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The zero-multiplicity of ternary recurrences

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Introduction

The purpose of this paper is to show that a non-degenerate ternary recurrence of rational numbers has zero-multiplicity at most six. To be more precise, consider the sequence $\{u_n\}_{n\in\mathbb{Z}}$ of rational numbers satisfying a relation of the form

$$u_{n+3} = Pu_{n+2} - Qu_{n+1} + Ru_n, \quad \forall n \in \mathbb{Z}$$

where $P, Q, R \in \mathbb{Q}$, $R \neq 0$ and $u_n \in \mathbb{Q}$ for all n and not all zero. The polynomial $X^3 - PX^2 + QX - R$ is called the *companion polynomial* of the recurrence. Let $\theta_1, \theta_2, \theta_3$ be its roots. We call the sequence *non-degenerate* if none of the ratios $\theta_i/\theta_j (i \neq j)$ is a root of unity. It is well known that the terms can now be written as

$$u_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3\theta_3^n \tag{1}$$

for suitable $\alpha_1, \alpha_2, \alpha_3$. We shall consider non-degenerate recurrences only and call the number of solutions $n \in \mathbb{Z}$ of $u_n = 0$ the zero multiplicity. We prove the following theorem

THEOREM 1. *Let $\{u_n\}_{n\in\mathbb{Z}}$ be a non-degenerate ternary recurrence of rational numbers. Then the zero-multiplicity is at most six.*

To give a little history of the problem, in 1957 it was conjectured by M. Ward [W] that the zero-multiplicity is at most five. That this conjecture is erroneous is shown by an example of Berstel [Ber] which has multiplicity six. Take $u_{n+3} = 2u_{n+2} - 4u_{n+1} + 4u_n$, $u_0 = u_1 = 0$, $u_2 = 1$, then $u_0 = u_1 = u_4 = u_6 = u_{13} = u_{52} = 0$. It is generally expected however, that this is essentially the only exception to Ward's conjecture. In more special cases the conjecture can be verified. For example, if $\theta_1, \theta_2, \theta_3$ are all real, Smiley [S] showed by a simple argument that the zero-multiplicity is at most three. In the special case when $\theta_3 = 1$, the recurrence can be written as $u_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3$ and the problem $u_n = 0$ is the same as $v_n = -\alpha_3$, where $v_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n$ satisfies the binary recurrence $v_{n+2} = (P-1)v_{n+1} - Rv_n$. When $P, R \in \mathbb{Z}$ the multiplicity of such integral binary recurrences has been treated by K. Kubota [K] and the author [Beu]. It turns out that Ward's conjecture is true in this case.

In [K, III] Kubota claims to be able to prove Theorem 1. However, this claim has not been substantiated up till now. In the meantime, B. Deshommes [D1] has found a complete solution for the case when $u_0 = u_1 = 0$ using p-adic methods. The methods used in this paper have been dormant for several years. In an earlier version of 1983 I proved a multiplicity upper bound of seven with the possibility of lowering it to six. The work involved looked so messy though, that I decided it would be better to wait for more elegant proofs. Unfortunately, several efforts in this direction were in vain. So at last I took up the original method again and tidied it up as much as possible. In particular I would like to thank B. Deshommes for her encouragement to get this work finished. For more information on the history of recurrent sequences in general we refer to [LP] or [T] and to [ST, Ch. 1, 2, 3] for very recent results.

The present paper is divided into two sections. The first one provides the necessary tools in a series of Lemmas which consider the equation $\lambda\alpha^x + \mu\beta^x = 1$ in $x \in \mathbb{Z}$, where α, β, λ, μ are given algebraic numbers. Obviously, this equation is a general version of expression (1) put equal to zero. Lemma 1 is taken from [BT], where it is proved by means of hypergeometric polynomials. This Lemma provides an upper bound for the second or third largest solution. Lemma 2, 3, 4 provide means to create large gaps between consecutive solutions, thus providing lower bounds for them. Then, in Lemma 5 and 6 we show that our equation has at most six solutions for a wide range of values of α and β. In section 2 we prove Theorem 1.

The interested reader may consult Table III at the end of this paper for examples of recurrences having zero-multiplicity at least four.

1. Some technical lemmas

In Lemmas 1 up to 6 we consider the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \tag{2}$$

in $x \in \mathbb{Z}$, where α, β, λ, μ are non-zero numbers in an algebraic number field K. We assume that none of α, β, α/β is a root of unity. We also assume that (2) has the solutions $x = 0, k, l, m$ with $0 < k < l < m$.

Let $\theta \in K$, $\theta \neq 0$ and let $a_0X^d + a_1X^{d-1} + \cdots + a_d$ be its minimal polynomial over $\mathbb{Z}$, with $gcd(a_0, a_1, \ldots, a_d) = 1$. We define the Mahler height $M(\theta)$ of θ to be $a_0\prod_i \max(1, |\theta_i|)$, where the product is taken over all conjugates θ_i of θ, including θ itself. In an earlier paper [BT] we used the canonical height $h(\theta)$ which is related to $M(\theta)$ by $M(\theta) = h(\theta)^d$.

Throughout this section we shall adhere to the assumptions and notations we just made.

LEMMA 1. *Suppose* $m \geqslant 10l$. *Let* $H = \max(h(\alpha), h(\beta), h(\beta/\alpha))$ *and suppose* $H > 1$. *Then*

$$l \leqslant 27\frac{\log 2}{\log H} + \frac{50}{3}k$$

Proof. See [BT, Lemma 7].

The purpose of this Lemma is clear, it gives a bound on the larger solutions of (2). The following Lemmas create large gaps between the solutions.

LEMMA 2. *Let* v *be a finite valuation on* K *such that* $|\alpha|_v < 1$ *and* $|\beta|_v = 1$. *Then there exists a positive integer* d *with the following properties,*

(i) $d > 1$ *and* d *divides both* $m - l$ *and* $l - k$.
(ii) *If there is an extra solution* n *of* (2) *with* $0 < n < k$, *then* $d \geqslant 4$.
(iii) *If* $|\alpha|_v^k|\beta - 1|_v < |p|_v^{1/p-1}$, *where* p *is the rational prime above* v, *then* $|(m-l)/d|_v \leqslant |\alpha|_v^{l-k}$ *and* $|(l-k)/d|_v = 1$.

Proof. Since $\lambda + \mu = 1$, it follows from $\lambda\alpha^x + \mu\beta^x = 1$ that $\beta^x - 1 = \lambda(\lambda-1)^{-1}(a^x - 1)$ and hence $|\beta^k - 1|_v = |\beta^l - 1|_v = |\beta^m - 1|_v$. Denote this value by Q and remember that $Q \leqslant |\beta - 1|_v$. After elimination of λ, μ from $\lambda\alpha^x + \mu\beta^x = 1$ with $x = 0, k, l$ we obtain

$$\beta^l - \beta^k = (\beta^l - 1)\alpha^k - (\beta^k - 1)\alpha^l$$

Hence

$$|\beta^l - \beta^k|_v = |\alpha|_v^k|\beta^l - 1|_v = |\alpha|_v^k Q$$

and since $|\beta|_v = 1$,

$$|\beta^{l-k} - 1|_v = |\alpha|_v^k Q \tag{3}$$

In the same way,

$$|\beta^{m-l} - 1|_v = |\alpha|_v^l Q \tag{4}$$

For d we choose the smallest natural number such that $|\beta^d - 1|_v \leqslant |\alpha|_v^k Q$. We must have $d > 1$, since $d = 1$ would imply $|\beta - 1|_v \leqslant |\alpha|_v^k Q < |\beta - 1|_v$, contradiction. Furthermore, $|\beta^x - 1|_v \leqslant |\alpha|_v^k Q$ implies $d \mid x$. In particular, $d \mid l - k$ and $d \mid m - l$. This proves our first assertion. Notice by the way, that (3) also implies $|\beta^d - 1|_v = |\alpha|_v^k Q$.

To prove our second assertion we derive in a similar way as we did for (3), that $|\beta^{k-n} - 1|_v = |\alpha|_v^n Q$.

Choose $e \in \mathbb{N}$ minimal such that $|\beta^e - 1|_v \leqslant |\alpha|_v^n Q$. Again, by the same arguments as above, $e > 1$, e divides $k - n$ and we have $|\beta^e - 1|_v = |\alpha|_v^n Q$. Since $n < k$ we now see that e must be a non-trivial divisor of d and hence $d \geqslant 4$.

To prove our third assertion, put $\beta^d = 1 + \pi$ and keep in mind that $|\pi|_v = |\alpha|_v^k Q$. Write $t = (m - l)/d$. Then

$$\beta^{m-l} - 1 = (1 + m)^t - 1 = t\pi + \binom{t}{2}\pi^2 + \cdots + \pi^t$$

$$= t\pi + t\pi\left\{\binom{t-1}{1}\frac{\pi}{2} + \binom{t-1}{2}\frac{\pi^2}{3} + \cdots + \binom{t-1}{t-1}\frac{\pi^{t-1}}{t}\right\}$$

The v-adic value of the sum between brackets can be estimated by

$$\max_{r\geqslant 2}\left|\frac{\pi^{r-1}}{r}\right|_v \leqslant \max_{s\geqslant 1}\left(|\pi|_v, \left|\frac{\pi^{p^s-1}}{p^s}\right|_v\right) \leqslant \max_{s\geqslant 1}\left(|\pi|_v, \left|\frac{\pi^{p-1}}{p}\right|_v^s\right)$$

and the last term is smaller than 1 according to our assumption $|\pi|_v = |\alpha|_v^k Q \leqslant |\alpha|_v^k|\beta - 1|_v < |p|_v^{1/p-1}$. Hence

$$|t\pi|_v = |\beta^{m-l} - 1|_v = |\alpha|_v^l Q$$

and thus

$$|t|_v = |\alpha|_v^l Q/|\pi|_v = |\alpha|_v^{l-k},$$

as asserted. The statement $|(l - k)/d|_v = 1$ follows from $|\beta^{l-k} - 1|_v = |\beta^d - 1|_v$. □

LEMMA 3. *With the assumptions above, let $\beta = \bar{\alpha}$, $\mu = \bar{\lambda}$, where the bar denotes complex conjugation. Suppose also $|\alpha| \geqslant 4/3$. The argument of a complex number z, denoted by* Arg z, *will be taken between $-\pi$ and π. Then,*

(i) *If $k > 50$ and $10^{-4} < |\mathrm{Arg}(\bar{\alpha}/\alpha)| < \pi - 10^{-4}$, then $m - k \geqslant |\alpha|^k$*
(ii) *If $|\alpha| \geqslant 2.1$ and $k \geqslant 2$, then $m - k > 2|\alpha|^k$.*

Proof. Eliminate λ, $\bar{\lambda}$ from $\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1$ with $x = 0, k, l, m$ to obtain

$$\frac{\bar{\alpha}^k - 1}{\alpha^k - 1} = \frac{\alpha^l - 1}{\alpha^l - 1} = \frac{\bar{\alpha}^m - 1}{a^m - 1}$$

If these quotients equal one, then α^k would be real and $\bar{\alpha}/\alpha$ a root of unity, contrary to our assumptions. Put

$$\eta = \frac{\bar{\alpha}^k - 1}{\alpha^k - 1} - 1.$$

Then $\eta \neq 0$, $|\eta| \leqslant 2$. Notice, that

$$\frac{\bar{\alpha}^k - 1}{\alpha^k - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^k + \frac{\eta}{\alpha^k}, \frac{\bar{\alpha}^l - 1}{a^l - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^l + \frac{\eta}{\alpha^l}, \frac{\bar{\alpha}^m - 1}{\alpha^m - 1} = \left(\frac{\bar{\alpha}}{\alpha}\right)^m + \frac{\eta}{\alpha^m}$$

Hence

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^{l-k} - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^{k}\left(\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right) \tag{5}$$

$$\left(\frac{\bar{\alpha}}{\alpha}\right)^{m-l} - 1 = \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^{l}\left(\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right)$$

A straightforward verification shows that the assumptions in either (i) or in (ii) ensure that both righthand sides in (5) are smaller than 1 in absolute value. For any complex number w with $|1 + w| = 1$ and $|w| \leqslant 1$ the inequalities $|w| \leqslant |\text{Arg}(1 + w)| \leqslant (\pi/3)|w|$ hold. Application of this principle with w equal to the righthand sides of (5) yields

$$(l - k)\text{Arg}(\bar{\alpha}/\alpha) + 2\pi r = \text{Arg}\left[1 + \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^{k}\left(\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right)\right] = \mu\left|\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right||\eta|$$

$$(m - l)\text{Arg}(\bar{\alpha}/\alpha) + 2\pi s = \text{Arg}\left[1 + \eta\left(\frac{\alpha}{\bar{\alpha}}\right)^{l}\left(\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right)\right] = \nu\left|\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right||\eta|$$

where $r, s \in \mathbb{Z}$ and μ, ν are real numbers with $1 \leqslant |\mu|, |\nu| \leqslant \pi/3$. We eliminate $\text{Arg}(\bar{\alpha})/\alpha)$ from these equalities and find that

$$E := (m - l)\mu\left|\frac{1}{\alpha^k} - \frac{1}{\alpha^l}\right||\eta| - (l - k)\nu\left|\frac{1}{\alpha^l} - \frac{1}{\alpha^m}\right||\eta|$$

is either zero or larger than 2π in absolute value.

We first show that $E \neq 0$. Suppose $E = 0$. Then, by $\eta \neq 0$

$$(m - l)\mu|\alpha^{l-k} - 1| = (l - k)\nu|1 - \alpha^{l-m}|.$$

So,

$$\frac{|\alpha^{l-k} - 1|}{l - k} = \frac{(1 - \alpha^{l-m}|}{m - l}\left|\frac{\nu}{\mu}\right|$$

and hence

$$\frac{|\alpha|^{l-k} - 1}{l - k} \leqslant \frac{1 + |\alpha|^{l-m}}{m - l}\cdot\frac{\pi}{3}. \tag{7}$$

Suppose we are in case (i). Because $|\alpha| \geqslant 4/3$, $k > 50$, the right hand sides of (6) are smaller than 10^{-6}. The condition on $\text{Arg}(\bar{\alpha}/\alpha)$ together with (6) now implies that $\min(l - k, m - l) \geqslant 3$. Because $l - k = m - l = 3$ cannot happen according to Lemma 4 we have either $l - k \geqslant 4$, $m - l \geqslant 3$ or $l - k \geqslant 3$, $m - l \geqslant 4$. Together with $|\alpha| \geqslant 4/3$ these inequalities give a contradiction when substitued in (7). Suppose we are in case (ii). Since $l - k \neq m - l$ we have either $l - k \geqslant 2$, $m - l \geqslant 1$ or $l - k \geqslant 1$, $m - l \geqslant 2$. Together with $|\alpha| \geqslant 2.1$, these inequalities give a contradiction when substituted in (7).

Having shown that $E \neq 0$, we now find that

$$2\pi \leqslant (m-l)\frac{\pi}{3}\left|\frac{1}{\alpha^k}-\frac{1}{\alpha^l}\right||\eta| + (l-k)\frac{\pi}{3}\left|\frac{1}{\alpha^l}-\frac{1}{\alpha^m}\right||\eta|$$

$$\leqslant (m-k)\frac{2\pi}{3}(1+|\alpha|^{-1})|\alpha|^{-k}.$$

Hence $m-k \geqslant 3(1+|\alpha|^{-1})^{-1}|\alpha|^k$, from which our assertions follow. □

LEMMA 4. *Suppose that the equation $\lambda\alpha^x + \mu\beta^x = 1$ with $\lambda\mu\alpha\beta \neq 0$ has the integer solutions p, $p+d$, q, $q+d$. Then α and β are roots of unity.*

Proof. From $1 = \lambda\alpha^p + \mu\beta^p = \lambda\alpha^{p+d} + \mu\beta^{p+d}$ and $1 = \lambda\alpha^q + \mu\beta^q = \lambda\alpha^{q+d} + \mu\beta^{q+d}$ it follows that the coefficient determinant of λ, μ vanishes,

$$\begin{vmatrix}\alpha^{p+d}-\alpha^p & \beta^{p+d}-\beta^p\\ \alpha^{q+d}-\alpha^q & \beta^{q+d}-\beta^q\end{vmatrix} = 0$$

This determinant equals $\beta^p\alpha^q(\alpha^d-1)(\beta^d-1)((\beta/\alpha)^{q-p}-1)$. Thus at least one of the factors vanishes, so that α or β is a d-th root of unity or β/α is $(q-p)$th root of unity. Going back to the original equation it is not very hard to see that actually both α and β are roots of unity. □

REMARK. In words, Lemma 4 states that a given difference between solutions of (2) can occur at most once, unless α and β are roots of unity. In particular, if $x_1 < x_2 < x_3 < \cdots < x_n$ are solutions, then $x_n - x_1 \geqslant \binom{n}{2}$ and $x_3 - x_2 \neq x_2 - x_1$.

LEMMA 5. *Let $\alpha \in \bar{\mathbb{Q}}$, $|\alpha| \geqslant 4$ and $h(\alpha) \geqslant 2^{1/3}$. Then*

$$\lambda\alpha^x + \bar{\lambda}\bar{\alpha}^x = 1 \quad in\ x \in \mathbb{Z}$$

has at most six solutions.

Proof. Suppose that the equation has seven solutions, which we may assume to be $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. By Lemma 3 (ii) we have $x_6 - x_4 > 2|\alpha|^{x_4}$. By Lemma 4, $x_4 \geqslant 10$, and so we certainly have $x_6 \geqslant 10x_4$. Application of Lemma 1 yields

$$x_4 < 81 + \frac{50}{3}x_1.$$

A lower bound for x_4 is obtained by application of Lemma 3 3(ii) $x_4 \geqslant x_2 + 2|\alpha|^{x_2}$ and so we find,

$$x_2 + 2|\alpha|^{x_2} < 81 + \frac{50}{3}x_1 \tag{8}$$

In particular, $x_1 + 2.4^{x_1+1} < 81 + \frac{50}{3}x_1$, which implies $x_1 = 1$. On the other hand, $x_2 \geqslant 3$ and so (8) implies $3 + 2.4^3 < 81 + \frac{50}{3}$, which is impossible. Hence there are at most six solutions. □

LEMMA 6. *Let K be a numberfield and $\lambda, \mu, \alpha, \beta \in K$ with $\lambda\mu\alpha\beta \neq 0$ and such that* $\max(h(\alpha), h(\beta), h(\alpha/\beta)) \geqslant 2^{1/6}$. *Let v be a valuation such that $|\alpha|_v < 1$ and $|\beta|_v = 1$. Let p be the rational prime above v and suppose that $v|p$ has ramification index at most 2. Then (2) has at most six solutions.*

Proof. Suppose that (2) has seven solutions, which we may assume to be $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. By Lemma 4 we know $x_2 \geqslant 3$. Since v has ramification index at most 2, we have automatically $|\alpha|_v^{x_2} \leqslant |\alpha|_v^3 < |p|_v^{1/p-1}$. Application of Lemma 2 with $k = x_2$, $l = x_3$, $m = x_4$ yields

$$\left|\frac{x_4 - x_3}{d}\right|_v \leqslant |\alpha|_v^{x_3 - x_2}$$

for some $d \geqslant 4$. Since v has ramification index $\leqslant 2$, we infer

$$x_4 - x_3 \geqslant d \cdot p^{1/2(x_3 - x_2)} \geqslant 4 \cdot 2^{1/2(x_3 - x_2)} \tag{9}$$

Similarly,

$$x_5 - x_4 \geqslant 4 \cdot 2^{1/2(x_4 - x_3)} x_6 - x_5 \geqslant 4 \cdot 2^{1/2(x_5 - x_4)} \tag{10}$$

By Lemma 2 we also have $d|x_3 - x_2$, hence $x_3 - x_2 \geqslant 4$. It is straightforward to verify that $x_6 - x_2 \geqslant 10(x_5 - x_2)$, so we can apply Lemma 1 with $k = x_3 - x_2$, $l = x_4 - x_2$, $m = x_5 - x_2$ and $H \geqslant 2^{1/6}$ to obtain,

$$x_5 - x_2 \leqslant 162 + \frac{50}{3}(x_3 - x_2) \tag{11}$$

A lower bound for $x_5 - x_2$ is obtained from (9) and (10) implying

$$x_5 - x_2 > x_5 - x_4 \geqslant 4 \cdot 2^{2 \cdot 2(1/2)(x_3 - x_2)} = 4^{1 + 2(1/2)(x_3 - x_2)}$$

This clearly contradicts (11) since $x_3 - x_2 \geqslant 4$. □

2. Proof of theorem 1

In this section we consider the equation

$$\alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3\theta_3^n = 0 \text{ in } n \in \mathbb{Z} \tag{12}$$

where θ_1, θ_2, θ_3 are the roots of a cubic polynomial $X^3 - PX^2 + QX - R \in \mathbb{Q}[X]$ with $R \neq 0$, and α_1, α_2, α_3 are such that all numbers $u_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3\theta_3^n$, $n \in \mathbb{Z}$ are rational and not all zero. We assume that none of the ratios θ_i/θ_j ($i \neq j$) is a root of unity. It is easy to see that the sequence $\{u_n\}_{n \in \mathbb{Z}}$ satisfies the recurrence $u_{n+3} = Pu_{n+2} - Qu_{n+1} + Ru_n$ and solving (12) is equivalent to solving $u_n = 0$ in $n \in \mathbb{Z}$.

LEMMA 7. *Let notations be as above and suppose that* $u_0 = 0$. *Suppose we can find* $a, b \in \mathbb{Q}$, *positive integers* d, δ *and a prime* p *such that*

(i) $\theta_i^d = a + b\theta_i^\delta$ $(i = 1, 2, 3)$.
(ii) $|b|_p \leqslant 1/4$, $|R|_p = |a|_p = 1$, $|u_n|_p \leqslant 1$ *for all* $n \in \mathbb{Z}$.
(iii) $|u_r|_p < 1$ *and* $0 \leqslant r \leqslant d + 2\delta$ *implies* $u_r = 0$.

Then $u_n = 0$ *implies either* $0 \leqslant n < d$ *or* $n = d + r$, *where* $0 \leqslant r < d$, $u_r = u_{r+\delta} = 0$.

REMARK. Condition (i) may seem outlandish at first glance, but as soon as we have three solutions $0, \delta, d$ of (12), the determinant

$$\begin{vmatrix} 1 & \theta_1^\delta & \theta_1^d \\ 1 & \theta_2^\delta & \theta_2^d \\ 1 & \theta_3^\delta & \theta_3^d \end{vmatrix}$$

vanishes, hence the columns are dependent and (i) readily follows. Notice also that, according to Lemma 4, $u_r = u_{r+\delta} = 0$ can happen for at most one r.

Proof. Suppose $u_n = 0$ and put $n = dq + r$ with $q \in \mathbb{Z}$, $0 \leqslant r < d$. Since

$$u_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3\theta_3^n$$

and $\theta_j^d = a + b\theta_j^h$, $u_n = 0$ can be written as

$$\sum_{i=1}^{3} \alpha_i\left(1 + \frac{b}{a}\,\theta_i^\delta\right)^q \theta_i^r = 0.$$

Taking binomial expansions, we get a p-adically converging series,

$$u_r + \sum_{t=1}^{\infty} \binom{q}{t}\left(\frac{b}{a}\right)^t \left(\sum_{i=1}^{3} \alpha_i\theta_i^{r+t\delta}\right) = 0,$$

Hence

$$u_r + \sum_{t=1}^{\infty} \binom{q}{t}\left(\frac{b}{a}\right)^t u_{r+t\delta} = 0. \tag{13}$$

Since $|u_{r+t\delta}|_p \leqslant 1$, $|b|_p < 1$, we see that (13) implies $|u_r|_p < 1$ and condition (iii) tells us that $u_r = 0$. So u_r can be dropped from (13). If $q = 0$ we are done. Suppose $q \neq 0$. After division by qb/a we are left with

$$u_{r+\delta} + \sum_{t=2}^{q} \binom{q-1}{t-1}\frac{1}{t}\left(\frac{b}{a}\right)^{t-1} u_{r+t\delta} = 0 \tag{14}$$

because $|b|_p \leqslant 1/4$, we have $|(b/a)^{t-1}/t|_p < 1$ for all $t \geqslant 2$ and (14) implies $|u_{r+\delta}|_p < 1$. Condition (iii) tells that $u_{r+\delta} = 0$. If $q = 1$ we are done. So suppose $q \neq 1$, put $u_{r+\delta} = 0$ in (14) and divide by $(q-1)b/a$ to obtain

$$u_{r+2\delta} + \sum_{t=3}^{q} \binom{q-2}{t-2}\frac{1}{t(t-1)}\left(\frac{b}{a}\right)^{t-2} u_{r+t\delta} = 0. \tag{15}$$

Since $|b|_p \leqslant 1/4$, we have $|(b/a)^{t-2}/t(t-1)|_p < 1$ for all $t \geqslant 3$ and (15) implies $|u_{r+2\delta}|_p < 1$. Condition (iii) tells us that $u_{r+2\delta} = 0$. According to Lemma 4 $u_r = u_{r+\delta} = u_{r+2\delta}$ cannot happen. Thus we have found either $q = 0, 0 \leqslant r < d$ or $q = 1$, $u_r = u_{r+\delta} = 0$, as asserted. □

Proof of theorem 1. Since the recurrence is non-degenerate, the roots $\theta_1, \theta_2, \theta_3$ of $X^3 - PX^2 + QX - R$ are all distinct and there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}(\theta_1, \theta_2, \theta_3)$ such that

$$u_n = \alpha_1\theta_1^n + \alpha_2\theta_2^n + \alpha_3\theta_3^n$$

We now study the equation (12) in the unknown $n \in \mathbb{Z}$. If one of the α_i is zero, the problem becomes trivial, so we assume $\alpha_1\alpha_2\alpha_3 \neq 0$ from now on. If the θ_i are all real, we know that there are at most three solutions according to a theorem of Smiley [S]. For completeness we prove that (13) has at most four solutions when the θ_i are real. Rewrite (12) as

$$\alpha\left(\frac{\theta_1}{\theta_3}\right)^n + \alpha_2\left(\frac{\theta_2}{\theta_3}\right)^n + \alpha_3 = 0.$$

Suppose that there are five solutions. Then there must be at least three solutions having the same parity, say even. This implies that $\alpha_1(\theta_1/\theta_3)^{2x} + \alpha_2(\theta_2/\theta_3)^{2x}$ considered as a function of $x \in \mathbb{R}$ has at least two stationary points, which is impossible, unless $(\theta_1/\theta_3)^2 = (\theta_2/\theta_3)^2 = 1$.

So we may now assume that we have one real root and two complex conjugate roots. Thus we have the possibilities that one of the roots is rational and the others are conjugates in an imaginary quadratic field or that the roots are conjugate cubic numbers. Consider the finite valuations of the field $K = \mathbb{Q}(\theta_1, \theta_2, \theta_3)$. If there exists a valuation v such that $|\theta_1|_v, |\theta_2|_v, |\theta_3|_v$ are all distinct, then it is trivial to see that (12) has at most three solutions. Suppose there exists a valuation v such that $|\theta_3|_v \neq |\theta_1|_v = |\theta_2|_v$. Consider the equation

$$\frac{\alpha_1}{\alpha_2}\left(\frac{\theta_1}{\theta_2}\right)^n + \frac{\alpha_3}{\alpha_2}\left(\frac{\theta_3}{\theta_2}\right)^n + 1 = 0$$

in $n \in \mathbb{Z}$. If $|\theta_3|_v < |\theta_2|_v$ we can apply Lemma 6, provided we show that v has ramification index $\leqslant 2$ over $\mathbb{Q}$. This is obvious however, since ramification index 3 would imply $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$. If $|\theta_3|_v > |\theta_2|_v$ we replace n by $-n$ and consider the equivalent equation

$$\frac{\alpha_1}{\alpha_2}\left(\frac{\theta_2}{\theta_1}\right)^n + \frac{\alpha_3}{\alpha_2}\left(\frac{\theta_2}{\theta_3}\right)^n + 1 = 0.$$

Thus we are left with $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$ for all finite valuations v of K. Let θ_3 be the real root. It is clear that θ_1, θ_2 cannot be conjugates in an imaginary quadratic field, since $|\theta_1|_v = |\theta_2|_v$ for all finite v would imply that θ_1/θ_2 is a root of unity. Hence θ_3 is a cubic number and θ_1, θ_2 are its algebraic conjugates.

Let us now make some normalisations. First of all, if $|\theta_3| > |\theta_1|$, we consider equation (12) with θ_i^{-1} instead of θ_i and replace n by $-n$. So we can assume $|\theta_3| < |\theta_1|$. Multiply $\theta_1, \theta_2, \theta_3$ by the same rational number to make sure that the new θ_i are algebraic integers, not all divisible by the same rational integer and $\theta_3 > 0$.

If $|\theta_1/\theta_3| \geqslant 4$ we can deal with (13) by using Lemma 5. Thus we assume from now on, $|\theta_1/\theta_3| < 4$. Together with $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$ for all finite v and our normalisation, this yields a finite list of θ_i which is reproduced in Table I, where we also give an explanation of how this table is compiled. Notice in particular, that always $|\theta_1/\theta_3| > 4/3$ and $10^{-4} < |\mathrm{Arg}(\theta_1/\theta_2)| < \pi - 10^{-4}$.

Suppose that (12) has seven solutions, which we may assume to be $0 = x_0 < x_1 < x_2 < \cdots < x_6$. Suppose first that $x_2 > 100$. We can apply Lemma 3(i) with $m = x_6$, $l = x_5$, $k = x_4$, $\alpha = \theta_1/\theta_3$ to obtain $x_6 - x_4 > |\theta_1/\theta_3|^{x_4} > (4/3)^{x_4}$. Since $x_4 > 100$, this certainly implies $x_6 > 10x_4$ and we can apply Lemma 1. Note that $H > (4/3)^{1/3}$ and hence

$$x_4 < 27 \frac{\log 2}{\log((4/3)^{1/3})} + \frac{50}{3} x_1 \leqslant 196 + \frac{50}{3} x_1 \tag{16}$$

On the other hand, we have from Lemma 3(i) the lower bound $x_4 - x_2 > |\theta_1/\theta_3|^{x_2} > (4/3)^{x_2}$ which certainly contradicts (16) when $x_2 > 100$. Hence $x_2 \leqslant 100$. To each entry in Table I there corresponds a recurrence relation given by $P, Q, R \in \mathbb{Z}$. We have determined all starting values u_0, u_1, u_2 such that $u_0 = 1$ and such that $u_n = 0$ has at least three solutions n, $0 \leqslant n < 250$. These recurrences are listed in Table II. In particular, the recurrence we are studying should be in this list. Suppose our recurrence has three or four solutions n with $0 \leqslant n \leqslant 250$. Hence $x_4 > 250$. Inequality (16) still holds. From Table II we infer $x_1 \leqslant 2$ with one exception, and (16) yields $x_4 < 250$, contradiction. The exceptional case corresponds to $P = -2$, $Q = 0$, $R = 4$. In this case $H > (1.8)^{1/3}$ and the 196 in (16) can be improved to 100. Then $x_1 = 4$ again yields a contradiction. So $x_4 < 250$, i.e. the recurrence has at least five solutions n with $n \leqslant 250$. Our list of recurrences has now dwindled to the following Table.

Nr.	P	Q	R	u_0	u_1	u_2	solutions n
1	2	4	4	0	0	1	0, 1, 4, 6, 13, 52
2	2	4	4	0	1	0	0, 3, 5, 12, 51
3	−1	0	1	0	1	0	0, 2, 3, 7, 16
4	−2	0	4	0	1	0	0, 2, 3, 8, 24

The second recurrence is a subsequence of the first. We deal with nrs 1, 3, 4 by using Lemma 7. For recurrence nr 1 we take $d = 52$, $\delta = 1$, $a = -206.2^{34}$, $b = 159.2^{34}$, $p = 53$, for nr 3 we take $d = 16$, $\delta = 2$, $a = 4$, $b = -7$, $p = 7$ and for nr 4, $d = 22$, $\delta = 1$, $a = 26.2^{14}$, $b = -23.2^{14}$, $p = 23$. □

Table I

Let θ_1, θ_2, θ_3 be the roots of the cubic polynomial $x^3 - Px^2 + Qx - R$, P, Q, $R \in \mathbb{Z}$. Suppose that the discriminant is negative and that θ_1, θ_2 are the complex conjugate roots. In this table we list all P, Q, R such that $\theta_3 > 0$, $|\theta_1/\theta_3| < 4$, $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$ for all finite valuations v and θ_1, θ_2, θ_3 have no common factor in $\mathbb{Z}$.

The table is compiled as follows. First note, that $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$ implies that either $|\theta_1|_v = 1$ or v ramifies of order three above the corresponding rational prime. Hence there exist units η_1, η_2, η_3 (conjugates) and a natural number M such that $\theta_i^3 = M\eta_i (i = 1, 2, 3)$. Notice that $|\theta_1/\theta_3| < 4$ implies $|\eta_1/\eta_3| < 64$ and together with $\eta_1\eta_2\eta_3 = 1$ this yields $1 < |\eta_1| = |\eta_2| < 4$, $|\eta_3| < 1$. Using these bounds it is not hard to compute all polynomials having zeros which satisfy these conditions. The discriminants of these polynomials give the possible rational primes which ramify of order three in $\mathbb{Q}(\eta_1, \eta_2, \eta_3)$ and thus the possibilities for M.

P	Q	R	$\|\theta_1/\theta_3\|$	P	Q	R	$\|\theta_1/\theta_3\|$	P	Q	R	$\|\theta_1/\theta_3\|$
2	3	1	3.545	2	4	4	1.356	0	6	6	2.944
5	25	25	3.677	−2	0	2	1.839	0	7	7	3.115
3	6	3	3.104	−3	0	9	1.762	0	2	1	3.276
2	4	2	2.769	−2	0	4	1.664	0	9	9	3.428
4	8	4	3.383	−2	1	1	3.148	0	10	10	3.574
−6	0	36	1.961	−1	0	1	1.525	0	11	11	3.713
3	9	9	1.961	0	1	1	1.774	0	12	12	3.847
6	18	18	1.961	0	2	2	2.089	0	13	13	3.977
−5	0	25	1.905	0	3	3	2.342	−1	1	1	2.494
1	2	1	2.325	0	4	4	2.562	−2	2	2	3.246
5	10	5	3.627	0	5	5	2.761	−2	4	4	3.528

Table II

For each P, Q, R from Table I we have determined u_0, u_1, u_2 in such a way that the sequence $\{u_n\}_{n \geqslant 0}$ given by $u_n = Pu_{n-1} - Qu_{n-2} + Ru_{n-3} (n \geqslant 3)$ has at least three zeros n with $0 \leqslant n < 250$ one of which is 0 itself, i.e. $u_0 = 0$. Clearly, if $\{u_n\}_{n \geqslant 0}$ is listed we do not list its multiples, although they satisfy the requirements. Moreover, if $\{u_n\}_{n \geqslant 0}$ is listed, we do not list its shifted versions $\{u_{n+k}\}_{n \geqslant 0}$ for any $k \in \mathbb{N}$.

The table is compiled as follows. For each m, $0 < m < 250$ we determine u_1

such that $u_m = 0$. For each recurrence thus obtained we check for all n, $0 < n < 250$ such that $u_n = 0$ and record all such instances.

P	Q	R	u_0	u_1	u_2	solutions n with $0 \leqslant n < 250$
−3	0	9	0	1	0	0, 2, 3, 9
−2	0	2	0	1	0	0, 2, 3, 26
−2	0	2	0	2	−1	0, 4, 12
−2	0	4	0	1	0	0, 2, 3, 8, 24
−1	0	1	0	1	0	0, 2, 3, 7, 16
−1	1	1	0	0	1	0, 1, 4, 17
0	1	1	0	0	1	0, 1, 3, 8
0	3	3	0	0	1	0, 1, 3, 10
0	6	6	0	0	1	0, 1, 3, 12
2	4	2	0	0	1	0, 1, 4, 12
2	4	4	0	0	1	0, 1, 4, 6, 13, 52
3	9	9	0	0	1	0, 1, 4, 9

Table III

Here we list P, Q, R, u_0, u_1, u_2 such that the corresponding recurrence sequence has at least four zeros. We call two recurrent sequences $\{u_n\}_{n\in\mathbb{Z}}$, $\{v_n\}_{n\in\mathbb{Z}}$ equivalent if there exist $\lambda \in \mathbb{Q}^*$, $k \in \mathbb{Z}$ and a choice of $\pm$ sign such that $v_n = \lambda^n u_{\pm n+k}$ for all $n \in \mathbb{Z}$. From each equivalence class we list at most one representant. We do not claim this list to be complete, although it seems hard to find other examples.

P	Q	R	u_0	u_1	u_2	solutions n
−1	0	1	0	1	0	0, 2, 3, 7, 16
−2	0	2	0	1	0	0, 2, 3, 26
−2	0	4	0	1	0	0, 2, 3, 8, 24
−3	0	9	0	1	0	0, 2, 3, 9
0	1	1	0	0	1	0, 1, 3, 8
0	3	3	0	0	1	0, 1, 3, 10
0	6	6	0	0	1	0, 1, 3, 12
0	20	100	0	0	1	0, 1, 3, 13
0	15	45	0	0	1	0, 1, 3, 15
−1	1	1	0	0	1	0, 1, 4, 17
2	4	2	0	0	1	0, 1, 4, 12
2	4	4	0	0	1	0, 1, 4, 6, 13, 52
3	9	9	0	0	1	0, 1, 4, 9
3	−9	18	0	0	1	0, 1, 4, 10
5	−25	50	0	0	1	0, 1, 4, 16

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