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## Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d \rangle$

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The aim of this paper is to classify map germs  $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$  and germs of curve singularities in  $\mathbb{C}^2$  given by an equation of the type  $f = (x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{ij} x^i y^j = 0$  with a fixed Milnor number  $\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$ . Here we always suppose  $p < q$  and  $\gcd(p, q) = 1$ .

The moduli space  $M_{p,q,\mu}$  of the map germs described above is an affine Zariski-open subset of  $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$  divided by a suitable action of  $\mu_{2pq}$  (the group of  $2pq$ -roots of unity) depending on  $\mu(f)$ .

The moduli space  $T_{p,q,\mu}$  of all plane curve singularities described above (which is the moduli space of all plane curve singularities with the semigroup  $\langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$  if  $\mu$  is even) is  $\mathbb{C}^{(p-2)(q-2) + [q/p] - 1}$  divided by a suitable action of  $\mu_d$ ,  $d = \mu - (2p-1)(2q-1)$ .

In both cases we also get an algebraic universal family. It turns out that the Tjurina-number  $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (f, \partial f / \partial x, \partial f / \partial y) = \mu(f) - (p-1)(q-1)$  depends only on  $\mu(f)$  and  $p$  and  $q$ .

Constructing the moduli spaces we use the graduation of  $\mathbb{C}[[x, y]]$  defined by  $p, q$ :  $\deg x^i y^j = iq + jp$ .

We use the following idea to construct the moduli spaces: Let  $\mu = (2p-1)(2q-1) + d$ . We prove that for all  $f$  of the above type we can choose the same monomial base of  $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$  (Lemma 2). We choose  $\alpha, \beta$  and that  $\alpha < p, \alpha q + \beta p = 2pq + d$ . Hence  $\mu(f_0) = \mu$  with  $f_0 = (x^p + y^q)^2 + x^\alpha y^\beta$ . Then we consider a universal  $\mu$ -constant unfolding of  $f_0$  as a “global” family (Lemma 3). The parameter space  $U$  of that unfolding is an affine open subset of  $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$ . The group  $\mu_{2pq}$  acts on  $U$  and  $M_{p,q,\mu} = U / \mu_{2pq}$ .

To construct  $T_{p,q,\mu}$  we consider the Kodaira-Spencer map of the universal  $\mu$ -constant unfolding. The Kernel of the Kodaira-Spencer map is a Lie-algebra acting on  $U$ . The integral manifolds of that Lie-algebra are the analytically trivial subfamilies of the unfolding.

We choose a suitable section transversal to those integral manifolds, which turns out to be isomorphic to  $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}$ . The group  $\mu_d$  acts on the corresponding family and we prove that  $T_{p,q,\mu} = \mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$ .

**1. A normal form for map germs  $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$  with initial term  $(x^p + y^q)^2$**

LEMMA 1. *Let*

$$f = \left( x^p + y^q + \sum_{iq+jp>pq} h_{ij}x^i y^j \right)^2 + \sum_{iq+jp \geq 2pq+d} w_{ij}x^i y^j$$

then  $\mu(f) \geq (2p - 1)(2q - 1) + d$ , and  $\mu(f) = (2p - 1)(2q - 1) + d$  iff

$$f_d := \sum_{iq+jp=2pq+d} (-1)^{[i/p]} w_{ij} \neq 0.$$

*Proof.* Either  $f$  is irreducible or the components of  $f$  have the same tangent direction. This implies that

$$\mu(\tilde{f}) = \mu(f) - 2p(2p - 1),$$

where  $\tilde{f}$  is the blowing up

$$\begin{aligned} \tilde{f} = \frac{f(xy, y)}{y^{2p}} &= \left( x^p + y^{q-p} + \sum h_{ij}x^i y^{i+j-p} \right)^2 + \sum w_{ij}x^i y^{i+j-2p} \\ &= \left( x^p + y^{q-p} + \sum_{i(q-p)+jp > (q-p)p} h_{i,j-i+p}x^i y^j \right)^2 + \\ &\quad + \sum_{i(q-p)+jp \geq 2(q-p)p+d} w_{i,j-i+2p}x^i y^j. \end{aligned}$$

Using induction we may assume that

$$\mu(\tilde{f}) \geq (2p - 1)(2(q - p) - 1) + d$$

and

$$\begin{aligned} \mu(\tilde{f}) &= (2p - 1)(2(q - p) - 1) + d \text{ iff } 0 \neq \sum_{i_q + j_p = 2pq + d} (-1)^{\lfloor i/p \rfloor} w_{ij} \\ &= \sum_{i(q-p) + j_p = 2(p-q)p + d} (-1)^{\lfloor i/p \rfloor} w_{i, j-i+2p}. \end{aligned}$$

This yields the if part of the result. Now if  $f$  is as above and  $\mu(f) > (2p - 1)(2q - 1) + d$ , the condition  $f_d = 0$  says that  $(x^p + y^q)$  divides  $\sum_{i_q + j_p = 2pq + d} w_{ij} x^i y^j$ , and adding  $-\frac{1}{2} \sum_{i_q + j_p = 2pq + d} w_{ij} x^i y^j$  to the first part of  $f$  one gets  $f = (x^p + y^q + \dots)^2 +$  terms of degree greater than  $2pq + d$ . Continuing this way, we get the result.

**LEMMA 2.** *Let  $f = (x^p + y^q)^2 + \sum_{i_q + j_p > 2pq} h_{ij} x^i y^j$  and  $\mu(f) = (2p - 1)(2q - 1) + d$ . Let  $\gamma, \delta$  such that  $\gamma q + \delta p = 3pq - q - p + d, \gamma < p$ . Let  $B = \{(i, j) \in N^2 / i < 2p - 1, j < q - 1\} \cup \{(i, j) \in N^2 / i < p, j < q\} \cup \{(i, j), (i, j) \in N^2 / i < \gamma, j < \delta + q\}$ . Then  $\{x^i y^j\}_{(i, j) \in B}$  is a base of  $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$ .*

*Proof.* We use the algorithm of Mora (cf. [3]) to compute a Groebner base of the ideal  $(\partial f / \partial x, \partial f / \partial y)$ . We consider  $\mathbb{C}[[x, y]]$  as a graded ring with  $\deg x = q, \deg y = p$ . Let  $f_1 = 1/2p(\partial f / \partial x)$  and  $f_2 = 1/2q(\partial f / \partial y)$ .

Consider  $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$  and let  $f_3$  be the reduction of  $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$  with respect to the initial terms  $x^{2p-1}$  resp.  $x^p y^{q-1}$  of  $f_1$  resp.  $f_2$ , i.e.

$$s(f_1, f_2) = f_3 + h_1 f_1 + h_2 f_2$$

$$f_3 = \sum_{\gamma_i < p} l_i x^{\gamma_i} y^{\delta_i}$$

$$q\gamma_i + p\delta_i = 3pq - q - p + i$$

and the initial terms of  $h_1$  resp.  $h_2$  have degree  $> pq - p$  resp.  $> pq - q$ .  $f_3 \neq 0$  because of  $\mu(f) < \infty$ . Let  $k$  be the minimal such that  $l_k \neq 0$ , i.e.  $l_k x^{\gamma_k} y^{\delta_k}$  is the initial term of  $f_3$ . Consider now

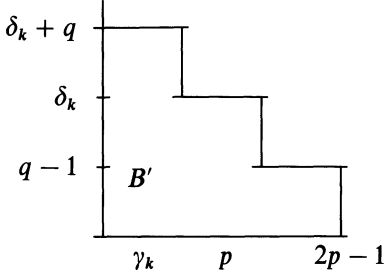
$$\begin{aligned} s(f_2, f_3) &= l_k y^{\delta_k - q + 1} f_2 - x^{p - \gamma_k} f_3 \\ &= l_k y^{\delta_k + q} + \text{terms of degree } > p\delta_k + pq \\ &=: f_4. \end{aligned}$$

It is not difficult to see that the reductions of  $s(f_1, f_3)$  and  $s(f_i, f_4) i = 1, 2, 3$  with respect to the initial terms of  $f_1, f_2, f_3, f_4$  are zero, i.e.  $f_1, f_2, f_3, f_4$  is a

Groebner base of  $(\partial f/\partial x, \partial f/\partial y)$ . This implies that

$$\{x^i y^j\}_{(i,j) \in B'}, B' = \{(i, j), i < 2p - 1, j < q - 1\} \cup \{(i, j), i < p, j < \delta_k\} \cup \{(i, j), i < \gamma_k, j < \delta_k + q\},$$

is a base of  $\mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$ .



This implies  $\mu(f) = (p - 1)(q - 1) + q\gamma_k + p\delta_k$  and therefore  $\gamma = \gamma_k$  and  $\delta = \delta_k$  and  $B = B'$ . □

**LEMMA 3.** Let  $f = (x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{ij} x^i y^j$  and  $\mu(f) = (2p - 1)(2q - 1) + d$ .

Let  $\gamma, \delta$  be defined by

$$\gamma < p \quad \text{and} \quad \gamma q + \delta p = 3pq - q - p + d$$

Let  $B_0 = \{(i, j), iq + jp > pq, i \leq p - 2, j \leq q - 2\}$ ;

$$B_1 = \{(i, j), iq + jp \geq 2pq + d, i < p, j < \delta\} \cup \{(i, j), iq + jp \geq 2pq + d, i < \gamma, j < \delta + q\}.$$

There is an automorphism  $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$  such that

$$f(\varphi) = \left( x^p + y^q + \sum_{(i,j) \in B_0} h_{ij} x^i y^j \right)^2 + \sum_{(i,j) \in B_1} w_{ij} x^i y^j$$

for suitable  $h_{ij}, w_{ij} \in \mathbb{C}$ .

*Proof.* Using Lemma 1 we may assume that

$$f = \left( x^p + y^q + \sum_{iq + jp > pq} b_{ij} x^i y^j \right)^2 + \sum_{iq + jp \geq 2pq + d} c_{ij} x^i y^j.$$

Assume that there is an automorphism  $\varphi^{(k)}$  such that

$$f(\varphi^{(k)}) = \left( x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq+jp \geq pq+k} b_{ij}^{(k)} x^i y^j \right)^2 + \\ + \sum_{(i,j) \in B_1} w_{ij}^{(k)} x^i y^j + \sum_{iq+jp \geq 2pq+k} c_{ij}^{(k)} x^i y^j,$$

$\varphi^{(1)} = \text{identity}$ .

Now

$$\sum_{iq+jp=2pq+k} c_{ij}^{(k)} x^i y^j = (x^p + y^q)H + \sum_{iq+jp=2pq+k} (-1)^{[i/p]} c_{ij}^{(k)} x^{i_0} y^{j_0}$$

for a suitable homogeneous  $H$  of degree  $pq + k$  and  $i_0 < p, i_0q + j_0p = 2pq + k$ .

If  $\sum_{iq+jp > 2pq+k} (-1)^{[i/p]} c_{ij}^{(k)} \neq 0$  and  $(i_0, j_0) \notin B_1$  then  $k \geq pq - q - p + d$  (Lemma 1) and  $j_0 \geq \delta, i_0 \geq \gamma$  or  $j_0 \geq \delta + q$ .

Let  $\alpha, \beta$  be defined by  $q\alpha + p\beta = 2pq + d, \alpha < p$ , then  $w_{\alpha\beta} \neq 0$ . Notice that  $\alpha - 1 \equiv \gamma \pmod p$  and  $\beta - 1 \equiv \delta \pmod q$ .

Let

$$g := x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq+jp \geq pq+k} b_{ij}^{(k)} x^i y^j$$

and

$$\omega := e \cdot x^\xi y^\eta \left( \frac{\partial g}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right), \quad \xi q + \eta p = k - pq + p + q - d$$

$$e := \frac{1}{(\alpha q + \beta p) w_{\alpha\beta}} \cdot \sum_{ip+jq=2pq+k} (-1)^{[i/p] - [\alpha - 1 + \xi/p] + 1} c_{ij}^{(k)}$$

with  $(\xi, \eta) = \begin{cases} (i_0 - \gamma, j_0 - \delta) & \text{if } j_0 \geq \delta, i_0 \geq \gamma \\ (i_0 - \gamma + p, j_0 - \delta - q) & \text{if } j_0 \geq \delta + q. \end{cases}$

Let  $\psi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$  the automorphisms corresponding to the vector field  $\omega$ , then  $g(\psi) = g$ .

Hence,

$$f(\psi \circ \varphi^{(k)}) = g^2 + \sum_{(i,j) \in B_0} w_{ij}^{(k)} x^i y^j + \sum_{iq+jp \geq 2pq+k} \bar{c}_{ij}^{(k)} x^i y^j$$

and

$$\begin{aligned} & \sum_{i_q + j_p = 2pq + k} (-1)^{i/p} \bar{c}_{ij}^{(k)} \\ &= \sum_{i_q + j_p = 2pq + k} (-1)^{i/p} c_{ij}^{(k)} + (-1)^{\alpha - 1 + \zeta/p} (\alpha q + \beta p) w_{\alpha\beta} \cdot e. \end{aligned}$$

If  $(i_0, j_0) \notin B_1$  we may assume now that  $\sum_{i_q + j_p = 2pq + k} (-1)^{i/p} c_{ij}^{(k)} = 0$ .

Let  $g_1 := g + \frac{1}{2}H$  and

$$\sum_{i_q + j_p = pq + k} b_{ij}^{(k)} x^i y^j + \frac{1}{2}H + m_k \frac{\partial g_1}{\partial x} + n_k \frac{\partial g_1}{\partial y} = \sum_{(i, j) \in B_0} d_{ij}^{(k)} x^i y^j.$$

The degree of the initial part of  $m_k$  resp.  $n_k$  is  $q + k$  resp.  $p + k$ .

We define  $\varphi^{(k+1)}$  by

$$\varphi^{(k+1)}(x) = \varphi^{(k)}(x) + m_k$$

$$\varphi^{(k+1)}(y) = \varphi^{(k)}(y) + n_k$$

and

$$h_{ij}^{(k+1)} = h_{ij}^{(k)} + d_{ij}^{(k)}$$

$$w_{ij}^{(k+1)} = w_{ij}^{(k)} \quad \text{if } (i, j) \neq (i_0, j_0)$$

$$w_{i_0, j_0}^{(k+1)} = w_{i_0, j_0}^{(k)} + \sum_{i_q + j_p = 2pq + k} (-1)^{i/p} c_{ij}^{(k)} \quad \text{if } (i_0, j_0) \in B_1.$$

Then

$$\begin{aligned} f(\varphi^{(k+1)}) &= \left( x^p + y^q + \sum_{(i, j) \in B_0} h_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq pq + k + 1} b_{ij}^{(k+1)} x^i y^j \right)^2 + \\ &+ \sum_{(i, j) \in B_1} w_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq 2pq + k + 1} c_{ij}^{(k+1)} x^i y^j \end{aligned}$$

for suitable  $b_{ij}^{(k+1)}, c_{ij}^{(k+1)}$ . □

LEMMA 4. *Let*

$$f_t = (x^p + y^q)^2 + \sum_{i_q + j_p > 2pq} a_{ij}(t) x^i y^j, \quad a_{ij}(t) \in \mathbb{C}[t]$$

and  $\mu(f_t) = (2p - 1)(2q - 1) + d$  for  $t \in \mathbb{C}$ .

Let  $\gamma, \delta, B_0, B_1$  be as in Lemma 3. There is a  $\mathbb{C}[t]$ -automorphism  $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$  such that

$$f_t(\varphi_t) = \left( x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}(t)x^i y^j \right)^2 + \sum_{(i,j) \in B_1} w_{ij}(t)x^i y^j$$

for suitable  $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$ .

The proof is similar to that of Lemma 3. □

Let us consider the family

$$F(x, y, H, W) = \left( x^p + y^q + \sum_{(i,j) \in B_0} H_{ij}x^i y^j \right)^2 + \sum_{(i,j) \in B_1} W_{ij}x^i y^j$$

depending on the parameters  $H = (H_{ij})_{(i,j) \in B_0}$ ,  $W = (W_{ij})_{(i,j) \in B_1}$  and define  $N = \# B_0 + \# B_1$ , then  $\mu(F) = (2p - 1)(2q - 1) + d$  on the open set  $U$  defined by  $W_{\alpha\beta} \neq 0$ ,  $\alpha q + \beta p = 2pq + d$ , in  $\mathbb{C}^N = \text{Spec } \mathbb{C}[H, W]$ . Notice that  $N = 2(p - 1)(q - 1) - p - q + 2 + [q/p]$  is not depending on  $d$ !

The group of  $2pq$ -roots of unity acts on  $U$ :

$$\lambda \in \mu_{2pq}, \lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})). \quad \square$$

**THEOREM 1.**  $U/\mu_{2pq}$  is the moduli space of all functions

$$f = (x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij}x^i y^j$$

with  $\mu(f) = (2p - 1)(2q - 1) + d$  and  $F$  is the universal family.

*Proof.* Using Lemma 3 we have to prove the following

**LEMMA 5.** Let  $\varphi$  be an automorphism  $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$  such that

$$F(\varphi(x), \varphi(y), \bar{h}, \bar{w}) = F(x, y, h, w) \quad (*)$$

for  $(\bar{h}, \bar{w}), (h, w) \in U \subseteq \mathbb{C}^N$  then  $\lambda \cdot (\bar{h}, \bar{w}) = (h, w)$  for a suitable  $\lambda \in \mu_{2pq}$ .

*Proof.* Let  $\bar{x} := \varphi(x)$ ,  $\bar{y} := \varphi(y)$  then grouping the squared part of (\*) one gets:

$$\begin{aligned} & \left( x^p + y^q + \bar{x}^p + \bar{y}^q + \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \times \\ & \times \left( x^p + y^q - \bar{x}^p - \bar{y}^q - \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \\ & = \sum \bar{w}_{ij} \bar{x}^i \bar{y}^j - \sum w_{ij} x^i y^j. \end{aligned}$$



This equation implies obviously that the degree of the initial term of  $\varphi(x)$  is  $\geq q$  and

$$\bar{x} = \lambda^q \left( x + \sum_{iq+jp>q} a_{ij}^{(1)} x^i y^j \right) \quad \bar{y} = \lambda^p \left( y + \sum_{iq+jp>p} a_{ij}^{(2)} x^i y^j \right), \lambda \in \mu_{2pq}.$$

We may assume that  $\lambda = 1$  and prove  $(\bar{h}, \bar{w}) = (h, w)$ .

Let the degree of the leading parts of both sides of the above equation be  $2pq + m$  and let  $r$  be the degree of  $\varphi$ , i.e.  $a_{ij}^{(1)} = 0$  if  $iq + jp < q + r$ ,  $a_{ij}^{(2)} = 0$  if  $iq + jp < p + r$  and  $a_{ij}^{(1)} \neq 0$  or  $a_{ij}^{(2)} \neq 0$  for suitable  $i, j$  with  $iq + jp = q + r$  resp.  $iq + jp = p + r$ .

1. Step. We prove that

(a)  $r \geq pq - p - q$

$$\sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} \cdot k, \quad \sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} \cdot k$$

(b)  $h = \bar{h}$  and  $w_{ij} = \bar{w}_{ij}$  if  $iq + jp < 3pq - p - q + d$ .

First of all  $m \geq d + r$  because the leading part of the left side of the equation is divisible by  $x^p + y^q$  and  $m < d + r$  would imply that the leading part of the right side is a monomial. This implies  $w_{ij} = \bar{w}_{ij}$  if  $iq + jp < 2pq + d + r$ . Now  $h_{ij} = \bar{h}_{ij}$  if  $iq + jp < pq + r$ . Otherwise the leading part of the left side of the equation would be  $2(x^p + y^q)(h_{ij} - \bar{h}_{ij})x^i y^j$  for some  $i, j$  with  $iq + jp < pq + r$  and therefore of degree  $2pq + r < 2pq + m$ .

Now suppose  $r < pq - p - q$ . Then there is at most one monomial of degree  $p + r$  resp.  $q + r$ .

If  $iq + jp = pq + r$  for some  $(i, j) \in B_0$  and

$$qi_0 + pj_0 = q + r$$

$$qi_1 + pj_1 = p + r$$

then

$$(h_{ij} - \bar{h}_{ij})x^i y^j - pa_{i_0 j_0}^{(1)} x^{i_0+p-1} y^{j_0} - qa_{i_1 j_1}^{(2)} x^{i_1} y^{j_1+q-1} = 0$$

otherwise the leading part of the left side of the equation would have degree  $2pq + r < 2pq + m$ .

But  $(i, j) \in B_0$ , i.e.  $i < p - 1$  and  $j < q - 1$ . This implies  $h_{ij} = \bar{h}_{ij}$ ,  $a_{i_0 j_0}^{(1)} = a_{i_1 j_1}^{(2)} = 0$  (because of  $r < pq - p - q$  we have  $i_1 < p - 1$ ). This is a contradiction since  $a_{i_0 j_0}^{(1)} \neq 0$  or  $a_{i_1 j_1}^{(2)} \neq 0$  by the definition of  $r$ .

Similarly one gets a contradiction if there is no  $(i, j) \in B_0$  with  $qi + pj = p + r$ , resp. no  $i_0, j_0$  with  $qi_0 + pj_0 = q + r$  resp. no  $i_1, j_1$  with  $qi_1 + pj_1 = p + r$ .

This proves that  $r \geq pq - p - q$ . With the same method we obtain

$$\sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} k \quad \text{and} \quad \sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} k.$$

(b) is clear now by the choice of  $B_0$  and the fact that  $r \geq pq - p - q$ .

2. Step. We prove that  $r \geq 2pq - p - q$ .

Assume that  $r < 2pq - p - q$ . Then  $\deg k < pq$ , i.e.,  $k$  is a monomial.

The leading part of the left side of the above equation is divisible by  $x^p + y^q$ .

The leading part  $L$  of the right side is

$$(\bar{w}_{ij} - w_{ij})x^i y^j + \bar{w}_{\alpha\beta} \cdot k \left( \frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if  $iq + jp = 2pq + m$  for some  $(i, j) \in B_1$  or

$$\bar{w}_{\alpha\beta} \cdot k \left( \frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if  $iq + jp \neq 2pq + m$  for  $(i, j) \in B_1$ .

Let  $k = \kappa \cdot x^\xi y^\eta$ . If  $\alpha + \xi - 1 < p$ , then  $i = \alpha + \xi - 1$  and  $j = \beta + \eta + q - 1$ . If  $\alpha + \xi - 1 \geq p$ , then  $i = \alpha + \xi - 1 - p$  and  $j = \beta + \eta + 2q - 1$ . But  $(\alpha + \xi - 1, \beta + \eta + q - 1) \notin B_1$  and  $(\alpha + \xi - 1 - p, \beta + \eta + 2q - 1) \notin B_1$ . This implies  $L = \bar{w}_{\alpha\beta} \cdot \kappa x^{\alpha-1} y^{\beta-1} ((\alpha/p)y^q - (\beta/p)x^p)$  which is not divisible by  $x^p + y^q$ . This is a contradiction and therefore  $r \geq 2pq - p - q$ .

Now  $iq + jp \leq 4pq - 2p - 2q + d$  for  $(i, j) \in B_1$  then  $w_{ij} = \bar{w}_{ij}$  for all  $(i, j) \in B_1$ . □

## 2. The construction of the moduli space

We will construct the moduli space of all plane curve singularities given by an equation  $(x^p + y^q)^2 + \sum_{iq+jp>2pq} a_{ij} x^i y^j = 0$  with fixed Milnor number  $\mu$ .

For  $\mu$  being even we get especially the moduli space for all irreducible plane curve singularities with the semigroup  $\Gamma = \langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$ .

We use the family

$$V(F) \subseteq U \times \mathbb{C}^2 \rightarrow U$$

constructed in Theorem 1.

$U$  admits a  $\mathbb{C}^*$ -action defined by

$$\lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})).$$

We get

$$F(\lambda^q x, \lambda^p y, h, w) = \lambda^{2pq} F(x, y, \lambda \circ (h, w)).$$

If  $\mu = (2p - 1)(2q - 1) + d$  and  $\alpha q + \beta p = 2pq + d$ ,  $\alpha < p$ , then  $U \subseteq \mathbb{C}^N$  was defined by  $W_{\alpha, \beta} \neq 0$ .

For the construction of the moduli space it is enough to consider the restriction of our family to the transversal section to the orbits of the  $C^*$ -action defined by  $W_{\alpha, \beta} = 1$ .

Let  $W'$  be defined by  $W = (W_{\alpha, \beta}, W')$  and  $G(x, y, H, W') = F(x, y, H, 1, W')$ . The parameter space of  $G$  is  $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[[H, W']$ .

The group  $\mu_d$  of  $d$ th roots of unity acts on the family

$$V(G) \subseteq \mathbb{C}^2 \times \mathbb{C}^{N-1} \rightarrow \mathbb{C}^{N-1}$$

induced by the above  $C^*$ -action

$$G(\lambda^q x, \lambda^p y, h, w') = \lambda^{2pq} G(x, y, \lambda \circ (h, w')) \quad \lambda \in \mu_d.$$

LEMMA 6. *Let  $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$  be an automorphism and  $u \in \mathbb{C}[[x, y]]$  a unit such that*

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

*then there is a  $\lambda \in \mu_d$  such that  $(h, w')$  and  $\lambda \circ (\bar{h}, \bar{w}')$  are contained in an analytically trivial subfamily of  $V(G) \rightarrow \mathbb{C}^{N-1}$ .*

*Proof.* Let

$$\varphi(x) = \sum a_{ij}^{(1)} x^i y^j \quad \text{and} \quad \varphi(y) = \sum a_{ij}^{(2)} x^i y^j, \quad u = \sum u_{ij} x^i y^j.$$

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

implies

- (1)  $a_{ij}^{(1)} = 0$  if  $iq + jp < q$
- (2)  $a_{1,0}^{(1)2p} = a_{0,1}^{(2)2q} = a_{1,0}^{(1)p} a_{0,1}^{(2)q} = u_{0,0}^{-1}$ .

Let  $a_{1,0}^{(1)} = \lambda^q$  and  $a_{0,1}^{(2)} = \lambda^p$ .

We will prove later that  $\lambda^d = 1$ .

Now we may assume that  $\lambda = 1$  and prove that  $(h, w')$  and  $(\bar{h}, \bar{w}')$  are contained in an analytically trivial subfamily of  $V(G) \rightarrow \mathbb{C}^{N-1}$ .

We choose

- (1)  $u(t) \in \mathbb{C}[t][[x, y]]$  with the following properties  $u(0) = 1$ ,  $u(1) = u$  and  $u$  is a unity for all  $t \in \mathbb{C}$ .
- (2)  $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$  with the following properties  $\varphi_0 = \text{identity}$ ,  $\varphi_1 = \varphi$  and  $\varphi_t$  is an automorphism of positive degree for all  $t \in \mathbb{C}$ .

Let  $H(t) := u(t)G(\varphi_t(x), \varphi_t(y), h, w')$  and apply Lemma 4. There is an  $\mathbb{C}[t]$ -automorphism  $\Phi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$  such that

$$H(\Phi_t) = F(x, y, h(t), w(t))$$

for suitable  $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$  with the property

$$\begin{aligned} h(0) &= h \\ w(0) &= (1, w'). \end{aligned}$$

$H(\Phi_t)$  has a constant Milnor number, i.e.  $w_{\alpha, \beta}(t)$  has to be constant.

This implies

$$H(\Phi_t) = G(x, y, h(t), w'(t)).$$

But,

$$G(x, y, h(1), w'(1)) = H(\Phi_1) = G(\Phi_1(x), \Phi_1(y), \bar{h}, \bar{w}').$$

Using Lemma 5 and the fact that  $\Phi_1$  has positive degree we get

$$\begin{aligned} \bar{h} &= h(1) \\ \bar{w}' &= w'(1), \end{aligned}$$

i.e.  $(h, w')$  and  $(\bar{h}, \bar{w}')$  are in the trivial family

$$G(x, y, h(t), w'(t)) = u(\Phi_t)G(\Phi_t \varphi_t, h, w').$$

To finish the proof of Lemma 6 we have to prove

LEMMA 7. *Let*

$$f_k = \left( x^p + y^q + \sum_{i_q + j_p > pq} a_{ij}^{(k)} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{i_q + j_p > 2pq + d} b_{ij}^{(k)} x^i y^j, \quad k = 1, 2$$

$$\alpha < p, \alpha q + \beta p = 2pq + d.$$

Let  $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$  be an automorphism with the property

$$\varphi(x) = \lambda^q x + \text{terms of degree } > q$$

$$\varphi(y) = \lambda^p y + \text{terms of degree } > p$$

and  $u$  a unit, such that

$$f_1(\varphi) = f_2 \cdot u$$

then  $\lambda^d = 1$ .

*Proof.*  $u = \lambda^{2pq} + \text{terms of higher degree}$ .

$$u \cdot f_2 = \lambda^{2pq} \left( x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(2)} x^i y^j \right)^2 + \lambda^{2pq} x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \bar{b}_{ij}^{(2)} x^i y^j$$

$$f_1(\varphi) = \lambda^{2pq} \left( x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(1)} x^i y^j \right)^2 + \lambda^{2pq+d} x^\alpha y^\beta + \sum_{iq+jp \geq 2pq+d} \bar{b}_{ij}^{(1)} x^i y^j$$

for suitable  $\bar{a}_{ij}^{(k)}, \bar{b}_{ij}^{(k)}$ .

This implies

$$\begin{aligned} & \left( 2x^p + 2y^q + \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} + \bar{a}_{ij}^{(2)}) x^i y^j \right) \cdot \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} - \bar{a}_{ij}^{(2)}) x^i y^j \\ &= (1 - \lambda^d) x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \lambda^{-2pq} (\bar{b}_{ij}^{(2)} - \bar{b}_{ij}^{(1)}) x^i y^j. \end{aligned}$$

Because the leading term of the left side of the equation is divisible by  $x^p + y^q$ , we get  $\lambda^d = 1$ . □

We consider now the Kodaira-Spencer map of the family

$$V(G) \rightarrow \mathbb{C}^{N-1}:$$

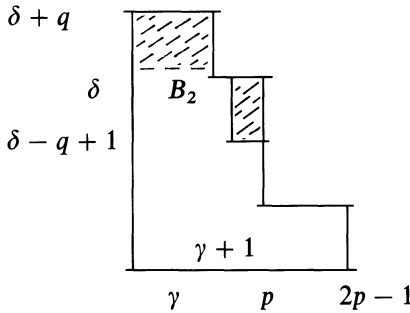
$$\rho: \text{Der}_{\mathbb{C}} \mathbb{C}[H, W'] \rightarrow \mathbb{C}[H, W'][[x, y]] / \left( G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

defined by

$$\rho(\delta) = \text{class}(\delta G).$$

The kernel of the Kodaira-Spencer map is a Lie-algebra  $L$  and along the integral manifolds of  $L$  the family is analytically trivial. We will choose a transversal section to the integral manifolds of  $L$  and divide by the action of  $\mu_d$  to get the moduli space. To describe this transversal section we choose a suitable subset of  $B_1$ :

$$B_2 = \{(i, j) \in B_1, i \leq \gamma, j \leq \delta\} \cup \{(i, j) \in B_1, j \leq \delta - q\}.$$



Let  $M := \#B_0 + \#B_2 = N - (p - 1)(q - 1) = (p - 2)(q - 2) + [q/p] - 1$ . Let  $W'' := (W_{ij})_{(i,j) \in B_2}$  and  $\mathbb{C}^M = \text{Spec } \mathbb{C}[H, W'']$

$$G_u(x, y, H, W'') := \left( x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{(i,j) \in B_2} W_{ij} x^i y^j$$

As before  $\mu_d$  acts on the family  $V(G_u) \subseteq \mathbb{C}^2 \times \mathbb{C}^M \rightarrow \mathbb{C}^M$ .

**THEOREM 2.**  $\mathbb{C}^M / \mu_d$  is the moduli space of all plane curve singularities defined by an equation

$$(x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{ij} x^i y^j = 0$$

with Milnor numbers  $\mu = (2p - 1)(2q - 1) + d$  and  $G_u$  is the corresponding universal family.

Especially the Tjurina number  $\tau = \mu - (p - 1)(q - 1)$  only depends on  $\mu$  for these singularities.

**COROLLARY.** Let  $\Gamma = \langle 2p, 2q, 2pq + d \rangle$ ,  $d$  odd, a semigroup.

Then  $\mathbb{C}^{(p-2)(q-2)+[q/p]-1} / \mu_d$  is the moduli space of all irreducible plane curve singularities with the semigroup  $\Gamma$ .

$G_u$  is the corresponding universal family.

*Proof.* To prove the theorem we compute generators of the kernel of the

Kodaira-Spencer map.

Let

$$G^{(0)} = x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j$$

$$G^{(1)} = x^\alpha y^\beta + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} W_{ij} x^i y^j, \text{ i.e.}$$

$$G = G^{(0)2} + G^{(1)}.$$

Let  $\delta \in \text{Der}_{\mathbb{C}} \mathbb{C}[H, W']$  be a vector field which belongs to the kernel of the Kodaira-Spencer map, i.e.

$$\delta G \in \left( G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Now

$$\delta G = 2G^{(0)} \sum_{(i,j) \in B_0} \delta H_{ij} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} \delta W_{ij} x^i y^j = S \cdot G \bmod \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$$

for a suitable  $S \in \mathbb{C}[H, W'][[x, y]]$ .

We will associate to any monomial  $x^a y^b$ ,  $(a, b) \neq (0, 0)$ , a vector field  $\delta_{a,b} \in \text{Der}_{\mathbb{C}}[H, W']$  such that

$$\delta_{a,b} G = x^a y^b G \bmod \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Obviously  $\{\delta_{a,b}\}$  generate the kernel of the Kodaira-Spencer map as  $\mathbb{C}[H, W']$ -module.

Now consider

$$x^a y^b G = x^a y^b G^{(0)2} + x^a y^b G^{(1)}.$$

$$\text{Let } x^a y^b G^{(0)} = \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + L_1 \frac{\partial G^{(0)}}{\partial x} + L_2 \frac{\partial G^{(0)}}{\partial y}$$

for suitable  $E_{ij}^{ab} \in \mathbb{C}[H, W']$ ,  $L_1, L_2 \in \mathbb{C}[H, W'][[x, y]]$ ,

$$L_1 = \frac{1}{p} x^{a+1} y^b + \text{terms of higher degree}$$

$$L_2 = \frac{1}{q} x^a y^{b+1} + \text{terms of higher degree,}$$

then

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \bmod \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

The leading term of

$$x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y}$$

is  $-(d/2pq)x^{\alpha+a}y^{\beta+b}$ .

Using Lemma 2 we get

$$\begin{aligned} & x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \\ &= \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right) \end{aligned}$$

for suitable  $D_{ij}^{ab} \in \mathbb{C}[H, W']$ .

This implies

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

We define for  $(a, b) \neq (0, 0)$

$$\delta_{a,b}(H_{ij}) := \frac{1}{2} E_{ij}^{ab}$$

$$\delta_{a,b}(W_{ij}) := D_{ij}^{ab}, \text{ i.e.,}$$

$$\delta_{a,b} = \frac{1}{2} \sum E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \sum D_{ij}^{ab} \frac{\partial}{\partial W_{ij}}$$

The vector fields  $\delta_{a,b}$  have the following properties:

- (1)  $\delta_{a,b}$  is zero if  $aq + bp > 2pq - 2p - 2q$
- (2)  $\delta_{a,b}(W_{ij}) = 0$  if  $iq + jp < 2pq + d + aq + bp$
- (3)  $\delta_{a,b}(W_{ij}) = -d/2pq$  if  $(i, j) = (\alpha + a, \beta + b)$  or  $(i, j) = (\alpha + a - p, \beta + b + q)$  (in this case  $iq + jp = 2pq + d + aq + bp$ ).



- (4)  $\delta_{a,b}(H_{ij}) = 0$  for all  $(i, j) \in B_0$  if  $aq + bp \geq pq - 2p - 2q$
- (5) For  $iq + jp \geq 3pq + d - q$  and  $(i, j) \in B_1$  there is  $(a', b')$  such that

$$(i, j) = (\alpha + a', \beta + b') \text{ or } (i, j) = (\alpha + a' - p, \beta + b' + q),$$

i.e.  $\delta_{a',b'}(W_{ij}) = -d/2pq.$

- (6) For any  $(a, b)$ ,  $aq + bp < pq - q$ , always  $(\alpha + a, \beta + b)$  or  $(\alpha + a - p, \beta + b + q) \in B_1$ , i.e.  $\delta_{a,b}(W_{ij}) = -d/2pq$  for the corresponding  $(i, j) \in B_1$ .

(1) and (4) hold because of the fact that

$$iq + jp \leq 2pq - 2p - 2q \text{ if } (i, j) \in B_0$$

$$iq + jp \leq 4pq + d - 2p - 2q \text{ if } (i, j) \in B_1.$$

(2) and (3) hold because of the fact that the leading term of  $G^{(1)}$  has degree  $2pq + d$  and because of Lemma 2. To prove (5) we consider two cases

1. Case  $i \geq \alpha - 1$

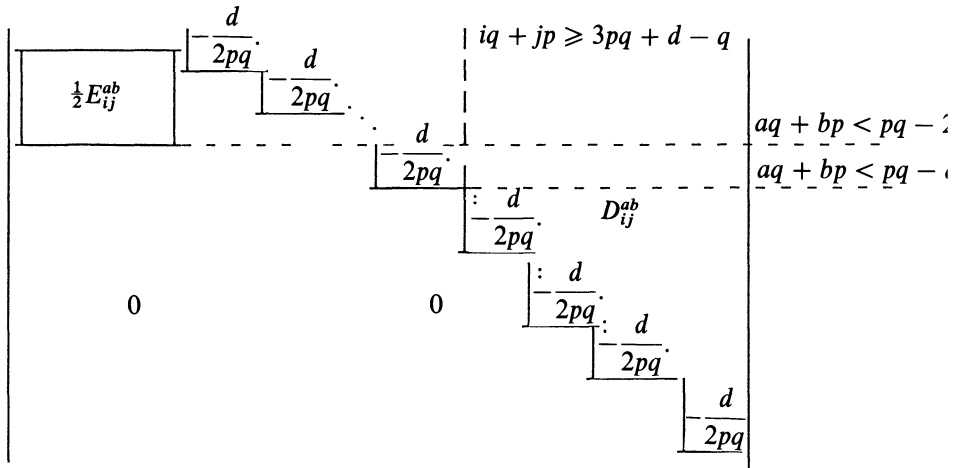
In this case  $(i, j) \in B_1$  implies  $i \leq p - 1$ . But  $iq + jp \geq 3pq + d - q$  implies  $j \geq \beta$ . Then  $a' = i - \alpha, b' = j - \beta$  have the required properties. Notice that  $i < \alpha - 1$  and  $iq + jp \geq 3pq + d - q$  implies  $(i, j) \notin B_1$

2 Case  $i < \alpha - 1$

Now  $iq + jp \geq 3pq + d - q$  implies  $j \geq \beta + q$  then  $a' = p + i - \alpha, b' = j - \beta - q$  have the required properties. (6) is similar to (5):

We may assume that  $(\alpha + a, \beta + b) \notin B_1$ . This implies  $2p - 3 \geq \alpha + a \geq p$  and  $\alpha \geq 2$  because  $b \leq q - 2, a \leq p - 2$ . Suppose  $(\alpha + a - p, \beta + b + q) \notin B_1$  then  $\beta + b + q \geq \beta + 2q - 1$ , i.e.  $b \geq q - 1$ , or  $\alpha + a - p \geq \alpha - 1$ , i.e.  $a \geq p - 1$ , but this is not possible. □

For the coefficients to the vectorfields  $\delta_{a,b}$  we get, because of (1)–(6), the following matrix:



This implies that the kernel of the Kodaira–Spencer map is generated (as  $\mathbb{C}[H, W']$ -module) by the vector fields

$$\delta'_{ij} := \frac{\partial}{\partial w_{ij}} \quad (i, j) \in B_1, \quad iq + jp \geq 3pq + d - q$$

and

$$\begin{aligned} \delta'_{l,m} = & -\frac{pq}{d} \sum_{(i,j) \in B_0} E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \frac{\partial}{\partial W_{l,m}} + \\ & + \sum_{\substack{(i,j) \in B_1 \\ 2pq + d + aq + bp < iq + jp < 3pq + d - q}} \left( -\frac{2pq}{d} \right) D_{ij}^{ab} \frac{\partial}{\partial W_{ij}} \\ & \times (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}), \quad lq + mp < 3pq + d - q, \end{aligned}$$

$$\text{with } (l, m) = \begin{cases} (\alpha + a, \beta + b) & \text{if } l \geq \alpha \\ (\alpha + a - p, \beta + b + q) & \text{else.} \end{cases}$$

The vectorfields  $\delta'_{l,m}$  act nilpotently on  $\mathbb{C}[H, W']$ . Namely, if we consider  $\mathbb{C}[H, W']$  as a graded algebra defined by  $\deg H_{ij} = pq - iq - jp < 0$ ,  $\deg W_{ij} = 2pq - iq - jp < 0$  then the  $E_{ij}^{ab}$  resp.  $D_{ij}^{ab}$  are polynomials in  $\mathbb{C}[H, W']$  of degree  $\geq aq + bp + pq - iq - jp$  resp.  $\geq aq + bp + 2pq - iq - jp$ . Notice that their degree is always  $\leq 0$ . Let  $A \in \mathbb{C}[H, W']$  be any polynomial of degree  $0 \geq \deg A = s$  ( $\deg A =$  minimum of the degrees of the monomials in  $A$ ). Then the degree of  $\delta'_{lm}(A) > s$ . Therefore there is some  $n$  with  $\delta'^n_{lm}(A) = 0$ .

LEMMA 8. Let  $A$  be a ring of finite type over a field  $k$ .  $L \subseteq \text{Der}_k(A)$  a Lie-Algebra.

Let  $\delta_1, \dots, \delta_r$  vector fields with the following properties:

- (1)  $\delta_1, \dots, \delta_r \in L$  and  $L \subseteq \sum \delta_i A$
- (2)  $[\delta_i, \delta_j] \in \sum_{k > \max\{i,j\}} \delta_k A$
- (3) There are  $x_1, \dots, x_r \in A$  such that

$$\delta_i(x_i) = 1 \text{ and } \delta_j(x_i) = 0 \quad j > i$$

- (4)  $\delta_1, \dots, \delta_r$  act nilpotently on  $A$ .

Then  $A^L[x_1, \dots, x_r] = A$ .

The Lemma is not difficult to prove. A similar lemma was used in the construction of the moduli space for curve singularities with the semi-group  $\langle p, q \rangle$  (cf. [1], [2]).

Obviously  $A^L$  is the ring of all elements of  $A$  being invariant under  $\delta_1, \dots, \delta_r$ .

Now  $A^{\delta_r}[x_r] = A$  and the conditions (2)–(4) of the lemma are satisfied for  $\delta_1, \dots, \delta_{r-1}$  acting on  $A^{\delta_r}$ .

Now we may apply the Lemma 8 to the kernel of the Kodaira-Spencer map and its generators  $\{\delta'_{lm}\}$ .

Because of the lemma the geometric quotient of  $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[H, W']$  by the action of the kernel of the Kodaira-Spencer map exist and is isomorphic to the transversal section to the maximal integral manifolds (which intersect therefore each of these integral manifolds exactly in one point) defined by

$$W_{l,m} = 0, \quad (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}).$$

Now we use Lemma 6 and get Theorem 2. Notice that

$$\{G^{(0)}x^i y^j\}_{(i,j) \in B_0} \cup \{x^i y^j\}_{(i,j) \in B_2} \cup \{x^i y^j, iq + jp \leq 2pq, (i, j) \in B\}$$

is a base of the free  $\mathbb{C}[H, W']$ -module  $\mathbb{C}[H, W'][[x, y]]/(G, \partial G/\partial x, \partial G/\partial y)$ . This implies  $\mu - \tau = \#(B_1 \setminus B_2) = (p - 1)(q - 1)$ .  $\square$

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