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Monads and cohomology modules of rank 2 vector bundles

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Introduction

Monads are a useful tool to construct and study rank 2 vector bundles on the complex projective space \mathbb{P}_n , $n \geq 2$ (compare [O-S-S]). Horrocks' technique of eliminating cohomology [Ho 2] represents a given rank 2 vector bundle \mathcal{E} as the cohomology of a monad

$$(M(\mathcal{E})) \quad \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$$

as follows.

First eliminate the graded $S = \mathbb{C}[z_0, \dots, z_n]$ -module
 $H^1 \mathcal{E}(\ast) = \bigoplus_{m \in \mathbb{Z}} H^1(\mathbb{P}_n, \mathcal{E}(m))$ by the universal extension

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow \tilde{L}_0 \rightarrow 0,$$

where

$$L_0 \rightarrow H^1 \mathcal{E}(\ast) \rightarrow 0.$$

is given by a minimal system of generators (\sim stands for sheafification).

If $n = 2$ take this extension as a monad with $\mathcal{A} = 0$.

If $n \geq 3$ eliminate dually $H^{n-1} \mathcal{E}(\ast)$ by the universal extension

$$0 \rightarrow \tilde{L}_0^\vee(c_1) \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$$

(where $c_1 = c_1(\mathcal{E})$ is the first Chern-class). Then notice, that the two extensions

* Partially supported by the DAAD.

can be completed to the display

$$\begin{array}{ccccccc}
 & & 0 & 0 & & & \\
 & & \downarrow & \downarrow & & & \\
 0 & \longrightarrow & \tilde{L}_0^\vee(c_1) & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \tilde{L}_0^\vee(c_1) & \xrightarrow{\varphi} & \mathcal{B} & \longrightarrow & \mathcal{L} \longrightarrow 0 \\
 & & & & \downarrow \psi & & \downarrow \\
 & & & & \tilde{L}_0 = \tilde{L}_0 & & \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

of a monad

$$\tilde{L}_0^\vee(c_1) \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \tilde{L}_0$$

for \mathcal{E} .

To get a better understanding for \mathcal{B} , φ and ψ consider first the case $n = 2, 3$. Then \mathcal{B} is a direct sum of line bundles by Horrocks' splitting criterion [Ho 1]. Taking cohomology we obtain a free presentation

$$B \xrightarrow{\psi} L_0 \longrightarrow H^1 \mathcal{E}(\ast) \longrightarrow 0$$

with $B = H^0 \mathcal{B}(\ast)$. The crucial point is that this is minimal [Ra]. Moreover, if $n = 3$, then B is self-dual [Ra]: $B^\vee(c_1) \simeq B$. We will see below that up to isomorphism φ is the dual map of ψ .

Let us summarize and slightly generalize. Consider an arbitrary graded S -module N of finite length with minimal free resolution (m.f.r. for short)

$$0 \longrightarrow L_{n+1} \xrightarrow{\alpha_n} L_n \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow N \longrightarrow 0.$$

If $n = 2$ then $N \simeq H^1 \mathcal{E}(\ast)$ for some rank 2 vector bundle \mathcal{E} on \mathbb{P}_2 iff $\widehat{\text{rk}} L_1 = \text{rk } L_0 + 2$ (compare [Ra]). In this case \mathcal{E} is uniquely determined as $\ker \alpha_0$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tilde{L}_0^\vee(c_1) & \xrightarrow{\alpha_0^\vee(c_1)} & \tilde{L}_1^\vee(c_1) & \xrightarrow{\alpha_1} & \tilde{L}_1 & \xrightarrow{\alpha_0} & \tilde{L}_0 & \longrightarrow & 0. \\
 & & & & \searrow & & \nearrow & & & & \\
 & & & & & & \mathcal{E} & & & & \\
 & & & & \nearrow & & \searrow & & & & \\
 & & & & 0 & & 0 & & & &
 \end{array}$$

with $B = H^0 \mathcal{B}(\ast)$, inducing the monad

$$(M(\mathcal{F})) \quad \tilde{L}_0^\vee(c_1) \rightarrow \mathcal{B} \rightarrow \tilde{L}_0$$

and the minimal free presentation

$$L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow H^1 \mathcal{F}(\ast) \longrightarrow 0.$$

The corresponding m.f.r. decomposes as

$$\begin{aligned} 0 \rightarrow L_5 \rightarrow L_4 \rightarrow L'_3 \oplus H_2 &\xrightarrow{\begin{pmatrix} \ast & 0 \\ \ast & \beta_1 \end{pmatrix}} L'_2 \oplus H_1 \xrightarrow{\begin{pmatrix} \ast & \beta'_0 \end{pmatrix}} L_1 \xrightarrow{\alpha_0} L_0 \rightarrow \\ &\rightarrow H^1 \mathcal{F}(\ast) \rightarrow 0 \end{aligned}$$

inducing the m.f.r.

$$0 \rightarrow \tilde{L}_5 \rightarrow \tilde{L}_4 \rightarrow \tilde{L}'_3 \rightarrow \tilde{L}'_2 \rightarrow \mathcal{F} \rightarrow 0.$$

$(M(\mathcal{F}))$ is the monad given in [H-M]. Using its display we can compute the above m.f.r.'s explicitly. Especially we reobtain the equations of the abelian surfaces in \mathbb{P}_4 ([Ma 1], [Ma 2]).

Of course we may deduce from \mathcal{F} some more bundles by pulling it back under finite morphisms $\pi: \mathbb{P}_4 \rightarrow \mathbb{P}_4$. The above result also holds for the bundles $\pi^* \mathcal{F}$ with $(M(\pi^* \mathcal{F})) = \pi^*(M(\mathcal{F}))$.

There is some evidence (but so far no complete proof), that a splitting as in Proposition 2 occurs for every indecomposable 2-bundle on \mathbb{P}_4 . This suggests a new construction principle for such bundles by constructing their H^2 -module first.

Proof of Proposition 1

Let $n = 3$ and N be a graded S -module of finite length with m.f.r.

$$0 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1 \xrightarrow{\alpha_0} L_0 \rightarrow N \rightarrow 0.$$

Suppose first that $N \simeq H^1 \mathcal{E}(\ast)$ for some 2-bundle \mathcal{E} on \mathbb{P}_3 (with first Chern-class c_1). As seen in the introduction, Horrocks' construction leads to a monad

$$(M(\mathcal{E})) \quad \tilde{L}_0^\vee(c_1) \xrightarrow{\varphi} \tilde{L}_1 \xrightarrow{\alpha_0} \tilde{L}_0$$

for \mathcal{E} . The dual sequence

$$\tilde{L}_0^\vee(c_1) \xrightarrow{\alpha_0^\vee(c_1)} \tilde{L}_1^\vee(c_1) \xrightarrow{\varphi^\vee(c_1)} \tilde{L}_0$$

is a monad for $\mathcal{E}^\vee(c_1) \simeq \mathcal{E}$. The induced presentation of N has to be isomorphic to that one given by the m.f.r.:

$$\begin{array}{ccccccc} L_1^\vee(c_1) & \xrightarrow{\varphi^\vee(c_1)} & L_0 & \rightarrow & N & \rightarrow & 0 \\ \Phi^\vee(c_1) \downarrow & \parallel & & & & & \parallel \\ L_1 & \xrightarrow{\alpha_0} & L_0 & \rightarrow & N & \rightarrow & 0. \end{array}$$

Dualizing gives (2) since $\alpha_0 \circ \varphi = 0$ and thus also a monad (M_Φ) for \mathcal{E} , isomorphic to $(M(\mathcal{E}))$ (replace φ by $\Phi \circ \alpha_0^\vee(c_1)$).

Conversely if N satisfies (2), we obtain a monad (M_Φ) by sheafification. (Since $\tilde{N} = 0$, α_0 is a bundle epimorphism. Dually $\alpha_0^\vee(c_1)$ is a bundle monomorphism.) Let \mathcal{E} be the cohomology bundle of (M_Φ) . Then $H^1\mathcal{E}(\ast) \simeq N$. \mathcal{E} has rank 2, if N satisfies (1). □

REMARK 1. (i) Let $N \simeq H^1\mathcal{E}(\ast)$ as above with induced splitting

$$0 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2' \oplus L_0^\vee(c_1) \rightarrow L_1 \rightarrow L_0 \rightarrow H^1\mathcal{E}(\ast) \rightarrow 0$$

as in the introduction. Recall that \mathcal{E} is stable iff $H^0(\mathbb{P}_3, \mathcal{E}(m)) = 0$ for $m \leq -c_1/2$. Thus \mathcal{E} is stable iff L_2' has no direct summand $S(m)$ with $m \geq c_1/2$. Notice that this condition only depends on N .

(ii) If N satisfies (1) and has only one generator, then (2) is obviously equivalent to the symmetry condition $L_1^\vee(c_1) \simeq L_1$. Thus [Ra, 3.1] is a special case of Proposition 1.

EXAMPLES. (i) The well-known Null correlation bundles are by definition the bundles corresponding to the S -module \mathbb{C} . Consider the Koszul-presentation

$$4S \xrightarrow{\alpha_0} S(1) \rightarrow \mathbb{C} \rightarrow 0, \quad \alpha_0 = (z_0, z_1, z_2, z_3).$$

The isomorphisms $4S \xrightarrow{\Phi} 4S$ with $\alpha_0 \circ \Phi \circ \alpha_0^\vee(c_1) = 0$ are precisely the 4×4 skew symmetric matrices with nonzero determinant. Two such matrices give isomorphic bundles iff they differ by a scalar (use [O-S-S, II, Corollary 1 to 4.1.3]). The moduli space of Null correlation bundles is thus isomorphic to $\mathbb{P}_5 \setminus \mathbb{G}$, where \mathbb{G} is the Plucker embedded Grassmanian of lines in \mathbb{P}_3 .

Unlike the case $n = 2$ the bundle is not uniquely determined by the module.

(ii) The S -module

$$6S \xrightarrow{\alpha_0} S(1) \oplus S(2) \rightarrow N \rightarrow 0, \alpha_0 = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & 0 & 0 \\ 0 & 0 & z_0^2 & z_1^2 & z_2^2 & z_3^2 \end{pmatrix}$$

satisfies (1) and the symmetry condition $L_1^\vee \simeq L_1$, i.e. the necessary conditions of [Ra]. But N does not satisfy (2).

Cohomology modules of the Horrocks-Mumford-bundle \mathcal{F}

We first recall the construction of \mathcal{F} [H-M]. Let

$$V = \text{Map}(\mathbb{Z}_5, \mathbb{C})$$

be the vector space of complex valued functions on \mathbb{Z}_5 . Denote by

$$H \subset N \subset \text{SL}(5, \mathbb{C})$$

the Heisenberg group and its normalizer in $\text{SL}(5, \mathbb{C})$ resp.

Let $V_0 = V, V_1, V_2, V_3$ and

$$W = \text{Hom}_H(V_1, \Lambda^2 V)$$

be defined as in [H-M]. The V_i are irreducible representations of H and N of degree 5. W is an irreducible representation of N/H of degree 2. It is unimodular, so it comes up with an invariant skew symmetric pairing.

Let $\mathbb{P}_4 = \mathbb{P}(V)$ be the projective space of lines in V . The Koszul-complex on $\mathbb{P}(V)$ is the exact sequence

$$(K) \quad \dots \rightarrow \mathcal{O}(-1) \otimes \Lambda^2 V \rightarrow \mathcal{O} \otimes \Lambda^3 V \rightarrow \dots,$$

$$\begin{array}{ccc} & \searrow & \nearrow \\ & (\Lambda^2 \mathcal{F})(-3) & \\ & \nearrow & \searrow \\ 0 & & 0 \end{array}$$

obtained by exterior multiplication with the tautological subbundle

$$\mathcal{O}(-1) \rightarrow V \otimes \mathcal{O}.$$

The exterior product provides (K) with a self-duality (with values in $\mathcal{O}(-1) \otimes \Lambda^5 V \simeq \mathcal{O}(-1)$).

This induces the natural pairings

$$(\Lambda^i \mathcal{F})(-i-1) \otimes (\Lambda^{4-i} \mathcal{F})(i-5) \xrightarrow{\wedge} (\Lambda^4 \mathcal{F})(-6) \simeq \mathcal{O}(-1)$$

and is compatible with the action of $\mathrm{SL}(5, \mathbb{C})$.

It can be extended to $(K) \otimes W$ by tensoring with the invariant form, then being compatible with the action of N .

As in the proof of [H-M, Lemma 2.4] it follows, that $(K) \otimes W$ decomposes as

$$\begin{array}{ccccc} \dots & \rightarrow & \begin{array}{c} \mathcal{O}(-1) \otimes V_1 \\ \oplus \\ \mathcal{O}(-1) \otimes V_1 \otimes U \end{array} & \xrightarrow{\begin{pmatrix} 0 & \alpha_0 \\ \alpha_0^\vee(-1) & \star \end{pmatrix}} & \begin{array}{c} \mathcal{O} \otimes V_3 \\ \oplus \\ \mathcal{O} \otimes V_3 \otimes U \end{array} & \rightarrow \dots, \\ & & & \searrow & \nearrow \\ & & & (\Lambda^2 T)(-3) \otimes W & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

given by the splitting into irreducible N -modules. Moreover the induced

$$\mathcal{O}(-1) \otimes V_1 \rightarrow (\Lambda^2 \mathcal{F})(-3) \otimes W \rightarrow \mathcal{O} \otimes V_3$$

is the self-dual Horrocks-Mumford-monad, whose cohomology is \mathcal{F} (normalized such that $c_1 \mathcal{F} = -1$).

To proof Proposition 2 consider the display

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{O}(-1) \otimes V_1 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F} \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{O}(-1) \otimes V_1 & \rightarrow & (\Lambda^2 \mathcal{F})(-3) \otimes W & \rightarrow & \mathcal{L} \rightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{O} \otimes V_3 & = & \mathcal{O} \otimes V_3 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

It first follows that $H^2 \mathcal{F}(\star) = W$ is a vector space, sitting in degree -2 (compare [H-M]). Its m.f.r. is the Koszul-complex obtained from $(K) \otimes W$ by taking global

sections. So it decomposes, inducing the presentation

$$S(-1) \otimes V_1 \otimes U \xrightarrow{\alpha_0} S \otimes V_3 \longrightarrow H^1 \mathcal{F}(\ast) \longrightarrow 0$$

and the Horrocks-Mumford-monad. But this is just

$$(M(\mathcal{F})) \quad \mathcal{O}(-1) \otimes V_1 \rightarrow \mathcal{B} \rightarrow \mathcal{O} \otimes V_3:$$

apply e.g. [O-S-S, II, Corollary 1 to 4.1.3] (notice that $H^0 \mathcal{F} = 0$ implies $H^0 \mathcal{B} = H^0 \mathcal{B}^\ast(-1) = 0$ by construction of $(M(\mathcal{F}))$).

It remains to show that α_0 is minimal and that the corresponding m.f.r. of $H^1 \mathcal{F}(\ast)$ decomposes, inducing the m.f.r., say,

$$0 \longrightarrow F_3 \xrightarrow{\gamma_2} F_2 \xrightarrow{\gamma_1} F_1 \xrightarrow{\gamma_0} F_0 \longrightarrow F \longrightarrow 0$$

of $F = H^0 \mathcal{F}(\ast)$.

From the second row of the display we obtain the m.f.r. of $Q = H^0 \mathcal{Q}(\ast)$:

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
 & & S(-3) \otimes W & = & & & S(-3) \otimes W \\
 & & \downarrow \beta_1 & & & & \downarrow \beta_1 \\
 & & S(-2) \otimes V \otimes W & = & & & S(-2) \otimes V \otimes W \\
 & & \downarrow \beta_0 & & & & \downarrow \beta_0'' \\
 0 \rightarrow & S(-1) \otimes V_1 \rightarrow & (S(-1) \otimes V_1) \oplus & (S(-1) \otimes V_1 \otimes U) \rightarrow & S(-1) \otimes V_1 \otimes U \rightarrow & 0 \\
 & \parallel & \downarrow & & \downarrow & \\
 0 \rightarrow & S(-1) \otimes V_1 \rightarrow & H^0(\Lambda^2 \mathcal{F})(\ast - 3) \otimes W & \longrightarrow & Q \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \\
 & & 0 & & 0 &
 \end{array}$$

Define

$$A = (a_{ij})_{\substack{0 \leq i \leq 4 \\ 0 \leq j \leq 1}} \text{ by}$$

$$\left. \begin{aligned} a_{i0} &= e_{i+2} \wedge e_{i+3} \\ a_{i1} &= e_{i+1} \wedge e_{i+4} \end{aligned} \right\} i \text{ mod } 5.$$

Then w_0, w_1 , given by $w_j(e_i) = a_{ij}$ is a basis of W . Identifying $W \simeq \mathbb{C}^2$, the invariant form on W becomes the standard symplectic form $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on \mathbb{C}^2 .

We thus may rewrite $(M(\mathcal{F}))$ as

$$(M(\mathcal{F})) \quad 5\mathcal{O}(-1) \xrightarrow{Q \cdot A} 2(\Lambda^2 \mathcal{F})(-3) \xrightarrow{A} 5\mathcal{O},$$

the matrices operating by exterior multiplication.

(ii) From the explicit form of $(M(\mathcal{F}))$ we can compute α_0 explicitly. Choose a convenient basis of $\Lambda^2 V \otimes W = (V_1 \oplus U)$. Then

$$15S(-1) \xrightarrow{\alpha_0} 5S \longrightarrow H^1 \mathcal{F}(\ast) \longrightarrow 0$$

is the matrix

$$\alpha_0 = \begin{pmatrix} 0 & z_3 & 0 & 0 & z_2 & 0 & 0 & z_1 & z_4 & 0 & z_0 & 0 & 0 & 0 & 0 \\ z_3 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & z_2 & z_0 & 0 & z_1 & 0 & 0 & 0 \\ 0 & z_4 & 0 & z_0 & 0 & z_1 & 0 & 0 & 0 & z_3 & 0 & 0 & z_2 & 0 & 0 \\ 0 & 0 & z_0 & 0 & z_1 & z_4 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 \\ z_2 & 0 & 0 & z_1 & 0 & 0 & z_0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & z_4 \end{pmatrix}.$$

Resolving it (use e.g. [B-S]), we obtain the m.f.r. of $H^1 \mathcal{F}(\ast)$. Its shape is

$$\begin{aligned} 0 \rightarrow 2S(-8) \rightarrow 20S(-6) \rightarrow 35S(-5) \oplus 2S(-3) \rightarrow (15S(-4) \oplus 4S(-3)) \oplus 10S(-2) \\ \rightarrow 15S(-1) \rightarrow 5S \rightarrow H^1 \mathcal{F}(\ast) \rightarrow 0. \end{aligned}$$

(iii) Consider the induced m.f.r. of F and its dual

$$\begin{array}{ccccccc} \dots \rightarrow 35S(-5) & \xrightarrow{\gamma_0} & 15S(-4) \oplus 4S(-3) & \xrightarrow{\Gamma} & 15S(3) \oplus 4S(2) & \xrightarrow{\gamma_0} & 35S(4) \rightarrow \dots \\ & & \searrow & & \nearrow & & \\ & & & F \simeq F^\vee(-1) & & & \\ & \nearrow & & & \searrow & & \\ 0 & & & & & & 0. \end{array}$$

Γ can be computed by resolving γ_0 (use again [B-S]). We thus obtain explicit

bases for the spaces of sections $H^0 \mathcal{F}(m)$. Especially we get the equations of the zero-schemes of sections of $\mathcal{F}(3)$, including the abelian surfaces in \mathbb{P}_4 .

(iv) Let $\pi: \mathbb{P}_4 \rightarrow \mathbb{P}_4$ be a finite morphism and d^4 its degree. Then $\pi^* \mathcal{F}$ is a stable 2-bundle with Chern-classes $c_1 = -d$, $c_2 = 4d^2$. Proposition 2 and the above remarks also hold for $\pi^* \mathcal{F}$: Replace (K) by $\pi^*(K)$, $(M(\mathcal{F}))$ by $\pi^*(M(\mathcal{F})) = (M(\pi^* \mathcal{F}))$ and z_0, \dots, z_4 in α_0 by f_0, \dots, f_4 , where f_0, \dots, f_4 are the forms of degree d defining π .

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