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Essentially different factorizations of a natural number

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Let $f(n)$ denote the number of essentially different factorizations of a natural number n. In this paper, we prove that for any given $A > 0$, $f(n) \leqslant C(n/\log^A n)$ for every odd number $n > 1$, where C is a constant only related to A.

Let $f(n)$ denote the number of ways to write n as the product of integers $\geqslant 2$, where we consider factorizations that differ only in the order of the factors to be the same. We define $f(1) = 1$.

In 1983, John F. Hughes and J.O. Shallit [1] proved that $f(n) \leqslant 2n^{\sqrt{2}}$.

In this note, we shall prove the following

THEOREM. *For any given* $A > 0$, we have

$$f(n) \leqslant C \frac{n}{\log^A n} \quad \text{for every odd number } n > 1,$$

where C is a constant only related to A.

In this note, let $P(n)$, $P_1(n)$ be the largest and the smallest prime divisor of n respectively.

To prove the theorem, we need the following

LEMMA. *If $n > 1$ and p is a prime divisor of n, then* $f(n) \leqslant \Sigma_{d|n/p} f(d)$.

Proof. Let $d \mid n/p$, and let $m_1 \ldots m_s$ be a factorization of d. Then $n/d\ (m_1 \ldots m_s)$ is a factorization of n. However, each factorization of n can be obtained in this way. Namely, let $n = n_1 \ldots n_k$ and suppose p divides n_1. Then choose $d = n/n_1$. Hence $f(n) \leqslant \Sigma_{d|n/p} f(d)$.

Proof of the theorem. For any given $A > 0$, take a sufficiently large $k_0 > 2$ such that $(1 - 2/k_0)^A > \frac{1}{2}$. Let $A_0 = \frac{1}{2}(k_0/k_0 - 2)^A$. Then $0 < A_0 < 1$.

It is well-known that $d(n) = 0(n^\delta)$ and $\log^A n = 0(n^\delta)$ for every positive δ, where $d(n)$ is the number of divisors of n. Hence we have

$$d(n) \leqslant C_0 n^{1/k_0}, \tag{1}$$

$$\log^A n \leqslant C_1 n^{1/k_0}, \tag{2}$$

where C_0, C_1 are constants and $C_0C_1 \geqslant \log^A 3/3$.

Let $n=\Pi_{i=1}^r p_i^{a_i}$, $p_1<p_2<\cdots<p_r$. It is easy to prove

$$\sum_{d\mid n/p_r} d \leqslant \frac{n}{p_1-1} \tag{3}$$

In fact, we either have

$$\sum_{d\mid n/p_1} d = \frac{p_1^{a_1}-1}{p_1-1} < \frac{n}{p_1-1} \quad (\gamma=1)$$

or

$$\sum_{d\mid n/p_r} d = \frac{p_r^{a_r}-1}{p_r-1}\prod_{i=1}^{\gamma-1}\frac{p_i^{a_i+1}-1}{p_i-1} = \frac{p_r^{a_r}-1}{p_1-1}\prod_{i=1}^{\gamma-1}\frac{p_i^{a_i+1}-1}{p_{i+1}-1}$$
$$\leqslant \frac{p_r^{a_r}}{p_1-1}\prod_{i=1}^{r-1}\frac{p_i^{a_i+1}}{p_i} = \frac{n}{p_1-1} \quad (r\geqslant 2).$$

Let $C=C_0C_1/1-A_0$, we shall prove that $f(n)\leqslant C(n/\log^A n)$ holds for every odd number $n>1$ by induction.

When $n=3$, we have $f(3)=1<C(3/\log^A 3)$.

Let n be any odd number larger than 3. Suppose that $f(d)\leqslant C(d/\log^A d)$ holds for all odd numbers $d<n$. We shall prove that $f(n)\leqslant C(n/\log^A n)$.

By the lemma, we have

$$f(n)\leqslant \sum_{d\mid n/p(n)} f(d) = \sum_{\substack{d\mid n/p(n)\\ d\leqslant n^{1-2/k_0}}} f(d) + \sum_{\substack{d\mid n/p(n)\\ d>n^{1-2/k_0}}} f(d) = S_1+S_2 \tag{4}$$

By means of induction on n, our lemma and (3), we immediately obtain

$$f(n)\leqslant n. \tag{5}$$

By (1), (2) and (5), we get

$$S_1 \leqslant n^{1-2/k_0}d(n) \leqslant C_0 n^{1-1/k_0} \leqslant C_0C_1\frac{n}{\log^A n}. \tag{6}$$

By (3), $p_1(n) > 2$ and the inductive hypothesis, we get

$$S_2 \leqslant \frac{C_0C_1}{1-A_0} \sum_{\substack{d|n/p(n)\\ d>n^{1-2/k_0}}} \frac{d}{\log^A d} \leqslant \frac{C_0C_1}{1-A_0}\left(\frac{k_0}{k_0-2}\right)^A \frac{1}{\log^A n} \sum_{d|n/p(n)} d$$

$$\leqslant \frac{C_0C_1}{1-A_0}\left(\frac{k_0}{k_0-2}\right)^A \frac{1}{p_1(n)-1}\frac{n}{\log^A n} \leqslant \frac{1}{2}\left(\frac{k_0}{k_0-2}\right)^A \frac{C_0C_1}{1-A_0}\frac{n}{\log^A n}$$

$$= \frac{A_0}{1-A_0} C_0C_1 \frac{n}{\log^A n}. \tag{7}$$

By (4), (6) and (7), we get

$$f(n) \leqslant C_0C_1 \frac{n}{\log^A n} + \frac{A_0}{1-A_0} C_0C_1 \frac{n}{\log^A n} = \frac{C_0C_1}{1-A_0}\frac{n}{\log^A n} = C\frac{n}{\log^A n}.$$

Our theorem is now proved by induction.

Finally, we point out that $f(n) = 0(n^\alpha)$, $\alpha < 1$, does not hold for every odd number $n > 1$. In fact, if $f(n) \leqslant Cn^\alpha$ for all odd number $n > 1$, then as the argument runs through the sequence formed by all odd numbers $n > 1$, we have

$$\overline{\lim_{n\to\infty}} \frac{\log f(n)}{\log n} \leqslant \alpha.$$

Let $B(n)$ denote the nth Bell number, and let $a_n = p_2p_3 \cdots p_{n+1}$, where p_i is ith prime, we have

$$\log f(a_n) = \log B(n) \sim n \log n$$

and

$$\log a_n = \sum_{i=2}^{n+1} \log p_i = \sum_{p \leqslant p_{n+1}} \log p - \log 2 \sim p_{n+1} \sim (n+1)\log(n+1).$$

It follows that

$$\lim_{n\to\infty} \frac{\log f(a_n)}{\log a_n} = 1.$$

This is a contradiction.

Reference

1. Hughes, J.F. and Shallit, J.O., On the number of multiplicative partitions, *Amer. Math. Monthly* 90 (1983), 468–471.