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Essentially different factorizations of a natural number

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Let $f(n)$ denote the number of essentially different factorizations of a natural number n . In this paper, we prove that for any given $A > 0$, $f(n) \leq C(n/\log^A n)$ for every odd number $n > 1$, where C is a constant only related to A .

Let $f(n)$ denote the number of ways to write n as the product of integers ≥ 2 , where we consider factorizations that differ only in the order of the factors to be the same. We define $f(1) = 1$.

In 1983, John F. Hughes and J.O. Shallit [1] proved that $f(n) \leq 2n^{\sqrt{2}}$.

In this note, we shall prove the following

THEOREM. For any given $A > 0$, we have

$$f(n) \leq C \frac{n}{\log^A n} \quad \text{for every odd number } n > 1,$$

where C is a constant only related to A .

In this note, let $P(n)$, $P_1(n)$ be the largest and the smallest prime divisor of n respectively.

To prove the theorem, we need the following

LEMMA. If $n > 1$ and p is a prime divisor of n , then $f(n) \leq \sum_{d|n/p} f(d)$.

Proof. Let $d|n/p$, and let $m_1 \dots m_s$ be a factorization of d . Then $n/d(m_1 \dots m_s)$ is a factorization of n . However, each factorization of n can be obtained in this way. Namely, let $n = n_1 \dots n_k$ and suppose p divides n_1 . Then choose $d = n/n_1$. Hence $f(n) \leq \sum_{d|n/p} f(d)$.

Proof of the theorem. For any given $A > 0$, take a sufficiently large $k_0 > 2$ such that $(1 - 2/k_0)^A > \frac{1}{2}$. Let $A_0 = \frac{1}{2}(k_0/k_0 - 2)^A$. Then $0 < A_0 < 1$.

It is well-known that $d(n) = O(n^\delta)$ and $\log^A n = O(n^\delta)$ for every positive δ , where $d(n)$ is the number of divisors of n . Hence we have

$$d(n) \leq C_0 n^{1/k_0}, \tag{1}$$

$$\log^A n \leq C_1 n^{1/k_0}, \tag{2}$$

where C_0, C_1 are constants and $C_0C_1 \geq \log^4 3/3$.

Let $n = \prod_{i=1}^r p_i^{a_i}$, $p_1 < p_2 < \dots < p_r$. It is easy to prove

$$\sum_{d|n/p_r} d \leq \frac{n}{p_1 - 1} \tag{3}$$

In fact, we either have

$$\sum_{d|n/p_1} d = \frac{p_1^{a_1} - 1}{p_1 - 1} < \frac{n}{p_1 - 1} \quad (\gamma = 1)$$

or

$$\begin{aligned} \sum_{d|n/p_r} d &= \frac{p_r^{a_r} - 1}{p_r - 1} \prod_{i=1}^{\gamma-1} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \frac{p_r^{a_r} - 1}{p_1 - 1} \prod_{i=1}^{\gamma-1} \frac{p_i^{a_i+1} - 1}{p_{i+1} - 1} \\ &\leq \frac{p_r^{a_r}}{p_1 - 1} \prod_{i=1}^{r-1} \frac{p_i^{a_i+1}}{p_i} = \frac{n}{p_1 - 1} \quad (r \geq 2). \end{aligned}$$

Let $C = C_0C_1/1 - A_0$, we shall prove that $f(n) \leq C(n/\log^4 n)$ holds for every odd number $n > 1$ by induction.

When $n = 3$, we have $f(3) = 1 < C(3/\log^4 3)$.

Let n be any odd number larger than 3. Suppose that $f(d) \leq C(d/\log^4 d)$ holds for all odd numbers $d < n$. We shall prove that $f(n) \leq C(n/\log^4 n)$.

By the lemma, we have

$$f(n) \leq \sum_{d|n/p(n)} f(d) = \sum_{\substack{d|n/p(n) \\ d \leq n^{1-2/k_0}}} f(d) + \sum_{\substack{d|n/p(n) \\ d > n^{1-2/k_0}}} f(d) = S_1 + S_2 \tag{4}$$

By means of induction on n , our lemma and (3), we immediately obtain

$$f(n) \leq n. \tag{5}$$

By (1), (2) and (5), we get

$$S_1 \leq n^{1-2/k_0} d(n) \leq C_0 n^{1-1/k_0} \leq C_0 C_1 \frac{n}{\log^4 n}. \tag{6}$$

By (3), $p_1(n) > 2$ and the inductive hypothesis, we get

$$\begin{aligned}
 S_2 &\leq \frac{C_0 C_1}{1 - A_0} \sum_{\substack{d|n/p(n) \\ d > n^{1-2/k_0}}} \frac{d}{\log^4 d} \leq \frac{C_0 C_1}{1 - A_0} \left(\frac{k_0}{k_0 - 2}\right)^4 \frac{1}{\log^4 n} \sum_{d|n/p(n)} d \\
 &\leq \frac{C_0 C_1}{1 - A_0} \left(\frac{k_0}{k_0 - 2}\right)^4 \frac{1}{p_1(n) - 1} \frac{n}{\log^4 n} \leq \frac{1}{2} \left(\frac{k_0}{k_0 - 2}\right)^4 \frac{C_0 C_1}{1 - A_0} \frac{n}{\log^4 n} \\
 &= \frac{A_0}{1 - A_0} C_0 C_1 \frac{n}{\log^4 n}.
 \end{aligned} \tag{7}$$

By (4), (6) and (7), we get

$$f(n) \leq C_0 C_1 \frac{n}{\log^4 n} + \frac{A_0}{1 - A_0} C_0 C_1 \frac{n}{\log^4 n} = \frac{C_0 C_1}{1 - A_0} \frac{n}{\log^4 n} = C \frac{n}{\log^4 n}.$$

Our theorem is now proved by induction.

Finally, we point out that $f(n) = O(n^\alpha)$, $\alpha < 1$, does not hold for every odd number $n > 1$. In fact, if $f(n) \leq Cn^\alpha$ for all odd number $n > 1$, then as the argument runs through the sequence formed by all odd numbers $n > 1$, we have

$$\lim_{n \rightarrow \infty} \frac{\log f(n)}{\log n} \leq \alpha.$$

Let $B(n)$ denote the n th Bell number, and let $a_n = p_2 p_3 \dots p_{n+1}$, where p_i is i th prime, we have

$$\log f(a_n) = \log B(n) \sim n \log n$$

and

$$\log a_n = \sum_{i=2}^{n+1} \log p_i = \sum_{p \leq p_{n+1}} \log p - \log 2 \sim p_{n+1} \sim (n + 1) \log(n + 1).$$

It follows that

$$\lim_{n \rightarrow \infty} \frac{\log f(a_n)}{\log a_n} = 1.$$

This is a contradiction.

Reference

1. Hughes, J.F. and Shallit, J.O., On the number of multiplicative partitions, *Amer. Math. Monthly* 90 (1983), 468–471.