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CHARLES ASMUTH

C. DAVID KEYS

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## Supercuspidal representations of $GSp_4$ over a local field of residual characteristic 2, part I

CHARLES ASMUTH<sup>1</sup> and C. DAVID KEYS<sup>2</sup>

<sup>1</sup>*David Sarnoff Research Center, Princeton, New Jersey 08540;* <sup>2</sup>*Department of Mathematics and Computer Science, Rutgers University, Newark, New Jersey 07102*

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### 1. Introduction

In this paper, and in one to follow, we analyze supercuspidal representations of  $G = GSp_4(F)$  where  $F$  is a local field of residual characteristic  $p = 2$ . We first classify these representations according to characters which occur in their restrictions to certain open compact subgroups. As in the case of  $GL_n(F)$ , these characters may be parametrized by  $4 \times 4$  matrices over  $F$ . Accordingly, we may classify certain supercuspidal representations as occurring in the “nilpotent” case. Theorem 3.9 states that these representations are all irreducibly induced from some open compact-by-center subgroup of  $G$ . The remaining cases will be treated in the sequel.

The methods used are suggested by the construction techniques for  $GL_n(F)$  developed by Howe ([H]) and exhaustion techniques developed by Kutzko and Manderscheid ([K] and [KM]). These ideas work well when no wildly ramified extensions of  $F$  are imbedded in the group in question. Moy ([M]) has used techniques involving Hecke algebra isomorphisms to classify supercuspidal representations of  $GSp_4(F)$  in the case of odd primes  $p$ .

When wild ramification is present (so  $p = 2$ ), these methods still provide a starting point. A representation  $(\pi, V)$  will fix a non-trivial subspace  $V(L)$  when restricted to a small enough open subgroup  $L$ . We will generally take  $L$  to be a member of some normal filtration of a maximal compact-by-center subgroup of  $G$ . We then consider certain groups  $A$  for which the quotient  $A/L$  is abelian. We classify  $\pi$  by characters occurring in the restriction  $(\pi|_A, V(L))$ . The occurrence of certain characters (cuspidal) immediately forces  $\pi$  to be irreducibly induced. Some characters are shown to be  $(G, A)$ -principal as defined in [K]. Some cases lead us to find an open subgroup  $L'$ , larger than  $L$ , having a fixed vector. The remaining cases are analogous to the alfalfa cases of [KM]. For these we must use the constructions in [H] to distinguish between representations which are irreducibly induced and those that contain principal characters.

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<sup>2</sup>Henry Rutgers Research Fellow.

Recall that the matrix coefficients of a supercuspidal representation have compact support mod the center of  $G$ . A  $(G, A)$ -principal representation of a subgroup  $A$  is one which cannot occur in the restriction to  $A$  of a supercuspidal representation of  $G$ . A representation of an open compact subgroup of  $G$  may be shown to be principal by proving that vectors in its representation space give rise to matrix coefficients of  $G$  whose support is not compact mod the center of  $G$ . The following is a restatement of some of the lemmas in [K] and gives the method we will use to demonstrate that certain representations are principal representations.

LEMMA 1.1. *Let  $(\pi, V)$  be an admissible representation of  $G$ . Let  $v_0 \in V$  and let  $L$  be an open compact subgroup fixing  $v_0$ . Suppose there exist elements  $x_{ij} \in G$  and constants  $N(i)$  and  $c_{ij}$ , for  $i = 1, 2, 3 \dots$  and  $1 \leq j \leq N(i)$ , such that*

(a) *each  $v_i \in V(L)$ , where*

$$v_i = \sum_{j=1}^{N(i)} c_{ij} \pi(x_{ij}) v_{i-1}$$

(b) *No subsequence of elements of the form*

$$s_n = \prod_{i=1}^n x_{i,j}$$

*with  $j = j(i, n)$ ,  $1 \leq j \leq N(i)$  for all  $i$ , and  $1 \leq i \leq n$ , for  $n = 1, 2, 3, \dots$ , is contained in a compact-by-center subset of  $G$ .*

*Then  $v_0$  produces a matrix coefficient of  $(\pi, V)$  whose support is not compact-by-center.*

*Proof.* Since  $\pi$  is admissible, the space  $V(L)$  of  $L$ -fixed vectors in  $V$  is finite-dimensional. Thus there exists a linear functional  $v'_0$  on  $V(L)$  such that  $\langle v_n, v'_0 \rangle \neq 0$  for infinitely many  $n$ . Extend  $v'_0$  to be zero on the complement of  $V(L)$  to define a smooth vector, also denoted  $v'_0$ , in the contragredient of  $V$ .

Consider the matrix coefficient  $g \mapsto \langle \pi(g)v_0, v'_0 \rangle$ . For  $v_n$  in the subsequence above,

$$0 \neq \langle v_n, v'_0 \rangle = \sum_{j=1}^{N(n)} c_{nj} \langle \pi(x_{nj})v_{n-1}, v'_0 \rangle$$

which inductively is a linear combination of terms of the form

$$\left\langle \pi \left( \prod_{i=1}^n x_{ij} \right) v_0, v'_0 \right\rangle$$

for  $j$  with  $1 \leq j \leq N(i)$ .

Thus at least one such term must be non-zero. Let  $s_n = \prod_{i=1}^n x_{ij}$  be such that  $\langle \pi(s_n)v_0, v'_0 \rangle \neq 0$  for each such  $n$ .

The support of the above matrix coefficient of  $\pi$  thus contains the set of such  $s_n$ , which is not compact-mod-center by hypothesis. Hence a representation  $(\pi, V)$  with these properties cannot be supercuspidal.

A representation of  $A$  with such a  $v_0$  in its space is thus a  $(G, A)$ -principal representation. In our applications of Lemma 1.1, the elements  $x_{ij}$  are generally products of some fixed translation in the affine Weyl group with appropriate unipotent elements. It will then be immediate to check that condition (b) of the Lemma holds.

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## 2. Structure and Notation

In this paper we will realize the group  $GSp_4(F)$  of symplectic similitudes as those elements  $g$  of  $GL_4(F)$  satisfying the relation  $gJ^t g = cJ$ , where  $c$  is a non-zero element of  $F$  and

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The group  $GSp_4(F)$  over a non-archimedean local field  $F$  has three conjugacy classes of maximal compact-mod-center subgroups. See, e.g., [I]. We will define a filtration of each of the maximal compact-mod-center subgroups of  $G$  consisting of compact open normal subgroups. First, for  $n > 0$ , let  $K_n$  denote the  $n$ th congruence subgroup of the open maximal compact subgroup  $K_0 = GSp_4(\mathcal{O})$ . We will use the following notation to describe  $K_n$ .

$$K_n = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n & n & n & n \\ n & n & n & n \end{bmatrix},$$

where the matrix entries denote the minimum valuation of entries of the difference between an element of  $K_n$  and the identity matrix.

We define the following groups in the same manner:

$$H_{n,0} = \begin{bmatrix} n & n+1 & n & n \\ n & n & n & n-1 \\ n & n+1 & n & n \\ n+1 & n+1 & n+1 & n \end{bmatrix}$$

$$H_{n,1} = \begin{bmatrix} n+1 & n+1 & n+1 & n \\ n & n+1 & n & n \\ n+1 & n+1 & n+1 & n \\ n+1 & n+2 & n+1 & n+1 \end{bmatrix}$$

$$J_{n,0} = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \end{bmatrix}$$

$$J_{n,1} = \begin{bmatrix} n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n+1 & n+1 \\ n+1 & n+1 & n+1 & n+1 \end{bmatrix}.$$

Let  $K = K_0Z(G)$ . Let  $H$  be the group generated by  $H_{0,0}$  and  $z_H$ , and let  $J$  be the group generated by  $J_{0,0}$  and  $z_J$ . Here

$$z_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix} \quad \text{and} \quad z_J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix}.$$

Then the groups  $K$ ,  $J$  and  $H$  represent the three conjugacy classes of maximal compact-by-center open subgroups of  $G$ .

Let  $W$  be the Weyl group of  $G$  and  $W_a$  the affine Weyl group. An element of  $W_a$  may be expressed in the form  $w t_1^r t_2^s$  where  $w \in W$ ,  $r$  and  $s$  are integers, and

$$t_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}, \quad t_2 = \begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Set  $t_0 = t_1 t_2$  as members of  $W_a$ . Then (modulo  $Z(G)$ )

$$t_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let  $\{\alpha, \beta\}$  be the base for the root system  $\Phi$  of  $G$  with  $\beta = e_2 - e_1$  taken to be the short root and  $\alpha = 2e_1$  the long root. For any root  $\gamma \in \Phi$ , let  $w_\gamma$  denote the reflection determined by  $\gamma$ , and let  $t_\gamma$  be the translation in the affine Weyl group associated to  $\gamma$ . Recall that  $t_\gamma$  acts on the standard apartment of the Bruhat-Tits building by sending a point  $x$  to  $x + 2\gamma^\vee$ .

Note that  $t_0 = t_{\alpha+2\beta}$  is the translation associated to  $\alpha + 2\beta$ , and further, that  $(t_1)^2 = t_\alpha t_{\alpha+2\beta} = t_{\alpha+\beta}$  and  $(t_2)^2 = t_\beta$ . Then we have the following double coset decompositions of  $GSp_4(F)$ .

LEMMA 2.1.

$$\begin{aligned} K \backslash G / K &= \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\} \\ J \backslash G / J &= \{t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geq 0\} \cup \{w_{\alpha+2\beta} t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geq 0\} \\ H \backslash G / H &= \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\} \cup \{w_{\alpha+\beta} t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\}. \end{aligned}$$

Let  $(\pi, V)$  be an irreducible admissible representation of a reductive p-adic group  $G$ . Let  $L$  be any open compact-by-center subgroup of  $G$ . Let  $(\tau, W)$  be an irreducible component of  $(\pi, V)$  restricted to  $L$ . Suppose that  $\tau$  does not induce irreducibly from  $L$  to  $G$ . Then, by Mackey theory, there exists an  $x$  in  $G$ , but not in  $L$ , which intertwines  $\tau$ , i.e., there is an  $x$  outside of  $L$  for which the restrictions of  $\tau$  and its conjugate  $\text{Ad}(x)\tau$  to the intersection  $L \cap {}^x L$  have an irreducible component in common.

Thus for some  $x \in G - L$ , the subgroup  $x \ker(\tau) x^{-1} \cap L$  of  $L$  has a fixed vector in  $W$  under  $\tau$ . Since  $\ker(\tau)$  is normal in  $L$ , the product  $\ker(\tau)(x \ker(\tau) x^{-1} \cap L) = \ker(\tau) x \ker(\tau) x^{-1} \cap L$  is a subgroup, which again has a fixed vector under  $\tau$ .

More generally, let  $M \subset \ker(\tau)$ . If  $\tau$  does not induce irreducibly to  $G$ , then some  $x \in G - L$  intertwines  $\tau$ , and we know that (the subgroup of  $L$  generated by)  $MxMx^{-1} \cap L$  has a fixed vector under  $\tau$ . If this (weaker) condition holds, we say that  $x$  *intertwines*  $\tau$  at  $M$ .

We assign a partial ordering to the elements of  $G$  by saying that  $x_1 < x_2$  when  $L \cap Mx_1Mx_1^{-1} \subset L \cap Mx_2Mx_2^{-1}$ . We may assume that such elements  $x$  belong to a chosen set of double coset representatives of  $L$  in  $G$ . Further, if  $x_2$  intertwines  $\tau$  at  $M$  and  $x_1 < x_2$  then  $x_1$  will also intertwine  $\tau$  at  $M$ .

Given such a group  $M$  and a set of double coset representatives of  $L$  in  $G$  we

need to know which representatives are minimal with respect to this ordering. If a representation  $\tau$  of  $L$  does not induce irreducibly to  $G$ , then some minimal  $x$  intertwines  $\tau$  at  $M$ . We will prove that certain representations  $\tau$  induce irreducibly to  $G$  by showing that no minimal representative  $x$  intertwines  $\tau$  at  $M$ .

For any admissible representation  $(\pi, V)$  of  $G$  and any open subgroup  $L$  we let  $V(L)$  denote the (necessarily) finite dimensional subspace of  $V$  fixed by  $L$ .

We now return to the case  $G = GSp_4(F)$

LEMMA 2.2. *Using the double coset representatives given in Lemma 2.1 we obtain the following maximal sets of minimal (with respect to " $<$ ") elements.*

- (a)  $\{t_0, t_1\}$  for  $L = K$  and  $M = K_n$ .
- (b)  $\{t_1, t_2, t_1 t_2, w_{\alpha+2\beta}\}$  for  $L = J$  and  $M = J_{n,j}$ .
- (c)  $\{t_0, t_1, w_{\alpha+\beta}\}$  for  $L = H$  and  $M = H_{n,j}$ .

LEMMA 2.3. *The following are abelian quotients:*

- (a)  $K_m/K_n$  for  $2m \geq n$ .
- (b)  $J_{m,i}/J_{n-1,0}$  for  $2m + i \geq n - 1$ .
- (c)  $H_{m,i}/H_{n-1,0}$  for  $2m + i \geq n - 1$ .

Let  $\phi$  be an additive character of  $F$  with conductor the ring of integers  $\mathcal{O}$ .

LEMMA 2.4. *Characters of the quotients in Lemma 2.4 are given by the formula*

$$\phi(I + y) = \phi(\text{trace}(x_\phi y))$$

where  $x_\phi$  may be taken in the form

$$x_\phi = \pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

subject to the following conditions:

- (a) For  $\phi \in (K_m/K_n)^\wedge$  all entries are in  $\mathcal{O}/\mathcal{P}^{n-m}$ ,
- (b) For  $\phi \in (J_{m,i}/J_{n-1,0})^\wedge$  we have

$$b, c, d \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$r, s, a, a' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$b', c', d' \in \mathcal{P}/\mathcal{P}^{n-m}$$

(c) For  $\varphi \in (H_{m,i}/H_{n-1,0})^\wedge$  we have

$$a, c \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$b \in \mathcal{O}/\mathcal{P}^{n-m-1-i}$$

$$r, s, d, d' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$a', c' \in \mathcal{P}/\mathcal{P}^{n-m}$$

$$b' \in \mathcal{P}^2/\mathcal{P}^{n-m+1-i}$$

*Proof.* Where convenient we may always tensor the original representation of  $G$  with a character of the determinant. We may also choose  $x_\varphi$  to be any convenient coset representative modulo the annihilator of these quotient groups which is given by elements of the form

$$\begin{bmatrix} u & w & 0 & y \\ x & v & -y & 0 \\ 0 & -z & u & x \\ z & 0 & w & v \end{bmatrix}$$

This gives the general form of  $x_\varphi$ . The conditions (a), (b) and (c) follow easily. We remark that we use the results of (b) extensively in the case  $2m + i = n - 1$ , so that  $n - m - 1 = m + i$ , etc.

Let  $L \subset G$  and let  $z \in G$ . If  $\tau$  is a representation of  $L$ , we denote by  $\text{Ad } z(\tau)$  the representation of  $zLz^{-1}$  which for  $x \in L$  is given by

$$\text{Ad } z(\tau)(zxz^{-1}) = \tau(x).$$

If  $\varphi$  is a character of  $L$  represented by  $x_\varphi$ , then  $\text{Ad } z(\varphi)$  is a character of  $zLz^{-1}$  represented by  $zx_\varphi z^{-1}$ .

In what follows, we assume that  $\pi$  is an irreducible supercuspidal representation of  $GSp_4$  of level  $n$ . That is,  $n$  is the smallest integer such that  $V(K_n)$  is non-zero. We also assume (after possibly tensoring  $\pi$  with a character through the determinant) that  $n$  is the minimal level among all such twists of  $\pi$ .

### 3. The Nilpotent Case

In this section, we consider the case in which  $\pi$  restricted to  $K_{n-1}$  contains a character  $\varphi$  for which  $\pi^n x_\varphi$  is nilpotent mod  $\mathcal{P}$ . We may then assume that  $x_\varphi$  is of



the form

$$\pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where entries are taken modulo  $\mathcal{P}$ .

We seen, then, that the subgroup

$$B_n = \begin{bmatrix} n & n & n-1 & n-1 \\ n-1 & n & n-1 & n-1 \\ n & n & n & n-1 \\ n & n & n & n \end{bmatrix}$$

which is normal in the Iwahori subgroup  $B$  has a non-zero fixed vector in  $V$ . Therefore  $(\pi, V)$ , when restricted to  $B$  has a component  $(\tau, Y)$  whose kernel contains  $B_n$ . Recall that for the rest of the paper, we assume that  $\pi$  is an irreducible supercuspidal representation of  $GS\mathcal{P}_4(F)$  of minimal level  $n$ .

**DEFINITION 3.1.** We say that such a representation  $(\tau, Y)$  of nilpotent type is obtuse if for every character  $\varphi$  contained in the restriction of  $\tau$  to  $B_{n-1}$ , both the entries  $a$  and  $d$  in  $x_\varphi$  are units.

Let

$$z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \\ \pi & 0 & 0 & 0 \end{bmatrix}$$

**LEMMA 3.2.** *If  $(\tau, V)$  is obtuse then  $(\pi, V)$  is irreducibly induced from  $\langle B, z \rangle$ .*

*Proof.* Referring to the remarks at the end of section 2., let  $L = \langle B, z \rangle$  and  $M = B_n$ . A systematic check of double coset representatives of  $\langle B, z \rangle$  in  $G$ , (i.e. elements of the affine Weyl group  $W_a$ ), shows that if any such element  $x$  intertwines  $\tau$  at  $B_n$ , then  $\tau$  cannot be obtuse.

**LEMMA 3.3.** *If  $(\tau, V)$  is not obtuse, then  $J_{n-1,0}$  has a non-zero fixed vector in  $V$ .*

*Proof.* Let  $v \in Y$  be an eigenvector for  $\varphi$  with  $x_\varphi$  as above. If  $(\tau, Y)$  is not obtuse then there are three possibilities.

If  $a = 0$  we are done.

If  $a \neq 0$  and  $c = 0$  then  $\pi(w_\alpha)v$  is fixed by  $J_{n-1,0}$ .

If  $a \neq 0$  and  $c \neq 0$  then  $\pi(u_{-\alpha}(ac^{-1}))v$  is fixed by  $J_{n-1,0}$ .

We may now assume that  $V(J_{n-1,0})$  is a non-zero subspace.

**DEFINITION 3.4.** Let  $L_{j,k}^{n-1} = (S_{\alpha+\beta} \cap J_{j,k})(S_{\beta} \cap J_{j,k})J_{n-1,0}$  where for any root  $\gamma$ ,  $S_{\gamma}$  denotes the corresponding imbedding of  $SL_2(F)$  in  $G$ . Let  $\mathcal{L}_{n-1}(J)$  denote the set of those groups  $L_{j,k}^{n-1}$  which are normal in  $J$ .

**LEMMA 3.5.**  $L_{j,k}^{n-1} \in \mathcal{L}_{n-1}(J)$  if and only if  $2j+k \geq n-1$  and  $j \geq n-1 - \text{val}(2)$ .

**LEMMA 3.6.** Let  $\tau$  be an irreducible representation of  $J$  occurring in the restriction of  $\pi$ . Then one of the following statements is true.

- (a)  $\pi = \text{Ind}_J^G \tau$ ,
- (b)  $V(H_{n-1,0}) \neq \{0\}$ ,
- (c)  $V(K_{n-1}) \neq \{0\}$ , or
- (d) The restriction of  $\pi$  to  $J_{m,i}$  contains a character  $\varphi$  for which we may assume that

$$x_{\varphi} = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \\ \pi & 0 & r & 0 \\ 0 & b' & 0 & s \end{bmatrix}$$

*Proof.* Let  $L_{j,k}^{n-1}$  be the largest member of  $\mathcal{L}_{n-1}(J)$  having a fixed vector. If  $2j+k = n-1$  then (d) holds. We may therefore assume that  $2j+k > n-1$  and that (a) does not hold. This means that some element  $x = w_{\alpha+2\beta}^{\sigma} t_1^{v_1} t_2^{v_2}$  intertwines  $\tau$  at  $L_{j,k}^{n-1}$  where  $\sigma \in \{0, 1\}$  and  $v_1, v_2 \geq 0$ . If neither (b) nor (c) holds, one can show that  $v_1 = v_2$  so that  $x = w_{\alpha+2\beta}^{\sigma} t_0^{v_0}$  with  $v_0 \geq 0$ .

Assume first that  $k = 0$ . Since  $x$  intertwines  $\tau$  at  $L_{j,0}^{n-1}$ , it can be shown that  $L_{j-1,1}^{n-1}$  has a fixed vector. Since  $L_{j,0}^{n-1}$  was the largest element of  $\mathcal{L}_{n-1}(J)$  having a fixed vector, it follows that  $j = n-1 - \text{val}(2)$ . We know that

$$x_{\varphi} = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & c \\ 0 & 0 & 0 & b \\ \pi & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

where  $r, s, a, a', c \in \mathcal{O}^{n-j}$ ,  $c' \in \mathcal{O}^{n-j+1}$ , and  $b, b' \in \mathcal{O}$ . A Hensel's Lemma type argument shows that conjugation by elements of the form  $u_{\beta}(x)$ ,  $u_{\alpha+\beta}(x)$ ,  $u_{-\beta}(x)$ , and  $u_{-(\alpha+\beta)}(x)$  in  $J$  will produce a matrix satisfying condition (d).

Now assume that  $k = 1$ . We know from Lemma 3.5 that  $\text{val}(2) \geq j - n + 1$ . If  $\text{val}(r \pm s) < j - n - 1$ , then the fact that  $x$  intertwines  $\tau$  at  $L_{j,0}^{n-1}$  would produce a vector fixed by the large group  $L_{j,0}^{n-1}$ . Therefore  $\text{val}(r \pm s) = j - n + 1$ . As in the previous case, this fact allows us to produce a character satisfying condition (d).

**LEMMA 3.7.** Assume that  $V(H_{n-1,0}) \neq \{0\}$ . Then either  $(\pi, V)$  is irreducibly

induced from  $H$  or  $(\pi, V)$  contains a  $K_{n-1}$ -fixed vector or  $(\pi, V)$  contains a principal character.

*Proof.* Assume that  $\pi$  is not irreducibly induced from  $H$ . From Lemma 2.2 and remarks preceding it, we see that  $\pi|_H$  must have a component  $\tau$  which is intertwined at  $H_{n-1,0}$  by one of  $t_0, t_1,$  or  $w_{\alpha+\beta}$ .

It is easy to see that if  $t_0$  intertwines  $\tau$  at  $H_{n-1,0}$ , then the intersection  $H_{n-1,0}t_0H_{n-1,0}t_0^{-1} \cap H$  is contained in  $\ker \tau$ . Then, conjugating by  $t_0^{-1}$ , we see that  $V(K_{n-1}) \neq \{0\}$ . This contradicts the assumption that  $\pi$  has minimal level  $n$ .

Suppose next that  $t_1$  or  $w_{\alpha+\beta}$  intertwines  $\tau$  at  $H_{n-1,0}$ . Then, in either case, the restriction of  $\tau$  to a suitable subgroup (see Lemma 2.3(c)) will contain a character  $\varphi$  for which

$$x_\varphi = \pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

with

$$a \in \mathcal{O} \quad r, s, b, c, a', c', d' \in \mathcal{P} \quad b', d \in \mathcal{P}^2$$

We may assume that  $\text{val}(a) = 0$  and that  $\text{val}(a') = 1$  or else we could easily produce a  $K_{n-1}$ -fixed vector. Applying Lemma 1.1, let  $H_{n-1,0}$  play the role of  $L$ . Let  $v_0 \in V(L)$  be an eigenvector of  $\varphi$ . Let  $v'_0 = \pi(t_0)v_0$ . There exist upper unipotent elements  $u_j$  and constants  $c_{1,j}$  such that  $v''_0 = \Sigma c_{1,j}\pi(u_j)v'_0$  is an eigenvector whose character is parametrized by a matrix in the form above with

$$d \in \mathcal{P} \quad b, c \in \mathcal{O} \quad c', d' \in \mathcal{P}^2 \quad b' \in \mathcal{P}^3$$

Then one can find  $u \in u_\beta(\mathcal{O})u_{\alpha+2\beta}(\mathcal{P}^{-1})u_{\alpha+\beta}(\mathcal{O})$  such that  $v_1 = \pi(u)v''_0$  is an eigenvector with a character satisfying the same conditions as the original  $\varphi$ . In the terms of Lemma 1.1, let  $x_{i,j} = uu_jt_0$  for  $i = 2, 3, \dots$ . Thus  $\varphi$  is a principal character.

It therefore remains to analyze the case (d) of Lemma 3.6. Let  $S_j = S_{\alpha+2\beta} \cap K_j$ . Let  $\tilde{S}_j = S_j J_{m,i}$ . When the quotient  $S_h/S_j$  is abelian, we may parametrize its characters by matrices of the form

$$\pi^{-j} \begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

with  $b, b'$ , and  $s \in \mathcal{O}$ . (When  $S_h \subseteq J_{m,i}$ , this is consistent with our parametrization of  $(J_{m,i})^\wedge$ ).

Recall that  $\phi$  is a fixed additive character of  $F$  with conductor the ring of

integers  $\mathcal{O}$ . Since the residual characteristic of  $F$  is 2, we may choose  $\rho \in \mathcal{O}^\times$  with the property that for  $x \in \mathcal{O}$ , we have  $\phi(\pi^{-1}x^2) = \phi(\pi^{-1}\rho x)$ .

LEMMA 3.8. *Let  $k = \min\{\text{val}(2), \text{val}(r \pm s)\}$ . Let  $j$  be the smallest integer such that  $\pi$  restricted to  $S_j$  contains a character  $\phi$  which is trivial on  $S_j$ . There are various cases that can occur.*

(a)  $j \geq 2k + 1$ :  $\phi$  may be extended to  $S_{j-2}/S_j$ . This situation may always be transformed into one of the following conditions.

- (i) If  $s \in \mathcal{O}^\times$  and  $b, b' \in \mathcal{P}$ , then  $\phi$  is principal.
- (ii) If  $s \in \mathcal{P}$ ,  $b \in \mathcal{O}^\times$ , and  $b' \in \mathcal{P} - \mathcal{P}^2$ , then  $\pi$  is irreducibly induced from  $J$ .
- (iii) If

$$\begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

is irreducible mod  $\mathcal{P}$ ,  $\pi$  is irreducibly induced from  $J$ .

(b)  $j = 2k$  and  $k = \text{val}(r + s) = \text{val}(r - s)$ : In this case, let

$$\tilde{s} = s + \frac{\rho^2 \pi^{2k}}{(r + s)^2}$$

If

$$\begin{pmatrix} 0 & b \\ b' & \tilde{s} \end{pmatrix}$$

is irreducible mod  $\mathcal{P}$ , then  $\pi$  is irreducibly induced from  $J$ . Otherwise  $\phi$  is principal and  $\pi$  is non-supercuspidal.

(c)  $j < 2k$  or  $j = 2k = \text{val}(2) < \text{val}(r \pm s)$ : Here  $(\pi, V)$  is never supercuspidal.

*Proof.* In this proof we assume for convenience that  $n - 1 = 2m$  and that  $i = 0$ . The other case is handled in a nearly identical fashion.

Since the illustration of principal characters in case (i) of (a) uses essentially the same arguments as in Section 4. of [KM], the proof is left to the reader.

Consider condition (ii) of case (a). Assume that some  $x \in G$  intertwines  $\text{Ind}_{S_{j-2}}^G \phi$ . We need to show that  $x \in J$ . Let  $t = t_1 t_2$  so that mod  $Z(G)$  we may take  $t$  to be the translation

$$t_{\alpha+2\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let  $\mathcal{D} = \{w_{\alpha+2\beta}t^v | v \geq 0\} \cup \{t^v | v > 0\}$ . It is enough to consider elements  $g \in J\mathcal{D}J$  since intertwining by elements of other double cosets of  $J$  would produce vectors fixed by subgroups already considered. We can write

$$J\mathcal{D}J = J_0\{I, z_J\}\mathcal{D}\{I, z_J\}J_0$$

where  $J_0 = J_{0,0}$ . The arguments for each piece of this double coset are essentially the same. We give the proof for  $g \in J_0t^vJ_0$ .

Let  $g = j_1t^vj_2 = n_1jt^vn_2$  where

$$j \in J_0 \cap t^vJ_0t^{-v}$$

$$n_1 \in J_0/(J_0 \cap t^vJ_0t^{-v})$$

$$n_2 \in (J_0 \cap t^{-v}J_0t^v) \setminus J_0.$$

This means that

$$n_1 \in [w_\beta u_{-\beta}(\mathcal{P}/\mathcal{P}^v) \cup u_{-\beta}(\mathcal{O}/\mathcal{P}^v)]u_{-(\alpha+2\beta)}(\mathcal{P}/\mathcal{P}^{1+2v})u_{-(\alpha+\beta)}(\mathcal{P}/\mathcal{P}^{1+v})$$

and

$$n_2 \in u_{\alpha+2\beta}(\mathcal{P}/\mathcal{P}^{2v})u_{\alpha+\beta}(\mathcal{P}/\mathcal{P}^v)[u_\beta(\mathcal{P}/\mathcal{P}^v)w_\beta \cup u_\beta(\mathcal{O}/\mathcal{P}^v)]$$

We claim that neither  $n_1$  nor  $n_2$  may have  $w_\beta$  as a factor. In general, if  $g$  intertwines  $\varphi$ , then  $g\varphi = \varphi$  on  $x\tilde{S}_{j-2}x^{-1} \cap \tilde{S}_{j-2}$ . Therefore  $t^vn_2\varphi = j^{-1}n_1^{-1}\varphi$  on  $J_{m,0} \cap t^vJ_{m,0}t^{-v}$ . If  $n_2$  has  $w_\beta$  as a factor, it is easy to see that  $t^vn_2\varphi$  has level  $n + 2v$  on  $u_{-(\alpha+2\beta)}$ . This is a contradiction since  $j^{-1}n_1^{-1}\varphi$  is trivial on  $J_{n-1,0}$ . Similar reasoning shows that  $n_1$  cannot have  $w_\beta$  as a factor either.

Since  $j$  may be factored, we may write  $x = n^-t^vn^+$  where  $n^+$  and  $n^-$  are respectively in the intersections of  $J_0$  with the positive and negative root Borel subgroups. Therefore, on  $J_{m,0} \cap t^vJ_{m,0}t^{-v}$  we have

$$t^vn^+\varphi = n^-\varphi$$

The left hand side of this equation is trivial on the  $\beta$  and  $\alpha + \beta$  root groups of  $J_{m,0}$ . We now write

$$n^-\varphi = u_{-\beta}(x)u_{-(\alpha+\beta)}(y)\varphi_1$$

where

$$\varphi_1 = u_{-\alpha}(p)u_{-(\alpha+2\beta)}(q)\varphi$$

One can verify that  $\varphi_1$  is also trivial on the  $\beta$  and  $\alpha + \beta$  root groups of  $J_{m,0}$ . It follows that  $x$  and  $y$  are in  $\mathcal{P}^{m-k+1}$ .

The next thing to see is that  $n^- \varphi_1$  still has precisely level  $j - 1$  on  $u_{\alpha+2\beta}$ . This is true because  $j \geq 2k + 1$  forces  $[u_{\alpha+2\beta}(z), u_{-\beta}(x)] \in \ker \varphi_1$  whenever  $z \in \mathcal{P}^{j-2}$  and  $x \in \mathcal{P}^{m-k+1}$ . A similar argument applies to the effect of  $u_{-(\alpha+\beta)}(y)$  on  $\varphi_1$ .

We conclude that  $n^- \varphi$  produces a ramified cuspidal character of  $S_{j-2}$  of the same level as that produced by  $\varphi$ . A similar check shows the same to be true for  $n^+ \varphi$ . It is evident that  $t^v$  cannot send one to the other. Similar arguments for the other kinds of double cosets of  $J$  show that the only elements that can intertwine  $\varphi$  are in  $J$  itself. Thus  $(\pi, V)$  is irreducibly induced from  $J$ . The proof for case (iii) of (a) is quite similar.

We now consider case (b). We may extend  $\varphi$  to  $\tilde{S}_{2k-2} u_{-(\alpha+\beta)}(\mathcal{P}^m)$  so that  $\varphi$  is trivial on  $u_{-(\alpha+\beta)}(\mathcal{P}^m)$ . This is a key point in the entire problem since this extension exists precisely because  $F$  has residual characteristic 2.

For  $p \in \mathcal{P}^m$  and  $q \in \mathcal{P}^{2k-1}$  we may write

$$\varphi(u_{-(\alpha+\beta)}(p)u_{-(\alpha+2\beta)}(q)) = \phi(\pi^{-n}ap + \pi^{-2k}bq)$$

where  $a$  and  $b$  are integers.

Conjugation of  $\varphi$  by  $u_{\alpha+\beta}(y)u_{\alpha+2\beta}(z)$  changes  $\varphi$  only on  $u_{-(\alpha+\beta)}$  and  $u_{-(\alpha+2\beta)}$ . This action changes the coefficient of  $p$  in the formula above by  $\pi^{-n}(r+s)y + \pi^{-(m+1)}\rho z^{1/2}$ . The  $q$ -coefficient is changed by the amount  $\pi^{-n+1}y^2 + \pi^{-2k}(b'z^2 + sz)$ .

$\varphi$  is a principal character when there exist elements  $y$  and  $z$  to make the  $q$ -coefficient zero without modifying the  $p$ -coefficient. This requires that

$$y^2 = \frac{\pi^{2m}\rho^2 z}{(r+s)^2}.$$

Thus the modification to the  $q$ -coefficient is

$$\pi^{-2k} \left[ \left( s + \frac{\pi^{2k}\rho^2}{(r+s)^2} \right) z + b'z^2 \right].$$

Thus the new  $q$ -coefficient will be

$$\pi^{-2k}(b + \tilde{s}z + b'z^2).$$

This leads immediately to the condition of case (b). When that condition fails, an argument similar to the one given in case (a) shows that  $(\pi, V)$  is irreducibly induced from  $J$ .

We now consider case (c). Let

$$\gamma = \begin{cases} \beta & \text{for } n - 1 \text{ odd;} \\ \alpha + \beta & \text{for } n - 1 \text{ even.} \end{cases}$$

Then  $\varphi$  may be extended to  $\tilde{S}_j u_{-\gamma}(\mathcal{P}^m)$  as before. As in the above cases, we give the proof for when  $n - 1$  is even and  $\gamma = \alpha + \beta$ .

We may assume that the extended  $\varphi$  is trivial on  $u_{-(\alpha+\beta)}(\mathcal{P}^m)$ . Note that conjugating  $u_{-(a+b)}(y)$  by  $u_{\alpha+2\beta}(z)$  gives

$$u_{-(\alpha+\beta)}(y)u_{\beta}(zy)u_{-\alpha}(zy^2).$$

When  $y \in \mathcal{P}^m$  and  $z \in \mathcal{O}$ , we have  $\varphi(u_{\beta}(zy)) = 1$ . Also (since  $F$  has residual characteristic 2) we find that the map  $y \mapsto \varphi(u_{-\alpha}(zy^2))$  is an additive character of  $\mathcal{P}^m/\mathcal{P}^{m+1}$ . It follows that  $z \in \mathcal{O}$  may be chosen so that  $\varphi_1 = \text{Ad}(u_{\alpha+2\beta}(z))(\varphi)$  is a character extending the original  $\varphi$  which is trivial on  $u_{-(\alpha+\beta)}(\mathcal{P}^m)$ .

Let

$$\tilde{t} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

We will show that  $\varphi_1$  is a principal character. Let  $\varphi_2$  be any extension to  $\tilde{S}_j u_{-(\alpha+\beta)}(\mathcal{P}^m)$  of  $\text{Ad}(\tilde{t})(\varphi_1)$ . Let  $P$  be the positive root Borel subgroup of  $S_{\alpha+2\beta}$  and set  $P_j = P \cap K_j$ . Then  $\varphi_2$  is trivial on  $\tilde{S}_{2k} P_j$ . We may choose  $y \in \mathcal{P}^{m-k}$  such that  $\text{Ad}(u_{\alpha+\beta}(y))(\varphi_2)$  is a character of  $\tilde{S}_j$  satisfying the initial conditions of the lemma. This demonstrates that  $\varphi$  is a principal character.

**THEOREM 3.9.** *All supercuspidal representations occurring in the nilpotent case are irreducibly induced from some open compact-by-center subgroup of  $G$ .*

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