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Quasi-Gorenstein Fano 3-folds with isolated non-rational loci

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1. Introduction

A quasi-Gorenstein Fano n-fold is, by definition, an n-dimensional normal projective variety X over $\mathbb{C}$ such that the anti-canonical divisor $-K_X$ is an ample Cartier divisor. Put $\Sigma_X = \{x \in X \mid x$ is not a rational singularity on $X\}$. Then Σ_X is a closed algebraic subset of X of codimension at least two. Brenton [B], Hidaka and Watanabe [HW] determine all quasi-Gorenstein Fano surfaces. As they show, both cases $\Sigma_X = \phi$ and $\Sigma_X \neq \phi$ occur.

In this paper we treat the latter case $\Sigma_X \neq \phi$. We try to clarify the structure of a quasi-Gorenstein Fano n-fold with $\dim \Sigma_X = 0$. If we assume the minimal model conjecture, it turns out to have the structure of a projective cone defined by an ample invertible sheaf $\mathscr{L}$ on a normal $(n-1)$-fold S satisfying $\mathcal{O}(K_S) \simeq \mathcal{O}_S$ with at worst rational singularities on S. Here the projective cone defined by $\mathscr{L}$ on S means the normal projective variety obtained by contracting the negative section of $\mathbb{P}(\mathcal{O}_S \oplus \mathscr{L})$. Since the minimal model conjecture is known to hold for surfaces, a quasi-Gorenstein Fano surface with $\Sigma_X \neq \phi$ (necessarily $\dim \Sigma_X = 0$) is the projective cone defined by an ample invertible sheaf $\mathscr{L}$ on an elliptic curve; this result is found in [B] and [HW].

For three-folds, the minimal model conjecture is proved in [M]. Therefore we have that a quasi-Gorenstein Fano 3-fold with $\dim \Sigma_X = 0$ is the projective cone defined by an ample invertible sheaf $\mathscr{L}$ on either an Abelian surface or a normal K3-surface (i.e. a normal surface with the trivial canonical sheaf whose minimal resolution is a K3-surface).

This work has been stimulated by discussions with Professors N. Nakayama, M. Tomari, K. Watanabe and K.-I. Watanabe.

Throughout this article, we use the notation and terminology appeared in [KMM] without comment.

2. The formulation and the proof of the theorem

PROPOSITION. *Let X be a quasi-Gorenstein Fano n-fold with* $\dim \Sigma_X = 0$.

Assume that the graded algebra $\bigoplus_{m\geqslant 0} f_*\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})$ is finitely generated $\mathcal{O}_X$-algebra, where $f: \tilde{X} \to X$ is a resolution of the singularities on X. Then X is the projective cone defined by an ample invertible sheaf $\mathscr{L}$ on a normal $(n-1)$-fold S satisfying $\mathcal{O}(K_S) \simeq \mathcal{O}_S$ with at worst rational singularities on S.

THEOREM. *Let X be a quasi-Gorenstein Fano 3-fold with* $\dim \Sigma_X = 0$. *Then*

(i) *X is the projective cone defined by an ample invertible sheaf on either an Abelian surface or a normal $K3$-surface S.*
(ii) *Furthermore, X is a Gorenstein variety if and only if S is a normal $K3$-surface.*

Proof of Theorem (*assuming Proposition*). If the minimal model conjecture holds, the assumption on the graded algebra of the proposition holds. For the three dimensional case, the minimal model conjecture was proved by Mori ([M]).

Therefore X is the projective cone on a normal surface S satisfying $\mathcal{O}(K_S) \simeq \mathcal{O}_S$ with at worst rational singularities on S. From the classification of surfaces, it follows that a normal surface S with $K_S = 0$ and only rational singularities is either an Abelian surface or a normal K3-surface (see, for example [U]). To prove (ii), let g: $Y = \mathbb{P}(\mathcal{O}_S \oplus \mathscr{L}) \to X$ be the contraction of the negative section. Clearly $R^1 g_* \mathcal{O}_Y \neq 0$ if g contracts an Abelian surface. So $R^1 g_* \mathcal{O}_Y = 0$ if and only if S is a normal K3-surface. Since Y has at worst rational singularities, $R^1 g_* \mathcal{O}_Y = 0$ means that the non-rational singularity of X is a Cohen–Macaulay (hence Gorenstein in our case) singularity. This completes the proof of the theorem.

Before stepping into the proof of Proposition, we prepare two lemmas, the first one of which is proved in the same way as the proof of [M, 1.3 and 2.3.2]. However, for the convenience of the readers, we write down the proof.

LEMMA 1 ([M, 1.3 and 2.3.2]). *Let Y be an n-dimensional projective variety with canonical singularities ($n \geqslant 2$). Let $R \subset \overline{NE}(Y)$ be an extremal ray such that the contraction morphism φ_R is birational, and D be the exceptional set. Assume that each fiber of the morphism $\varphi_R|_D : D \to \varphi_R(D)$ is of dimension one. Then,*

(i) *each fiber of $\varphi_R|_D$ is a union of $\mathbb{P}^1$'s whose configuration is a tree,*
(ii) $0 > K_Y \cdot l \geqslant -1$ *for a component l* of a fiber of *$\varphi_R|_D$ which contains a Gorenstein point of Y.*

Proof. By the assumption, $R^2 \varphi_{R*} \mathscr{F} = 0$ for a coherent sheaf $\mathscr{F}$ on Y. The Grauert–Riemenschneider vanishing theorem and a result of Kawamata [K2, Theorem 1.2] yield $R^i \varphi_{R*} \mathcal{O}_Y = R^i \varphi_{R*} \omega_Y = 0$ ($i > 0$). Therefore, for an arbitrary ideal J on Y such that Supp $\mathcal{O}_Y/J$ is contained in a fiber of φ_R, one sees $H^1(\mathcal{O}_Y/J) = H^1(\omega_Y/J\omega_Y) = H^1((\omega_Y/J\omega_Y)/\mathscr{T}\!or) = 0$. Take J the ideal of the reduced fiber. Then the vanishing $H^1(\mathcal{O}_Y/J) = 0$ implies (i).

Now consider (ii). Because l contains a Gorenstein point of Y, $\omega_Y \otimes \mathcal{O}_l/\mathscr{T}\!or$ is an invertible sheaf. Let m be a positive integer such that $\omega_Y^{[m]}$ is invertible. Then we have an inclusion $\alpha\colon (\omega_Y \otimes \mathcal{O}_l/\mathcal{T}or)^{\otimes m} \to \omega_Y^{[m]} \otimes \mathcal{O}_l$ of invertible sheaves on l which is an isomorphism on a general point of l. Denote $\omega_Y \otimes \mathcal{O}_l/\mathcal{T}or$ by $\mathcal{O}_{\mathbb{P}^1}(a)$. Since $\deg \omega_Y^{[m]} \otimes \mathcal{O}_l < 0$, the inclusion α yields that $a < 0$. By the vanishing $H^1(l, \omega_Y \otimes \mathcal{O}_l/\mathcal{T}or) = 0$ shown above, the negative number a must be -1. Therefore by the Riemann–Roch theorem, $mK_Y \cdot l = \deg \omega_Y^{[m]} \otimes \mathcal{O}_l = -m + \dim_{\mathbb{C}} \mathrm{Coker}\, \alpha$ which show the assertion (ii).

LEMMA 2. *Let X be a normal $\mathbb{Q}$-Gorenstein variety and Σ be the locus of non-canonical singularities on X. Let $g\colon Y \to X$ be a projective birational morphism such that Y has at worst canonical singularities, g is an isomorphism away from Σ and K_Y is relatively ample with respect to g. Then the reduced inverse image $g^{-1}(\Sigma)_{\mathrm{red}}$ is of pure codimension one. Moreover, if we denote $g^{-1}(\Sigma)_{\mathrm{red}}$ by the sum $\Sigma_{i=1}^r E_i$ of irreducible components E_i, the canonical divisor on Y is represented as $K_Y = g^*K_X - \Sigma_{i=1}^r a_iE_i$ with $a_i > 0$ for every i. In particular if X is a quasi-Gorenstein variety (i.e. K_X is a Cartier divisor), all the a_i are positive integers.*

Proof. As is well known, a projective birational morphism $g\colon Y \to X$ is obtained by the blowing up of some ideal sheaf on X. Therefore there are positive numbers $m_i\,(i = 1, 2, \ldots, r, r+1, \ldots, t)$ such that $L = -\Sigma_{i=1}^t m_iE_i$ is a relatively very ample Cartier divisor with respect to g, where all E_i's $(i = 1, 2, \ldots, r)$ are the irreducible Weil divisors contained in $g^{-1}(\Sigma)$ and E_i's are the ones not contained in $g^{-1}(\Sigma)$. Since K_Y is relatively ample. $K_Y + aL$ $(a \geqslant 0, a \in \mathbb{Q})$ is relatively ample with respect to g. Denote the canonical divisor K_Y by $g^*K_X - \Sigma_{i=1}^r a_iE_i$ with $a_i \in \mathbb{Q}$. If there exists a non-positive a_i, we let a be the non-negative number $-\min_{1 \leqslant i \leqslant r}\{a_i/m_i\}$. Then we have:

$$K_Y + aL = g^*K_X - \sum_{i=r+1}^{t} am_iE_i - \sum_{i=1}^{r} \beta_iE_i,$$

where $\beta_i = 0$ for the i's such that a_i/m_i attain the minimal value, and $\beta_i > 0$ for other i's. Here, there exists i $(1 \leqslant i \leqslant r)$ for which a_i/m_i does not attain the minimal value $-a$, otherwise the singularities in Σ would become canonical singularities. Let C be an irreducible curve on E_i $(1 \leqslant i \leqslant r)$ with $\beta_i = 0$ and be contained in a fiber $g^{-1}(x)$ $x \in \Sigma$. We can take C such that $C \not\subseteq \bigcup_{j \neq i} E_j$. Then $(K_Y - aL) \cdot C \leqslant 0$, which is a contradiction. Now it remains to show that $g^{-1}(\Sigma)_{\mathrm{red}} = \Sigma_{i=1}^r E_i$. Let C' be a curve contained in the intersection of a fiber $g^{-1}(x)$ and an irreducible component of $g^{-1}(\Sigma)_{\mathrm{red}}$ of codimension greater than one. We can assume the curve C' is not contained in $\Sigma_{i=1}^r E_i$. Then $K_Y \cdot C' = (g^*K_X - \Sigma_{i=1}^r a_iE_i) \cdot C' \leqslant 0$, since $a_i > 0$ for $i = 1, 2, \ldots, r$. This is a contradiction to that K_Y is relatively ample. Therefore $g^{-1}(\Sigma)_{\mathrm{red}}$ must coincide with $\Sigma_{i=1}^r E_i$. Now we show the last statement. The canonical divisor $K_Y = g^*K_X - \Sigma_{i=1}^r a_iE_i$ is a Cartier divisor on the non-singular locus on Y.

Therefore the divisor $\Sigma_{i=1}^r a_iE_i$ is also a Cartier divisor there, if K_X is a Cartier divisor on X. In this case, all a_i's are integers.

Proof of Proposition. By the hypothesis of the proposition, $Y = \mathrm{Proj} \oplus_{m \geqslant 0} f_*\mathcal{O}(mK_{\tilde{X}})$ is a projective variety over $\mathbb{C}$. Denote the canonical morphism by $g: Y \to X$. Then,

(1) Y has canonical singularities,
(2) K_Y is relatively ample $\mathbb{Q}$-Cartier divisor with respect to g,
(3) denoting $g^{-1}(\Sigma_X)_{\mathrm{red}}$ by E, E is of pure codimension one and the restricted morphism $g|_{Y-E}: Y - E \to X - \Sigma_X$ is an isomorphism, and
(4) decomposing E into irreducible components $E_1, \ldots, E_r$, K_Y is represented by $g^*K_X - \Delta$, where $\Delta = \Sigma_{i=1}^r a_iE_i$ ($a_i \in \mathbb{N}$ for every $i = 1, 2, \ldots, r$).

In facts, (1) and (2) are well known. The second assertion of (3) follows from the fact that the singularities on $X - \Sigma_X$ are canonical singularities. The first statement of (3) and (4) follow from Lemma 2. Besides, the statement (2) can be replaced by the following:

(2)′ $\Delta \cdot C < 0$ for an arbitrary irreducible curve C in E.

Now we claim that there exists an extremal ray $R \subset \overline{NE}(Y)$ such that $\Delta \cdot R > 0$. To prove the claim, let $\overline{NE}_{K_Y}(Y)$ be the subset $\{[C] \in \overline{NE}(Y) \mid K_Y \cdot C \geqslant 0\}$. Then, by the cone theorem ([K1]),

$$\overline{NE}(Y) = \sum_i R_i + \overline{NE}_{K_Y}(Y),$$

where R_i's are extremal rays. Since $-\Delta = K_Y - g^*K_X$, it is clear that $-\Delta \geqslant 0$ on $\overline{NE}_{K_Y}$. If $\Delta \cdot R \leqslant 0$ for every extremal ray, then it follows by the theorem on the cone that $-\Delta$ is nef, a contradiction. Now let R be the extremal ray with $\Delta \cdot R > 0$ and $\varphi: Y \to S$ be the contraction of R. Then we show

(6) $\varphi|_E: E \to \varphi(E)$ is a finite morphism,
(7) $\dim S = n - 1$,
(8) $\Delta = E = E_1$ (i.e. Δ is irreducible and reduced) which is isomorphic to S by the restricted morphism $\varphi|_E$,
(9) Σ_X consists of one point x and there exists a member H of $|-K_X|$ which does not pass through the point x, and
(10) $\tilde{H} = g^*H$ is isomorphic to S by the restricted morphism $\varphi|_{\tilde{H}}$.

To show (6), we assume that there exists a curve C in E mapped to a point by φ, and deduce a contradiction. By the assumption, there exists a curve C in E mapped to a point by φ. Since the morphism φ is the contraction of R, the class of an irreducible curve mapped to a point by φ belongs to R. Thus $[C] \in R$, which implies $\Delta \cdot C > 0$ by the definition of R. But the contradiction $\Delta \cdot C < 0$ follows from (2)′ since $C \subset E$.

Next we prove (7). By (6), we have $\dim S \geqslant \dim \varphi(E) = n-1$. Assuming $\dim S = n$, we deduce a contradiction. By the assumption, φ is birational. Denote the exceptional set by D. Then every fiber of the restricted morphism $\varphi|_D: D \to \varphi(D)$ is of dimension one. Otherwise, there would exist a fiber $\varphi^{-1}(s)$ containing a curve C disjoint from finite points set $\varphi^{-1}(s) \cap E$. By $C \cap E = \phi$, we would have $\Delta \cdot C = 0$. However C is contracted to a point s, which implies $[C] \in R$ therefore $\Delta \cdot C > 0$, a contradiction. Now we have a situation for which we can apply Lemma 1. Let l be an irreducible component of a fiber $\varphi^{-1}(s)$. Then l contains a Gorenstein point of Y, because the non-Gorenstein locus is contained in E and l intersects E at only finite points. Therefore, by (ii) of Lemma 1, $K_Y \cdot l \geqslant -1$. On the other hand, $K_Y \cdot l = g^*K_X \cdot l - \Delta \cdot l$, where $\Delta \cdot l > 0$ because of $[l] \in R$. Here $g^*K_X \cdot l < 0$ because of the ampleness of $-K_X$ and $l \not\subset E$. Furthermore, since g^*K_X is a Cartier divisor, $g^*K_X \cdot l$ must be an integer. So, we have $g^*K_X \cdot l \leqslant -1$. In conclusion, $K_Y \cdot l = g^*K_X \cdot l - \Delta \cdot l < -1$, which is a contradiction. This completes the proof of (7).

Now we prove (8). By (7), $\varphi: Y \to S$ is a fiber space of relative dimension one, with general fiber l isomorphic to $\mathbb{P}^1$ by a result of Kawamata. Therefore $-2 = K_Y \cdot l = g^*K_X \cdot l - \Delta \cdot l$. Here we may assume that l intersects each E_i at points where E_i is a Cartier divisor, because $\varphi|_{E_i}: E_i \to S$ is a finite surjective morphism, and so the image of the non-Cartier locus of E_i by φ is a proper closed subset of S. Thus we have $\Delta \cdot l \geqslant 1$. Since we already got $g^*K_X \cdot l \leqslant -1$ in the argument of (7), the equality $-2 = g^*K_X \cdot l - \Delta \cdot l$ yields the two equalities:

(11) $\Delta \cdot l = 1$ and $g^*K_X \cdot l = -1$.

By the choice of l, we get $E_i \cdot l \geqslant 1$ for every $i = 1, 2, \dots, r$. So $\Delta \cdot l = \Sigma_{i=1}^r a_i(E_i \cdot l) \geqslant \Sigma_{i=1}^r a_i \geqslant 1$ $(a_i \in \mathbb{N})$. Therefore the first equality of (11) implies that $r=1$ and $a_1 = 1$, i.e. Δ is irreducible and reduced. Turn to the statements (9). The first statement of (9) follows from the fact that Σ_X is the image of an irreducible divisor; denote the single point of Σ_X by x. In order to show the existence of H as claimed in (9), it is sufficient to show the injection $\iota: \Gamma(X, \mathcal{M}_x\mathcal{O}(-K_X)) \to \Gamma(X, \mathcal{O}(-K_X))$ is not a bijection, where $\mathcal{M}_x$ is the maximal $\mathcal{O}_X$-ideal defining the point x. Since $g_*\mathcal{O}(K_Y - 2g^*K_X) = \mathcal{M}_x\mathcal{O}(-K_X)$ it is sufficient to show $H^1(X, g_*\mathcal{O}_Y(K_Y^2 - g^*K_X)) = 0$. Consider the Leray spectral sequence:

$$E_2^{p,q} = H^p(X, R^q g_*\mathcal{O}(K_Y - 2g^*K_X)) \Rightarrow H^{p+q}(Y, \mathcal{O}(K_Y - 2g^*K_X)).$$

Then there exists an injection:

$$H^1(X, g_*\mathcal{O}(K_Y - 2g^*K_X)) \to H^1(Y, \mathcal{O}(K_Y - 2g^*K_X)).$$

Here the right-hand side is zero because $-2g^*K_X$ is nef and big (the Kawamata–Viehweg vanishing theorem). This completes the proof of (9). In order to prove the assertion (10), we take $H \in |-K_X|$ which does not pass through the point x. Denote g^*H by $\tilde{H}$. Then the restricted morphism $\varphi|_{\tilde{H}}: \tilde{H} \to S$ is a finite morphism. In fact, if there is a curve C on $\tilde{H}$ mapped to a point by φ, the curve must satisfy $\Delta \cdot C > 0$ which is however a contradiction to $\tilde{H} \cap E = \phi$. On the other hand, by the second equality of (11), one sees $\tilde{H} \cdot l = -g^*K_X \cdot l = 1$ for a general fiber l of φ. So, $\varphi|_{\tilde{H}}$ is a finite birational morphism onto a normal variety S. Thus, by Zariski's Main Theorem, $\varphi|_{\tilde{H}}: \tilde{H} \to S$ is an isomorphism. This completes the proof of all the statements (6)–(10). Now we are going to show that $\varphi: Y \to S$ is a $\mathbb{P}^1$-bundle over S defined by an ample invertible sheaf. Denote $\mathcal{O}_Y(\tilde{H})$ by $\mathscr{L}$. Then $\mathscr{L}$ is relatively ample with respect to φ. In fact, for an arbitrary irreducible component C of a fiber of φ, we get $\tilde{H} \cdot C = -g^*K_X \cdot C > 0$ since $C \not\subset E$ and $-K_X$ is ample on X. By an exact sequence: $0 \to \mathcal{O}_Y \to \mathscr{L} \to \mathscr{L} \otimes \mathcal{O}_{\tilde{H}} \to 0$, we have the following exact sequence on S:

$$\begin{array}{ccccccccc} 0 \to & \varphi_*\mathcal{O}_Y & \to & \varphi_*\mathscr{L} & \to & \varphi_*(\mathscr{L} \otimes \mathcal{O}_{\tilde{H}}) & \to & R^1\varphi_*\mathcal{O}_Y & \\ & \| & & & & & & \| & , \\ & \mathcal{O}_X & & & & & & 0 & \end{array} \tag{12}$$

where the vanishing of the last term comes from [K2, Theorem 1.2]. Since $\varphi|_{\tilde{H}}: \tilde{H} \to S$ is an isomorphism, the third term is an invertible sheaf on S. Hence $\varphi_*\mathscr{L}$ is a locally free sheaf of rank two. In the commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_Y & \to & \mathscr{L} & \to & \mathscr{L} \otimes \mathcal{O}_{\tilde{H}} & \to 0 \\ & \uparrow^{\alpha} & & \uparrow^{\beta} & & \uparrow^{\gamma} & , \\ & \varphi^*\varphi_*\mathcal{O}_Y & \to & \varphi^*\varphi_*\mathscr{L} & \to & \varphi^*\varphi_*(\mathscr{L} \otimes \mathcal{O}_{\tilde{H}}) & \to 0 \end{array}$$

α is bijective and γ is surjective. Therefore $\beta: \varphi^*\varphi_*\mathscr{L} \to \mathscr{L}$ is surjective, which yields that the rational map $\Phi_{|\mathscr{L}|}: Y \to \mathbb{P}(\varphi_*\mathscr{L})$ becomes a well-defined morphism on whole Y. As $\mathscr{L}$ is relatively ample, $\Phi_{|\mathscr{L}|}$ is a finite morphism. On the other hand, $\deg \mathscr{L}|_l = \tilde{H} \cdot l = 1$ for a general fiber l of φ which means that $\Phi_{|\mathscr{L}|}$ is birational. Hence $\Phi_{|\mathscr{L}|}$ is an isomorphism by Zariski's Main Theorem. So Y is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}(\varphi_*\mathscr{L})$ and has a disjoint section E from $\tilde{H}$ by (6) and by the definition of $\tilde{H}$. Therefore the exact sequence (12) splits. So, $Y = \mathbb{P}(\mathcal{O}_S \oplus \mathscr{L}_S)$, where $\mathscr{L}_S = \varphi_*(\mathscr{L} \otimes \mathcal{O}_{\tilde{H}})$. Now we have that X is obtained by contracting the negative section E, which means that X is a projective cone defined by $\mathscr{L}_S$ over S. The property $\mathcal{O}(K_S) \simeq \mathcal{O}_S$ which S is required to satisfy follows from that $S \simeq \tilde{H} \simeq H$ and $H \in |-K_X|$. The singularities on S are all rational, since S is obtained by the contraction of an extremal ray. This completes the proof of Proposition.

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