f_2$ with respect to the initial terms x^{2p-1} resp. x^py^{q-1} of f_1 resp. f_2, i.e.

$$s(f_1, f_2) = f_3 + h_1f_1 + h_2f_2$$

$$f_3 = \sum_{\gamma_i < p} l_i x^{\gamma_i} y^{\delta_i}$$

$$q\gamma_i + p\delta_i = 3pq - q - p + i$$

and the initial terms of h_1 resp. h_2 have degree $> pq - p$ resp. $> pq - q$. $f_3 \neq 0$ because of $\mu(f) < \infty$. Let k be the minimal such that $l_k \neq 0$, i.e. $l_kx^{\gamma_k}y^{\delta_k}$ is the initial term of f_3. Consider now

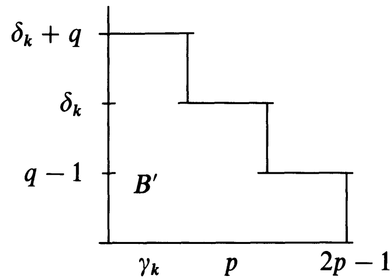
$$\begin{aligned} s(f_2, f_3) &= l_k y^{\delta_k - q + 1} f_2 - x^{p-\gamma_k} f_3 \\ &= l_k y^{\delta_k + q} + \text{terms of degree} > p\delta_k + pq \\ &=: f_4. \end{aligned}$$

It is not difficult to see that the reductions of $s(f_1, f_3)$ and $s(f_i, f_4)\, i = 1,2,3$ with respect to the initial terms of f_1, f_2, f_3, f_4 are zero, i.e. f_1, f_2, f_3, f_4 is a

Groebner base of $(\partial f/\partial x, \partial f/\partial y)$. This implies that

$$\{x^i y^j\}_{(i,j)\in B'}, B' = \{(i,j), i < 2p-1, j < q-1\} \cup \{(i,j), i < p, j < \delta_k\} \cup \{(i,j), i < \gamma_k, j < \delta_k + q\},$$

is a base of $\mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$.



This implies $\mu(f) = (p-1)(q-1) + q\gamma_k + p\delta_k$ and therefore $\gamma = \gamma_k$ and $\delta = \delta_k$ and $B = B'$. □

LEMMA 3. *Let* $f = (x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij}x^i y^j$ *and* $\mu(f) = (2p-1)(2q-1) + d$.

Let γ, δ *be defined by*

$$\gamma < p \quad \text{and} \quad \gamma q + \delta p = 3pq - q - p + d$$

Let $B_0 = \{(i,j), iq + jp > pq, i \leqslant p-2, j \leqslant q-2\}$;

$$B_1 = \{(i,j), iq + jp \geqslant 2pq + d, i < p, j < \delta\} \cup \{(i,j), iq + jp \geqslant 2pq + d, i < \gamma, j < \delta + q\}.$$

There is an automorphism $\varphi\colon \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ *such that*

$$f(\varphi) = \left(x^p + y^q + \sum_{(i,j)\in B_0} h_{ij}x^i y^j\right)^2 + \sum_{(i,j)\in B_1} w_{ij}x^i y^j$$

for suitable $h_{ij}, w_{ij} \in \mathbb{C}$.

Proof. Using Lemma 1 we may assume that

$$f = \left(x^p + y^q + \sum_{iq+jp>pq} b_{ij}x^i y^j\right)^2 + \sum_{iq+jp\geqslant 2pq+d} c_{ij}x^i y^j.$$

Assume that there is an automorphism $\varphi^{(k)}$ such that

$$f(\varphi^{(k)}) = \Bigg(x^p + y^q + \sum_{(i,j)\in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq+jp\geqslant pq+k} b_{ij}^{(k)} x^i y^j\Bigg)^2 +$$

$$+ \sum_{(i,j)\in B_1} w_{ij}^{(k)} x^i y^j + \sum_{iq+jp\geqslant 2pq+k} c_{ij}^{(k)} x^i y^j,$$

$\varphi^{(1)} =$ identity.

Now

$$\sum_{iq+jp=2pq+k} c_{ij}^{(k)} x^i y^j = (x^p + y^q)H + \sum_{iq+jp=2pq+k} (-1)^{[i/p]} c_{ij}^{(k)} x^{i_0} y^{j_0}$$

for a suitable homogeneous H of degree $pq + k$ and $i_0 < p, i_0 q + j_0 p = 2pq + k$.

If $\Sigma_{iq+jp>2pq+k}(-1)^{[i/p]} c_{ij}^{(k)} \neq 0$ and $(i_0, j_0) \notin B_1$ then $k \geqslant pq - q - p + d$ (Lemma 1) and $j_0 \geqslant \delta, i_0 \geqslant \gamma$ or $j_0 \geqslant \delta + q$.

Let α, β be defined by $q\alpha + p\beta = 2pq + d, \alpha < p$, then $w_{\alpha\beta} \neq 0$. Notice that $\alpha - 1 \equiv \gamma \bmod p$ and $\beta - 1 \equiv \delta \bmod q$.

Let

$$g := x^p + y^q + \sum_{(i,j)\in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq+jp\geqslant pq+k} b_{ij}^{(k)} x^i y^j$$

and

$$\omega := e \cdot x^{\xi} y^{\eta} \left(\frac{\partial g}{\partial y}\frac{\partial}{\partial x} - \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\right), \ \xi q + \eta p = k - pq + p + q - d$$

$$e := \frac{1}{(\alpha q + \beta p) w_{\alpha\beta}} \cdot \sum_{ip+jq=2pq+k} (-1)^{[i/p]-[\alpha-1+\xi/p]+1} c_{ij}^{(k)}$$

with $(\xi, \eta) = \begin{cases} (i_0 - \gamma, j_0 - \delta) & \text{if } j_0 \geqslant \delta, \quad i_0 \geqslant \gamma \\ (i_0 - \gamma + p, j_0 - \delta - q) & \text{if } j_0 \geqslant \delta + q. \end{cases}$

Let $\psi: \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ the automorphisms corresponding to the vector field ω, then $g(\psi) = g$.

Hence,

$$f(\psi \circ \varphi^{(k)}) = g^2 + \sum_{(i,j)\in B_0} w_{ij}^{(k)} x^i y^j + \sum_{iq+jp\geqslant 2pq+k} \bar{c}_{ij}^{(k)} x^i y^j$$

and

$$\sum_{iq+jp=2pq+k} (-1)^{[i/p]}\bar{c}_{ij}^{(k)}$$
$$= \sum_{iq+jp=2pq+k} (-1)^{[i/p]}c_{ij}^{(k)} + (-1)^{[\alpha-1+\xi/p]}(\alpha q + \beta p)w_{\alpha\beta}\cdot e.$$

If $(i_0, j_0)\notin B_1$ we may assume now that $\sum_{iq+jp=2pq+k}(-1)^{[i/p]}c_{ij}^{(k)} = 0$.

Let $g_1 := g + \frac{1}{2}H$ and

$$\sum_{iq+jp=pq+k} b_{ij}^{(k)}x^i y^j + \tfrac{1}{2}H + m_k\frac{\partial g_1}{\partial x} + n_k\frac{\partial g_1}{\partial y} = \sum_{(i,j)\in B_0} d_{ij}^{(k)}x^i y^j.$$

The degree of the initial part of m_k resp. n_k is $q + k$ resp. $p + k$.

We define $\varphi^{(k+1)}$ by

$$\varphi^{(k+1)}(x) = \varphi^{(k)}(x) + m_k$$
$$\varphi^{(k+1)}(y) = \varphi^{(k)}(y) + n_k$$

and

$$h_{ij}^{(k+1)} = h_{ij}^{(k)} + d_{ij}^{(k)}$$
$$w_{ij}^{(k+1)} = w_{ij}^{(k)} \quad \text{if } (i,j) \neq (i_0, j_0)$$
$$w_{i_0,j_0}^{(k+1)} = w_{i_0,j_0}^{(k)} + \sum_{iq+jp=2pq+k} (-1)^{[i/p]}c_{ij}^{(k)} \quad \text{if } (i_0, j_0)\in B_1.$$

Then

$$f(\varphi^{(k+1)}) = \left(x^p + y^q + \sum_{(i,j)\in B_0} h_{ij}^{(k+1)}x^i y^j + \sum_{iq+jp\geqslant pq+k+1} b_{ij}^{(k+1)}x^i y^j\right)^2 +$$
$$+ \sum_{(i,j)\in B_1} w_{ij}^{(k+1)}x^i y^j + \sum_{iq+jp\geqslant 2pq+k+1} c_{ij}^{(k+1)}x^i y^j$$

for suitable $b_{ij}^{(k+1)}, c_{ij}^{(k+1)}$. □

LEMMA 4. *Let*

$$f_t = (x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij}(t)x^i y^j, a_{ij}(t)\in\mathbb{C}[t]$$

and $\mu(f_t) = (2p-1)(2q-1) + d$ *for* $t\in\mathbb{C}$.

Let γ, δ, B_0, B_1 be as in Lemma 3. There is a $\mathbb{C}[t]$-automorphism $\varphi_t\colon \mathbb{C}[t][[x, y]] \to \mathbb{C}[t][[x, y]]$ such that

$$f_t(\varphi_t) = \left(x^p + y^q + \sum_{(i,j)\in B_0} h_{ij}(t)x^i y^j\right)^2 + \sum_{(i,j)\in B_1} w_{ij}(t)x^i y^j$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$.

The proof is similar to that of Lemma 3. □

Let us consider the family

$$F(x, y, H, W) = \left(x^p + y^q + \sum_{(i,j)\in B_0} H_{ij}x^i y^j\right)^2 + \sum_{(i,j)\in B_1} W_{ij}x^i y^j$$

depending on the parameters $H = (H_{ij})_{(i,j)\in B_0}$, $W = (W_{ij})_{(i,j)\in B_1}$ and define $N = \# B_0 + \# B_1$, then $\mu(F) = (2p-1)(2q-1) + d$ on the open set U defined by $W_{\alpha\beta} \neq 0$, $\alpha q + \beta p = 2pq + d$, in $\mathbb{C}^N = \operatorname{Spec} \mathbb{C}[H, W]$. Notice that $N = 2(p-1)(q-1) - p - q + 2 + [q/p]$ is not depending on d!

The group of $2pq$-roots of unity acts on U:

$$\lambda \in \mu_{2pq},\ \lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq}h_{ij}), (\lambda^{iq+jp-2pq}w_{ij})). \qquad \square$$

THEOREM 1. *U/μ_{2pq} is the moduli space of all functions*

$$f = (x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij}x^i y^j$$

with $\mu(f) = (2p-1)(2q-1) + d$ and F is the universal family.

Proof. Using Lemma 3 we have to prove the following

LEMMA 5. *Let φ be an automorphism $\varphi\colon \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ such that*

$$F(\varphi(x), \varphi(y), \bar{h}, \bar{w}) = F(x, y, h, w) \tag{$*$}$$

for $(\bar{h}, \bar{w}), (h, w) \in U \subseteq \mathbb{C}^N$ then $\lambda \cdot (\bar{h}, \bar{w}) = (h, w)$ for a suitable $\lambda \in \mu_{2pq}$.

Proof. Let $\bar{x} := \varphi(x)$, $\bar{y} := \varphi(y)$ then grouping the squared part of $(*)$ one gets:

$$\left(x^p + y^q + \bar{x}^p + \bar{y}^q + \sum \bar{h}_{ij}\bar{x}^i\bar{y}^j + \sum h_{ij}x^i y^j\right) \times$$
$$\times \left(x^p + y^q - \bar{x}^p - \bar{y}^q - \sum \bar{h}_{ij}\bar{x}^i\bar{y}^j + \sum h_{ij}x^i y^j\right)$$
$$= \sum \bar{w}_{ij}\bar{x}^i\bar{y}^j - \sum w_{ij}x^i y^j.$$

This equation implies obviously that the degree of the initial term of $\varphi(x)$ is $\geqslant q$ and

$$\bar{x} = \lambda^q\left(x + \sum_{iq+jp>q} a_{ij}^{(1)} x^i y^j\right) \qquad \bar{y} = \lambda^p\left(y + \sum_{iq+jp>p} a_{ij}^{(2)} x^i y^j\right), \lambda \in \mu_{2pq}.$$

We may assume that $\lambda = 1$ and prove $(\bar{h}, \bar{w}) = (h, w)$.

Let the degree of the leading parts of both sides of the above equation be $2pq+m$ and let r be the degree of φ, i.e. $a_{ij}^{(1)} = 0$ if $iq + jp < q + r$, $a_{ij}^{(2)} = 0$ if $iq + jp < p + r$ and $a_{ij}^{(1)} \neq 0$ or $a_{ij}^{(2)} \neq 0$ for suitable i, j with $iq + jp = q + r$ resp. $iq + jp = p + r$.

1. Step. We prove that

(a) $r \geqslant pq - p - q$

$$\sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} \cdot k, \qquad \sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} \cdot k$$

(b) $h = \bar{h}$ and $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 3pq - p - q + d$.

First of all $m \geqslant d + r$ because the leading part of the left side of the equation is divisible by $x^p + y^q$ and $m < d + r$ would imply that the leading part of the right side is a monomial. This implies $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 2pq + d + r$. Now $h_{ij} = \bar{h}_{ij}$ if $iq + jp < pq + r$. Otherwise the leading part of the left side of the equation would be $2(x^p + y^q)(h_{ij} - \bar{h}_{ij})x^i y^j$ for some i, j with $iq + jp < pq + r$ and therefore of degree $2pq + r < 2pq + m$.

Now suppose $r < pq - p - q$. Then there is at most one monomial of degree $p + r$ resp. $q + r$.

If $iq + jp = pq + r$ for some $(i, j) \in B_0$ and

$$qi_0 + pj_0 = q + r$$

$$qi_1 + pj_1 = p + r$$

then

$$(h_{ij} - \bar{h}_{ij})x^i y^j - pa_{i_0 j_0}^{(1)} x^{i_0+p-1} y^{j_0} - qa_{i_1 j_1}^{(2)} x^{i_1} y^{j_1+q-1} = 0$$

otherwise the leading part of the left side of the equation would have degree $2pq + r < 2pq + m$.

But $(i, j) \in B_0$, i.e. $i < p - 1$ and $j < q - 1$. This implies $h_{ij} = \bar{h}_{ij}$, $a_{i_0 j_0}^{(1)} = a_{i_1 j_1}^{(2)} = 0$ (because of $r < pq - p - q$ we have $i_1 < p - 1$). This is a contradiction since $a_{i_0 j_0}^{(1)} \neq 0$ or $a_{i_1 j_1}^{(2)} \neq 0$ by the definition of r.

Similarly one gets a contradiction if there is no $(i, j) \in B_0$ with $qi + pj = p + r$, resp. no i_0, j_0 with $qi_0 + pj_0 = q + r$ resp. no i_1, j_1 with $qi_1 + pj_1 = p + r$.
This proves that $r \geqslant pq - p - q$. With the same method we obtain

$$\sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} k \quad \text{and} \quad \sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} k.$$

(b) is clear now by the choice of B_0 and the fact that $r \geqslant pq - p - q$.

2. *Step.* We prove that $r \geqslant 2pq - p - q$.
Assume that $r < 2pq - p - q$. Then $\deg k < pq$, i.e., k is a monomial.
The leading part of the left side of the above equation is divisible by $x^p + y^q$.
The leading part L of the right side is

$$(\bar{w}_{ij} - w_{ij}) x^i y^j + \bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp = 2pq + m$ for some $(i, j) \in B_1$ or

$$\bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp \neq 2pq + m$ for $(i, j) \in B_1$.

Let $k = \kappa \cdot x^\xi y^\eta$. If $\alpha + \xi - 1 < p$, then $i = \alpha + \xi - 1$ and $j = \beta + \eta + q - 1$. If $\alpha + \xi - 1 \geqslant p$, then $i = \alpha + \xi - 1 - p$ and $j = \beta + \eta + 2q - 1$. But $(\alpha + \xi - 1, \beta + \eta + q - 1) \notin B_1$ and $(\alpha + \xi - 1 - p, \beta + \eta + 2q - 1) \notin B_1$. This implies $L = \bar{w}_{\alpha\beta} \cdot k x^{\alpha-1} y^{\beta-1}((\alpha/p) y^q - (\beta/p) x^p)$ which is not divisible by $x^p + y^q$. This is a contradiction and therefore $r \geqslant 2pq - p - q$.

Now $iq + jp \leqslant 4pq - 2p - 2q + d$ for $(i, j) \in B_1$ then $w_{ij} = \bar{w}_{ij}$ for all $(i, j) \in B_1$. □

2. The construction of the moduli space

We will construct the moduli space of all plane curve singularities given by an equation $(x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij} x^i y^j = 0$ with fixed Milnor number μ.

For μ being even we get especially the moduli space for all irreducible plane curve singularities with the semigroup $\Gamma = \langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$.

We use the family

$$V(F) \subseteq U \times \mathbb{C}^2 \to U$$

constructed in Theorem 1.

U admits a $\mathbb{C}^*$-action defined by

$$\lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})).$$

We get

$$F(\lambda^q x, \lambda^p y, h, w) = \lambda^{2pq} F(x, y, \lambda \circ (h, w)).$$

If $\mu = (2p-1)(2q-1) + d$ and $\alpha q + \beta p = 2pq + d, \alpha < p$, then $U \subseteq \mathbb{C}^N$ was defined by $W_{\alpha,\beta} \neq 0$.

For the construction of the moduli space it is enough to consider the restriction of our family to the transversal section to the orbits of the C^*-action defined by $W_{\alpha,\beta} = 1$.

Let W' be defined by $W = (W_{\alpha,\beta}, W')$ and $G(x, y, H, W') = F(x, y, H, 1, W')$. The parameter space of G is $\mathbb{C}^{N-1} = \operatorname{Spec} \mathbb{C}[H, W']$.

The group μ_d of dth roots of unity acts on the family

$$V(G) \subseteq \mathbb{C}^2 \times \mathbb{C}^{N-1} \to \mathbb{C}^{N-1}$$

induced by the above C^*-action

$$G(\lambda^q x, \lambda^p y, h, w') = \lambda^{2pq} G(x, y, \lambda \circ (h, w)) \quad \lambda \in \mu_d.$$

LEMMA 6. *Let $\varphi: \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ be an automorphism and $u \in \mathbb{C}[[x, y]]$ a unit such that*

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

then there is a $\lambda \in \mu_d$ such that (h, w') and $\lambda \circ (\bar{h}, \bar{w}')$ are contained in an analytically trivial subfamily of $V(G) \to \mathbb{C}^{N-1}$.

Proof. Let

$$\varphi(x) = \sum a_{ij}^{(1)} x^i y^j \quad \text{and} \quad \varphi(y) = \sum a_{ij}^{(2)} x^i y^j, \quad u = \sum u_{ij} x^i y^j.$$
$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

implies

(1) $a_{ij}^{(1)} = 0$ if $iq + jp < q$

(2) $a_{1,0}^{(1)2p} = a_{0,1}^{(2)2q} = a_{1,0}^{(1)p} a_{0,1}^{(2)q} = u_{0,0}^{-1}$.

Let $a_{1,0}^{(1)} = \lambda^q$ and $a_{0,1}^{(2)} = \lambda^p$.

We will prove later that $\lambda^d = 1$.

Now we may assume that $\lambda = 1$ and prove that (h, w') and $(\bar{h}, \bar{w}')$ are contained in an analytically trivial subfamily of $V(G) \to \mathbb{C}^{N-1}$.

We choose

(1) $u(t) \in \mathbb{C}[t][[x, y]]$ with the following properties $u(0) = 1$, $u(1) = u$ and u is a unity for all $t \in \mathbb{C}$.

(2) φ_t: $\mathbb{C}[t][[x, y]] \to \mathbb{C}[t][[x, y]]$ with the following properties $\varphi_0 =$ identity, $\varphi_1 = \varphi$ and φ_t is an automorphism of positive degree for all $t \in \mathbb{C}$.

Let $H(t) := u(t)G(\varphi_t(x), \varphi_t(y), h, w')$ and apply Lemma 4. There is an $\mathbb{C}[t]$-automorphism Φ_t: $\mathbb{C}[t][[x, y]] \to \mathbb{C}[t][[x, y]]$ such that

$$H(\Phi_t) = F(x, y, h(t), w(t))$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$ with the property

$$\begin{aligned} h(0) &= h \\ w(0) &= (1, w'). \end{aligned}$$

$H(\Phi_t)$ has a constant Milnor number, i.e. $w_{\alpha,\beta}(t)$ has to be constant.

This implies

$$H(\Phi_t) = G(x, y, h(t), w'(t)).$$

But,

$$G(x, y, h(1), w'(1)) = H(\Phi_1) = G(\Phi_1(x), \Phi_1(y), \bar{h}, \bar{w}').$$

Using Lemma 5 and the fact that Φ_1 has positive degree we get

$$\begin{aligned} \bar{h} &= h(1) \\ \bar{w}' &= w'(1), \end{aligned}$$

i.e. (h, w') and $(\bar{h}, \bar{w}')$ are in the trivial family

$$G(x, y, h(t), w'(t)) = u(\Phi_t)G(\Phi_t\varphi_t, h, w').$$

To finish the proof of Lemma 6 we have to prove

LEMMA 7. *Let*

$$f_k = \left(x^p + y^q + \sum_{iq + jp > pq} a_{ij}^{(k)} x^i y^j\right)^2 + x^\alpha y^\beta + \sum_{iq + jp > 2pq + d} b_{ij}^{(k)} x^i y^j, \quad k = 1, 2$$

$\alpha < p$, $\alpha q + \beta p = 2pq + d$.

Let $\varphi\colon \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ *be an automorphism with the property*

$$\varphi(x) = \lambda^q x + \textit{terms of degree} > q$$
$$\varphi(y) = \lambda^p y + \text{terms of degree} > p$$

and u a unit, such that

$$f_1(\varphi) = f_2 \cdot u$$

then $\lambda^d = 1$.

Proof. $u = \lambda^{2pq}$ + terms of higher degree.

$$u \cdot f_2 = \lambda^{2pq}\Big(x^p + y^q + \sum_{iq+jp>pq} \bar{a}_{ij}^{(2)} x^i y^j\Big)^2 + \lambda^{2pq} x^\alpha y^\beta + \sum_{iq+jp>2pq+d} \bar{b}_{ij}^{(2)} x^i y^j$$

$$f_1(\varphi) = \lambda^{2pq}\Big(x^p + y^q + \sum_{iq+jp>pq} \bar{a}_{ij}^{(1)} x^i y^j\Big)^2 +$$
$$+ \lambda^{2pq+d} x^\alpha y^\beta + \sum_{iq+jp\geqslant 2pq+d} \bar{b}_{ij}^{(1)} x^i y^j$$

for suitable $\bar{a}_{ij}^{(k)}, \bar{b}_{ij}^{(k)}$.

This implies

$$\Big(2x^p + 2y^q + \sum_{iq+jp>pq} (\bar{a}_{ij}^{(1)} + \bar{a}_{ij}^{(2)}) x^i y^j\Big) \cdot \sum_{iq+jp>pq} (\bar{a}_{ij}^{(1)} - \bar{a}_{ij}^{(2)}) x^i y^j$$
$$= (1 - \lambda^d) x^\alpha y^\beta + \sum_{iq+jp>2pq+d} \lambda^{-2pq} (\bar{b}_{ij}^{(2)} - \bar{b}_{ij}^{(1)}) x^i y^j.$$

Because the leading term of the left side of the equation is divisible by $x^p + y^q$, we get $\lambda^d = 1$. □

We consider now the Kodaira-Spencer map of the family

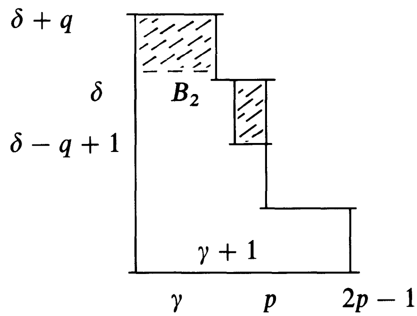
$$V(G) \to \mathbb{C}^{N-1}:$$
$$\rho\colon \mathrm{Der}_{\mathbb{C}}\mathbb{C}[H, W'] \to \mathbb{C}[H, W'][[x, y]] \Big/ \Big(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\Big)$$

defined by

$$\rho(\delta) = \mathrm{class}(\delta G).$$

The kernel of the Kodaira-Spencer map is a Lie-algebra L and along the integral manifolds of L the family is analytically trivial. We will choose a transversal section to the integral manifolds of L and divide by the action of μ_d to get the moduli space. To describe this transversal section we choose a suitable subset of B_1:

$$B_2 = \{(i,j) \in B_1, i \leqslant \gamma, j \leqslant \delta\} \cup \{(i,j) \in B_1, j \leqslant \delta - q\}.$$



Let $M := \# B_0 + \# B_2 = N - (p-1)(q-1) = (p-2)(q-2) + [q/p] - 1$. Let $W'' := (W_{ij})_{(ij) \in B_2}$ and $\mathbb{C}^M = \operatorname{Spec} \mathbb{C}[H, W''\}$

$$G_u(x, y, H, W'') := \left(x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j\right)^2 + x^\alpha y^\beta + \sum_{(i,j) \in B_2} W_{ij} x^i y^j$$

As before μ_d acts on the family $V(G_u) \subseteq \mathbb{C}^2 \times \mathbb{C}^M \to \mathbb{C}^M$.

THEOREM 2. *$\mathbb{C}^M/\mu_d$ is the moduli space of all plane curve singularities defined by an equation*

$$(x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{\overline{ij}} x^i y^j = 0$$

with Milnor numbers $\mu = (2p-1)(2q-1) + d$ and G_u is the corresponding universal family.

Especially the Tjurina number $\tau = \mu - (p-1)(q-1)$ only depends on μ for these singularities.

COROLLARY. *Let $\Gamma = \langle 2p, 2q, 2pq + d \rangle$, d odd, a semigroup.*

Then $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$ is the moduli space of all irreducible plane curve singularities with the semigroup Γ.

G_u is the corresponding universal family.

Proof. To prove the theorem we compute generators of the kernel of the

Kodaira-Spencer map.

Let

$$G^{(0)} = x^p + y^q + \sum_{(i,j)\in B_0} H_{ij}x^iy^j$$

$$G^{(1)} = x^\alpha y^\beta + \sum_{\substack{(i,j)\in B_1 \\ iq+jp>2pq+d}} W_{ij}x^iy^j, \quad \text{i.e.}$$

$$G = G^{(0)2} + G^{(1)}.$$

Let $\delta \in \mathrm{Der}_{\mathbb{C}}\,\mathbb{C}[H, W']$ be a vector field which belongs to the kernel of the Kodaira-Spencer map, i.e.

$$\delta G \in \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right).$$

Now

$$\delta G = 2G^{(0)} \sum_{(i,j)\in B_0} \delta H_{ij}x^iy^j + \sum_{\substack{(i,j)\in B_1 \\ iq+jp>2pq+d}} \delta W_{ij}x^iy^j = S\cdot G \bmod \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$$

for a suitable $S \in \mathbb{C}[H, W'][[x, y]]$.

We will associate to any monomial x^ay^b, $(a, b) \neq (0, 0)$, a vector field $\delta_{a,b} \in \mathrm{Der}_{\mathbb{C}}[H, W']$ such that

$$\delta_{a,b}G = x^ay^bG \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right).$$

Obviously $\{\delta_{a,b}\}$ generate the kernel of the Kodaira-Spencer map as $\mathbb{C}[H, W']$-module.

Now consider

$$x^ay^bG = x^ay^bG^{(0)2} + x^ay^bG^{(1)}.$$

Let $x^ay^bG^{(0)} = \sum_{(i,j)\in B_0} E_{ij}^{ab}x^iy^j + L_1\frac{\partial G^{(0)}}{\partial x} + L_2\frac{\partial G^{(0)}}{\partial y}$

for suitable $E_{ij}^{ab} \in \mathbb{C}[H, W']$, $L_1, L_2 \in \mathbb{C}[H, W'][[x, y]]$,

$$L_1 = \frac{1}{p}x^{a+1}y^b + \text{terms of higher degree}$$

$$L_2 = \frac{1}{q}x^ay^{b+1} + \text{terms of higher degree},$$

then

$$x^a y^b G = G^{(0)} \sum_{(i,j)\in B_0} E_{ij}^{ab} x^i y^j + x^a y^b G^{(1)} - \tfrac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} -$$

$$-\tfrac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

The leading term of

$$x^a y^b G^{(1)} - \tfrac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \tfrac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y}$$

is $-(d/2pq)x^{\alpha+a}y^{\beta+b}$.

Using Lemma 2 we get

$$x^a y^b G^{(1)} - \tfrac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \tfrac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y}$$

$$= \sum_{\substack{(i,j)\in B_1 \\ iq+jp \geqslant 2pq+d+aq+bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

for suitable $D_{ij}^{ab} \in \mathbb{C}[H, W']$.

This implies

$$x^a y^b G = G^{(0)} \sum_{(i,j)\in B_0} E_{ij}^{ab} x^i y^j + \sum_{\substack{(i,j)\in B_1 \\ iq+jp \geqslant 2pq+d+aq+bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

We define for $(a, b) \neq (0,0)$

$$\delta_{a,b}(H_{ij}) := \tfrac{1}{2} E_{ij}^{ab}$$

$\delta_{a,b}(W_{ij}) := D_{ij}^{ab}$, i.e.,

$$\delta_{a,b} = \frac{1}{2} \sum E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \sum D_{ij}^{ab} \frac{\partial}{\partial W_{ij}}$$

The vector fields $\delta_{a,b}$ have the following properties:

(1) $\delta_{a,b}$ is zero if $aq + bp > 2pq - 2p - 2q$

(2) $\delta_{a,b}(W_{ij}) = 0$ if $iq + jp < 2pq + d + aq + bp$

(3) $\delta_{a,b}(W_{ij}) = -d/2pq$ if $(i,j) = (\alpha + a, \beta + b)$ or $(i,j) = (\alpha + a - p, \beta + b + q)$ (in this case $iq + jp = 2pq + d + aq + bp$).

(4) $\delta_{a,b}(H_{ij}) = 0$ for all $(i,j) \in B_0$ if $aq + bp \geqslant pq - 2p - 2q$

(5) For $iq + jp \geqslant 3pq + d - q$ and $(i,j) \in B_1$ there is (a', b') such that

$$(i,j) = (\alpha + a', \beta + b') \text{ or } (i,j) = (\alpha + a' - p, \beta + b' + q),$$

i.e. $\delta_{a',b'}(W_{ij}) = -d/2pq$.

(6) For any (a,b), $aq + bp < pq - q$, always $(\alpha + a, \beta + b)$ or $(\alpha + a - p, \beta + b + q) \in B_1$, i.e. $\delta_{a,b}(W_{ij}) = -d/2pq$ for the corresponding $(i,j) \in B_1$.

(1) and (4) hold because of the fact that

$$iq + jp \leqslant 2pq - 2p - 2q \quad \text{if } (i,j) \in B_0$$
$$iq + jp \leqslant 4pq + d - 2p - 2q \quad \text{if } (i,j) \in B_1.$$

(2) and (3) hold because of the fact that the leading term of $G^{(1)}$ has degree $2pq + d$ and because of Lemma 2. To prove (5) we consider two cases

1. *Case* $i \geqslant \alpha - 1$

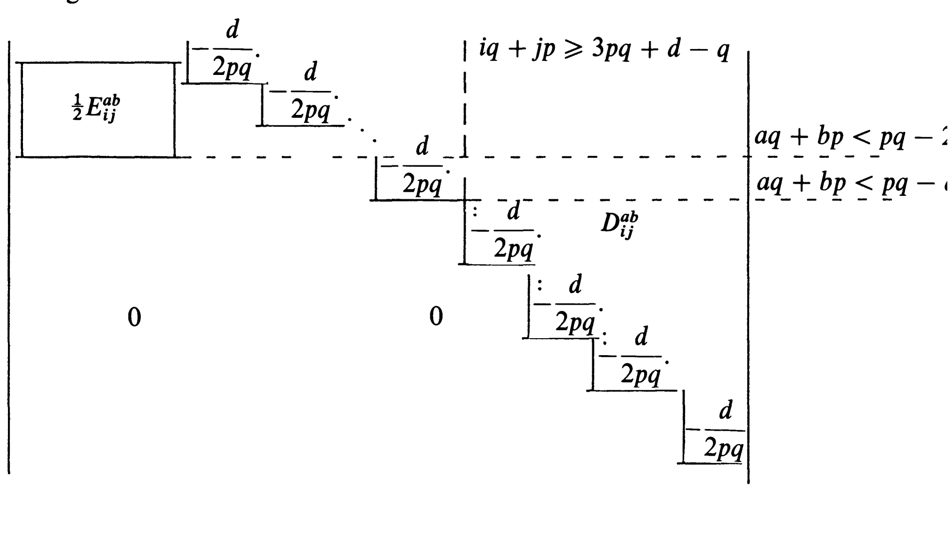
In this case $(i,j) \in B_1$ implies $i \leqslant p - 1$. But $iq + jp \geqslant 3pq + d - q$ implies $j \geqslant \beta$. Then $a' = i - \alpha, b' = j - \beta$ have the required properties. Notice that $i < \alpha - 1$ and $iq + jp \geqslant 3pq + d - q$ implies $(i,j) \notin B_1$

2 *Case* $i < \alpha - 1$

Now $iq + jp \geqslant 3pq + d - q$ implies $j \geqslant \beta + q$ then $a' = p + i - \alpha$, $b' = j - \beta - q$ have the required properties. (6) is similar to (5):

We may assume that $(\alpha + a, \beta + b) \notin B_1$. This implies $2p - 3 \geqslant \alpha + a \geqslant p$ and $\alpha \geqslant 2$ because $b \leqslant q - 2$, $a \leqslant p - 2$. Suppose $(\alpha + a - p, \beta + b + q) \notin B_1$ then $\beta + b + q \geqslant \beta + 2q - 1$, i.e. $b \geqslant q - 1$, or $\alpha + a - p \geqslant \alpha - 1$, i.e. $a \geqslant p - 1$, but this is not possible. □

For the coefficients to the vectorfields $\delta_{a,b}$ we get, because of (1)–(6), the following matrix:



This implies that the kernel of the Kodaira–Spencer map is generated (as $\mathbb{C}[H, W']$-module) by the vector fields

$$\delta'_{ij} := \frac{\partial}{\partial w_{ij}} \quad (i, j) \in B_1, \quad iq + jp \geqslant 3pq + d - q$$

and

$$\delta'_{l,m} = -\frac{pq}{d} \sum_{(i,j)\in B_0} E^{ab}_{ij} \frac{\partial}{\partial H_{ij}} + \frac{\partial}{\partial W_{l,m}} +$$

$$+ \sum_{\substack{(i,j)\in B_1 \\ 2pq+d+aq+bp<iq+jp<3pq+d-q}} \left(-\frac{2pq}{d}\right) D^{ab}_{ij} \frac{\partial}{\partial W_{ij}}$$

$$\times\ (l, m) \in B_1 \backslash (B_2 \cup \{(\alpha, \beta)\}, lq + mp < 3pq + d - q,$$

with $(l, m) = \begin{cases} (\alpha + a, \beta + b) & \text{if } l \geqslant \alpha \\ (\alpha + a - p, \beta + b + q) & \text{else.} \end{cases}$

The vectorfields $\delta'_{l,m}$ act nilpotentely on $\mathbb{C}[H, W']$. Namely, if we consider $\mathbb{C}[H, W']$ as a graded algebra defined by $\deg H_{ij} = pq - iq - jp < 0$, $\deg W_{ij} = 2pq - iq - jp < 0$ then the E^{ab}_{ij} resp. D^{ab}_{ij} are polynomials in $\mathbb{C}[H, W']$ of degree $\geqslant aq + bp + pq - iq - jp$ resp. $\geqslant aq + bp + 2pq - iq - jp$. Notice that their degree is always $\leqslant 0$. Let $A \in \mathbb{C}[H, W']$ be any polynomial of degree $0 \geqslant \deg A = s$ ($\deg A$ = minimum of the degrees of the monomials in A). Then the degree of $\delta'_{lm}(A) > s$. Therefore there is some n with $\delta'^{n}_{lm}(A) = 0$.

LEMMA 8. *Let A be a ring of finite type over a field k. $L \subseteq \mathrm{Der}_k(A)$ a Lie-Algebra. Let $\delta_1, \ldots, \delta_r$ vector fields with the following properties:*

(1) *$\delta_1, \ldots, \delta_r \in L$ and $L \subseteq \sum \delta_i A$*

(2) $[\delta_i, \delta_j] \in \sum_{k>\max\{i,j\}} \delta_k A$

(3) *There are $x_1, \ldots, x_r \in A$ such that*

$$\delta_i(x_i) = 1 \text{ and } \delta_j(x_i) = 0 \quad j > i$$

(4) *$\delta_1, \ldots, \delta_r$ act nilpotentely on A.*

Then $A^L[x_1, \ldots, x_r] = A$.

The Lemma is not difficult to prove. A similar lemma was used in the construction of the moduli space for curve singularities with the semi-group $\langle p, q \rangle$ (cf. [1], [2]).

Obviously A^L is the ring of all elements of A being invariant under $\delta_1, \ldots, \delta_r$.

Now $A^{\delta_r}[x_r] = A$ and the conditions (2)–(4) of the lemma are satisfied for $\delta_1, \ldots, \delta_{r-1}$ acting on A^{δ_r}.

Now we may apply the Lemma 8 to the kernel of the Kodaira-Spencer map and its generators $\{\delta'_{lm}\}$.

Because of the lemma the geometric quotient of $\mathbb{C}^{N-1} = \operatorname{Spec} \mathbb{C}[H, W']$ by the action of the kernel of the Kodaira-Spencer map exist and is isomorphic to the transversal section to the maximal integral manifolds (which intersect therefore each of these integral manifolds exactly in one point) defined by

$$W_{l,m} = 0, \quad (l,m) \in B_1 \backslash (B_2 \cup \{(\alpha, \beta)\}).$$

Now we use Lemma 6 and get Theorem 2. Notice that

$$\{G^{(0)} x^i y^j\}_{(i,j) \in B_0} \cup \{x^i y^j\}_{(i,j) \in B_2} \cup \{x^i y^j, iq + jp \leqslant 2pq, (i,j) \in B\}$$

is a base of the free $\mathbb{C}[H, W']$-module $\mathbb{C}[H, W'][[x, y]]/(G, \partial G/\partial x, \partial G/\partial y)$. This implies $\mu - \tau = \#(B_1 \backslash B_2) = (p-1)(q-1)$. □

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