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# Supercuspidal representations of $GSp_4$ over a local field of residual characteristic 2, part I

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## 1. Introduction

In this paper, and in one to follow, we analyze supercuspidal representations of $G = GSp_4(F)$ where $F$ is a local field of residual characteristic $p = 2$. We first classify these representations according to characters which occur in their restrictions to certain open compact subgroups. As in the case of $Gl_n(F)$, these characters may be parametrized by $4 \times 4$ matrices over $F$. Accordingly, we may classify certain supercuspidal representations as occurring in the "nilpotent" case. Theorem 3.9 states that these representations are all irreducibly induced from some open compact-by-center subgroup of $G$. The remaining cases will be treated in the sequel.

The methods used are suggested by the construction techniques for $Gl_n(F)$ developed by Howe ([H]) and exhaustion techniques developed by Kutzko and Manderscheid ([K] and [KM]). These ideas work well when no wildly ramified extensions of $F$ are imbedded in the group in question. Moy ([M]) has used techniques involving Hecke algebra isomorphisms to classify supercuspidal representations of $GSp_4(F)$ in the case of odd primes $p$.

When wild ramification is present (so $p = 2$), these methods still provide a starting point. A representation $(\pi, V)$ will fix a non-trivial subspace $V(L)$ when restricted to a small enough open subgroup $L$. We will generally take $L$ to be a member of some normal filtration of a maximal compact-by-center subgroup of $G$. We then consider certain groups $A$ for which the quotient $A/L$ is abelian. We classify $\pi$ by characters occurring in the restriction $(\pi|_A, V(L))$. The occurrence of certain characters (cuspidal) immediately forces $\pi$ to be irreducibly induced. Some characters are shown to be $(G, A)$-principal as defined in [K]. Some cases lead us to find an open subgroup $L'$, larger than $L$, having a fixed vector. The remaining cases are analogous to the alfalfa cases of [KM]. For these we must use the constructions in [H] to distinguish between representations which are irreducibly induced and those that contain principal characters.

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Recall that the matrix coefficients of a supercuspidal representation have compact support mod the center of $G$. A $(G, A)$-principal representation of a subgroup $A$ is one which cannot occur in the restriction to $A$ of a supercuspidal representation of $G$. A representation of an open compact subgroup of $G$ may be shown to be principal by proving that vectors in its representation space give rise to matrix coefficients of $G$ whose support is not compact mod the center of $G$. The following is a restatement of some of the lemmas in [K] and gives the method we will use to demonstrate that certain representations are principal representations.

LEMMA 1.1. *Let $(\pi, V)$ be an admissible representation of $G$. Let $v_0 \in V$* and let $L$ *be an open compact subgroup fixing $v_0$. Suppose there exist elements $x_{ij} \in G$ and constants $N(i)$ and $c_{ij}$, for $i = 1, 2, 3 \ldots$ and $1 \leqslant j \leqslant N(i)$, such that*

(a) *each $v_i \in V(L)$, where*

$$v_i = \sum_{j=1}^{N(i)} c_{ij} \pi(x_{ij}) v_{i-1}$$

(b) *No subsequence of elements of the form*

$$s_n = \prod_{i=1}^{n} x_{i,j}$$

*with $j = j(i, n)$, $1 \leqslant j \leqslant N(i)$ for all $i$, and $1 \leqslant i \leqslant n$, for $n = 1, 2, 3, \ldots$, is contained in a compact-by-center subset of $G$.*

*Then $v_0$ produces a matrix coefficient of $(\pi, V)$ whose support is not compact-by-center.*

*Proof.* Since $\pi$ is admissible, the space $V(L)$ of $L$-fixed vectors in $V$ is finite-dimensional. Thus there exists a linear functional $v_0'$ on $V(L)$ such that $\langle v_n, v_0' \rangle \neq 0$ for infinitely many $n$. Extend $v_0'$ to be zero on the complement of $V(L)$ to define a smooth vector, also denoted $v_0'$, in the contragredient of $V$.

Consider the matrix coefficient $g \mapsto \langle \pi(g) v_0, v_0' \rangle$. For $v_n$ in the subsequence above,

$$0 \neq \langle v_n, v_0' \rangle = \sum_{j=1}^{N(n)} c_{nj} \langle \pi(x_{nj}) v_{n-1}, v_0' \rangle$$

which inductively is a linear combination of terms of the form

$$\left\langle \pi \left( \prod_{i=1}^{n} x_{ij} \right) v_0, v_0' \right\rangle$$

for $j$ with $1 \leqslant j \leqslant N(i)$.

Thus at least one such term must be non-zero. Let $s_n = \Pi_{i=1}^{n} x_{ij}$ be such that $\langle \pi(s_n)v_0, v_0' \rangle \neq 0$ for each such $n$.

The support of the above matrix coefficient of $\pi$ thus contains the set of such $s_n$, which is not compact-mod-center by hypothesis. Hence a representation $(\pi, V)$ with these properties cannot be supercuspidal.

A representation of $A$ with such a $v_0$ in its space is thus a $(G, A)$-principal representation. In our applications of Lemma 1.1, the elements $x_{ij}$ are generally products of some fixed translation in the affine Weyl group with appropriate unipotent elements. It will then be immediate to check that condition (b) of the Lemma holds.

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## 2. Structure and Notation

In this paper we will realize the group $GSp_4(F)$ of symplectic similitudes as those elements $g$ of $GL_4(F)$ satisfying the relation $gJ^tg = cJ$, where $c$ is a non-zero element of $F$ and

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The group $GSp_4(F)$ over a non-archimedean local field $F$ has three conjugacy classes of maximal compact-mod-center subgroups. See, e.g., [I]. We will define a filtration of each of the maximal compact-mod-center subgroups of $G$ consisting of compact open normal subgroups. First, for $n > 0$, let $K_n$ denote the $n$th congruence subgroup of the open maximal compact subgroup $K_0 = GSp_4(\mathcal{O})$. We will use the following notation to describe $K_n$.

$$K_n = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n & n & n & n \\ n & n & n & n \end{bmatrix},$$

where the matrix entries denote the minimum valuation of entries of the difference between an element of $K_n$ and the identity matrix.

We define the following groups in the same manner:

$$H_{n,0} = \begin{bmatrix} n & n+1 & n & n \\ n & n & n & n-1 \\ n & n+1 & n & n \\ n+1 & n+1 & n+1 & n \end{bmatrix}$$

$$H_{n,1} = \begin{bmatrix} n+1 & n+1 & n+1 & n \\ n & n+1 & n & n \\ n+ & \mathrm{n}+1 & n+1 & n \\ n+1 & n+2 & n+1 & n+1 \end{bmatrix}$$

$$J_{n,0} = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \end{bmatrix}$$

$$J_{n,1} = \begin{bmatrix} n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n+1 & n+1 \\ n+1 & n+1 & n+1 & n+1 \end{bmatrix}.$$

Let $K = K_0Z(G)$. Let $H$ be the group generated by $H_{0,0}$ and $z_H$, and let $J$ be the group generated by $J_{0,0}$ and $z_J$. Here

$$z_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix} \quad \text{and} \quad z_J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix}.$$

Then the groups $K$, $J$ and $H$ represent the three conjugacy classes of maximal compact-by-center open subgroups of $G$.

Let $W$ be the Weyl group of $G$ and $W_a$ the affine Weyl group. An element of $W_a$ may be expressed in the form $wt_1^r t_2^s$ where $w \in W$, $r$ and $s$ are integers, and

$$t_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}, \quad t_2 = \begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Set $t_0 = t_1 t_2$ as members of $W_a$. Then (modulo $Z(G)$)

$$t_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let $\{\alpha, \beta\}$ be the base for the root system $\Phi$ of $G$ with $\beta = e_2 - e_1$ taken to be the short root and $\alpha = 2e_1$ the long root. For any root $\gamma \in \Phi$, let $w_\gamma$ denote the reflection determined by $\gamma$, and let $t_\gamma$ be the translation in the affine Weyl group associated to $\gamma$. Recall that $t_\gamma$ acts on the standard apartment of the Bruhat-Tits building by sending a point $x$ to $x + 2\gamma^\vee$.

Note that $t_0 = t_{\alpha+2\beta}$ is the translation associated to $\alpha + 2\beta$, and further, that $(t_1)^2 = t_\alpha t_{\alpha+2\beta} = t_{\alpha+\beta}$ and $(t_2)^2 = t_\beta$. Then we have the following double coset decompositions of $GSp_4(F)$.

LEMMA 2.1.

$$K\backslash G/K = \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geqslant 0\}$$

$$J\backslash G/J = \{t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geqslant 0\} \cup \{w_{\alpha+2\beta} t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geqslant 0\}$$

$$H\backslash G/H = \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geqslant 0\} \cup \{w_{\alpha+\beta} t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geqslant 0\}.$$

Let $(\pi, V)$ be an irreducible admissible representation of a reductive p-adic group $G$. Let $L$ be any open compact-by-center subgroup of $G$. Let $(\tau, W)$ be an irreducible component of $(\pi, V)$ restricted to $L$. Suppose that $\tau$ does not induce irreducibly from $L$ to $G$. Then, by Mackey theory, there exists an $x$ in $G$, but not in $L$, which intertwines $\tau$, i.e., there is an $x$ outside of $L$ for which the restrictions of $\tau$ and its conjugate $\mathrm{Ad}(x)\tau$ to the intersection $L \cap {}^xL$ have an irreducible component in common.

Thus for some $x \in G - L$, the subgroup $x\ker(\tau)x^{-1} \cap L$ of $L$ has a fixed vector in $W$ under $\tau$. Since $\ker(\tau)$ is normal in $L$, the product $\ker(\tau)(x\ker(\tau)x^{-1} \cap L) = \ker(\tau)x\ker(\tau)x^{-1} \cap L$ is a subgroup, which again has a fixed vector under $\tau$.

More generally, let $M \subset \ker(\tau)$. If $\tau$ does not induce irreducibly to $G$, then some $x \in G - L$ intertwines $\tau$, and we know that (the subgroup of $L$ generated by) $MxMx^{-1} \cap L$ has a fixed vector under $\tau$. If this (weaker) condition holds, we say that $x$ *intertwines* $\tau$ *at* $M$.

We assign a partial ordering to the elements of $G$ by saying that $x_1 \prec x_2$ when $L \cap Mx_1Mx_1^{-1} \subset L \cap Mx_2Mx_2^{-1}$. We may assume that such elements $x$ belong to a chosen set of double coset representatives of $L$ in $G$. Further, if $x_2$ intertwines $\tau$ at $M$ and $x_1 \prec x_2$ then $x_1$ will also intertwine $\tau$ at $M$.

Given such a group $M$ and a set of double coset representatives of $L$ in $G$ we need to know which representatives are minimal with respect to this ordering. If a representation $\tau$ of $L$ does not induce irreducibly to $G$, then some minimal $x$ intertwines $\tau$ at $M$. We will prove that certain representations $\tau$ induce irreducibly to $G$ by showing that no minimal representative $x$ intertwines $\tau$ at $M$.

For any admissible representation $(\pi, V)$ of $G$ and any open subgroup $L$ we let $V(L)$ denote the (necessarily) finite dimensional subspace of $V$ fixed by $L$.

We now return to the case $G = GSp_4(F)$

LEMMA 2.2. *Using the double coset representatives given in Lemma* 2.1 *we obtain the following maximal sets of minimal* (*with respect to* "$\prec$") *elements.*

(a) $\{t_0, t_1\}$ *for* $L = K$ *and* $M = K_n$.

(b) $\{t_1, t_2, t_1t_2, w_{\alpha+2\beta}\}$ *for* $L = J$ *and* $M = J_{n,j}$.

(c) $\{t_0, t_1, w_{\alpha+\beta}\}$ *for* $L = H$ *and* $M = H_{n,j}$.

LEMMA 2.3. *The following are abelian quotients:*

(a) $K_m/K_n$ *for* $2m \geqslant n$.

(b) $J_{m,i}/J_{n-1,0}$ *for* $2m + i \geqslant n - 1$.

(c) $H_{m,i}/H_{n-1,0}$ *for* $2m + i \geqslant n - 1$.

Let $\phi$ be an additive character of $F$ with conductor the ring of integers $\mathcal{O}$.

LEMMA 2.4. *Characters of the quotients in Lemma* 2.4 *are given by the formula*

$$\varphi(I + y) = \phi(trace(x_\varphi y))$$

*where $x_\varphi$ may be taken in the form*

$$x_\varphi = \pi^{-n}\begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

*subject to the following conditions:*

(a) *For* $\varphi \in (K_m/K_n)^\wedge$ *all entries are in* $\mathcal{O}/\mathcal{P}^{n-m}$,

(b) *For* $\varphi \in (J_{m,i}/J_{n-1,0})^\wedge$ *we have*

$$b, c, d \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$r, s, a, a' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$b', c', d' \in \mathcal{P}/\mathcal{P}^{n-m}$$

(c) *For* $\varphi \in (H_{m,i}/H_{n-1,0})^{\wedge}$ *we have*

$$a,\ c \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$b \in \mathcal{O}/\mathcal{P}^{n-m-1-i}$$

$$r,\ s,\ d,\ d' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$a',\ c' \in \mathcal{P}/\mathcal{P}^{n-m}$$

$$b' \in \mathcal{P}^2/\mathcal{P}^{n-m+1-i}$$

*Proof.* Where convenient we may always tensor the original representation of $G$ with a character of the determinant. We may also choose $x_\varphi$ to be any convenient coset representative modulo the annihilator of these quotient groups which is given by elements of the form

$$\begin{bmatrix} u & w & 0 & y \\ x & v & -y & 0 \\ 0 & -z & u & x \\ z & 0 & w & v \end{bmatrix}$$

This gives the general form of $x_\varphi$. The conditions (a), (b) and (c) follow easily. We remark that we use the results of (b) extensively in the case $2m + i = n - 1$, so that $n - m - 1 = m + i$, etc.

Let $L \subset G$ and let $z \in G$. If $\tau$ is a representation of $L$, we denote by $\operatorname{Ad} z(\tau)$ the representation of $zLz^{-1}$ which for $x \in L$ is given by

$$\operatorname{Ad} z(\tau)(zxz^{-1}) = \tau(x).$$

If $\varphi$ is a character of $L$ represented by $x_\varphi$, then $\operatorname{Ad} z(\varphi)$ is a character of $zLz^{-1}$ represented by $zx_\varphi z^{-1}$.

In what follows, we assume that $\pi$ is an irreducible supercuspidal representation of $GSp_4$ of level $n$. That is, $n$ is the smallest integer such that $V(K_n)$ is nonzero. We also assume (after possibly tensoring $\pi$ with a character through the determinant) that $n$ is the minimal level among all such twists of $\pi$.

## 3. The Nilpotent Case

In this section, we consider the case in which $\pi$ restricted to $K_{n-1}$ contains a character $\varphi$ for which $\pi^n x_\varphi$ is nilpotent mod $\mathcal{P}$. We may then assume that $x_\varphi$ is of the form

$$\pi^{-n}\begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where entries are taken modulo $\mathscr{P}$.

We seen, then, that the subgroup

$$B_n = \begin{bmatrix} n & n & n-1 & n-1 \\ n-1 & n & n-1 & n-1 \\ n & n & n & n-1 \\ n & n & n & n \end{bmatrix}$$

which is normal in the Iwahori subgroup $B$ has a non-zero fixed vector in $V$. Therefore $(\pi, V)$, when restricted to $B$ has a component $(\tau, Y)$ whose kernel contains $B_n$. Recall that for the rest of the paper, we assume that $\pi$ is an irreducible supercuspidal representation of $GSp_4(F)$ of minimal level $n$.

DEFINITION 3.1. We say that such a representation $(\tau, Y)$ of nilpotent type is obtuse if for every character $\varphi$ contained in the restriction of $\tau$ to $B_{n-1}$, both the entries $a$ and $d$ in $x_\varphi$ are units.

Let

$$z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \\ \pi & 0 & 0 & 0 \end{bmatrix}$$

LEMMA 3.2. *If $(\tau, V)$ is obtuse then $(\pi, V)$ is irreducibly induced from $\langle B, z\rangle$.*

*Proof.* Referring to the remarks at the end of section 2., let $L = \langle B, z\rangle$ and $M = B_n$. A systematic check of double coset representatives of $\langle B, z\rangle$ in $G$, (i.e. elements of the affine Weyl group $W_a$), shows that if any such element $x$ intertwines $\tau$ at $B_n$, then $\tau$ cannot be obtuse.

LEMMA 3.3. *If $(\tau, V)$ is not obtuse, then $J_{n-1,0}$ has a non-zero fixed vector in $V$.*

*Proof.* Let $v \in Y$ be an eigenvector for $\varphi$ with $x_\varphi$ as above. If $(\tau, Y)$ is not obtuse then there are three possibilities.

If $a = 0$ we are done.

If $a \neq 0$ and $c = 0$ then $\pi(w_\alpha)v$ is fixed by $J_{n-1,0}$.

If $a \neq 0$ and $c \neq 0$ then $\pi(u_{-\alpha}(ac^{-1}))v$ is fixed by $J_{n-1,0}$.

We may now assume that $V(J_{n-1,0})$ is a non-zero subspace.

DEFINITION 3.4. *Let* $L_{j,k}^{n-1} = (S_{\alpha+\beta} \cap J_{j,k})(S_\beta \cap J_{j,k})J_{n-1,0}$ where for any root $\gamma$, $S_\gamma$ denotes the corresponding imbedding of $\mathrm{SL}_2(F)$ in $G$. Let $\mathscr{L}_{n-1}(J)$ denote the set of those groups $L_{j,k}^{n-1}$ which are normal in $J$.

LEMMA 3.5. $L_{j,k}^{n-1} \in \mathscr{L}_{n-1}(J)$ *if and only if* $2j+k \geqslant n-1$ *and* $j \geqslant n-1-\mathrm{val}(2)$.

LEMMA 3.6. *Let $\tau$ be an irreducible representation of $J$ occurring in the restriction of $\pi$. Then one of the following statements is true.*

(a) $\pi = \mathrm{Ind}_J^G \tau$,

(b) $V(H_{n-1,0}) \neq \{0\}$,

(c) $V(K_{n-1}) \neq \{0\}$, *or*

(d) *The restriction of $\pi$ to $J_{m,i}$ contains a character $\varphi$ for which we may assume that*

$$x_\varphi = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \\ \pi & 0 & r & 0 \\ 0 & b' & 0 & s \end{bmatrix}$$

*Proof.* Let $L_{j,k}^{n-1}$ be the largest member of $\mathscr{L}_{n-1}(J)$ having a fixed vector. If $2j+k = n-1$ then (d) holds. We may therefore assume that $2j+k > n-1$ and that (a) does not hold. This means that some element $x = w_{\alpha+2\beta}^{\sigma} t_1^{v_1} t_2^{v_2}$ intertwines $\tau$ at $L_{j,k}^{n-1}$ where $\sigma \in \{0, 1\}$ and $v_1, v_2 \geqslant 0$. If neither (b) nor (c) holds, one can show that $v_1 = v_2$ so that $x = w_{\alpha+2\beta}^{\sigma} t_0^{v_0}$ with $v_0 \geqslant 0$.

Assume first that $k = 0$. Since $x$ intertwines $\tau$ at $L_{j,0}^{n-1}$, it can be shown that $L_{j-1,1}^{n-1}$ has a fixed vector. Since $L_{j,0}^{n-1}$ was the largest element of $\mathscr{L}_{n-1}(J)$ having a fixed vector, it follows that $j = n-1-\mathrm{val}(2)$. We know that

$$x_\varphi = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & c \\ 0 & 0 & 0 & b \\ \pi & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

where $r, s, a, a', c \in \mathscr{P}^{n-j}$, $c' \in \mathscr{P}^{n-j+1}$, and $b, b' \in \mathscr{P}$. A Hensel's Lemma type argument shows that conjugation by elements of the form $u_\beta(x)$, $u_{\alpha+\beta}(x)$, $u_{-\beta}(x)$, and $u_{-(\alpha+\beta)}(x)$ in $J$ will produce a matrix satisfying condition (d).

Now assume that $k = 1$. We know from Lemma 3.5 that $\mathrm{val}(2) \geqslant j-n+1$. If $\mathrm{val}(r \pm s) < j-n-1$, then the fact that $x$ intertwines $\tau$ at $L_{j,0}^{n-1}$ would produce a vector fixed by the large group $L_{j,0}^{n-1}$. Therefore $\mathrm{val}(r \pm s) = j-n+1$. As in the previous case, this fact allows us to produce a character satisfying condition (d).

LEMMA 3.7. *Assume that $V(H_{n-1,0}) \neq \{0\}$. Then either $(\pi, V)$ is irreducibly* *induced from $H$ or $(\pi, V)$ contains a $K_{n-1}$-fixed vector or $(\pi, V)$ contains a principal character.*

*Proof.* Assume that $\pi$ is not irreducibly induced from $H$. From Lemma 2.2 and remarks preceding it, we see that $\pi|_H$ must have a component $\tau$ which is intertwined at $H_{n-1,0}$ by one of $t_0$, $t_1$, or $w_{\alpha+\beta}$.

It is easy to see that if $t_0$ intertwines $\tau$ at $H_{n-1,0}$, then the intersection $H_{n-1,0}t_0H_{n-1,0}t_0^{-1} \cap H$ is contained in $\ker \tau$. Then, conjugating by $t_0^{-1}$, we see that $V(K_{n-1}) \neq \{0\}$. This contradicts the assumption that $\pi$ has minimal level $n$.

Suppose next that $t_1$ or $w_{\alpha+\beta}$ intertwines $\tau$ at $H_{n-1,0}$. Then, in either case, the restriction of $\tau$ to a suitable subgroup (see Lemma 2.3(c)) will contain a character $\varphi$ for which

$$x_\varphi = \pi^{-n}\begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

with

$$a \in \mathcal{O} \qquad r, s, b, c, a', c', d' \in \mathcal{P} \qquad b', d \in \mathcal{P}^2$$

We may assume that $\mathrm{val}(a) = 0$ and that $\mathrm{val}(a') = 1$ or else we could easily produce a $K_{n-1}$-fixed vector. Applying Lemma 1.1, let $H_{n-1,0}$ play the role of $L$. Let $v_0 \in V(L)$ be an eigenvector of $\varphi$. Let $v_0' = \pi(t_0)v_0$. There exist upper unipotent elements $u_j$ and constants $c_{1,j}$ such that $v_0'' = \Sigma c_{1,j}\pi(u_j)v_0'$ is an eigenvector whose character is parametrized by a matrix in the form above with

$$d \in \mathcal{P} \qquad b, c \in \mathcal{O} \qquad c', d' \in \mathcal{P}^2 \qquad b' \in \mathcal{P}^3$$

Then one can find $u \in u_\beta(\mathcal{O})u_{\alpha+2\beta}(\mathcal{P}^{-1})u_{\alpha+\beta}(\mathcal{O})$ such that $v_1 = \pi(u)v_0''$ is an eigenvector with a character satisfying the same conditions as the original $\varphi$. In the terms of Lemma 1.1, let $x_{i,j} = uu_jt_0$ for $i = 2, 3, \ldots$. Thus $\varphi$ is a principal character.

It therefore remains to analyze the case (d) of Lemma 3.6. Let $S_j = S_{\alpha+2\beta} \cap K_j$. Let $\tilde{S}_j = S_jJ_{m,i}$. When the quotient $S_h/S_j$ is abelian, we may parametrize its characters by matrices of the form

$$\pi^{-j}\begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

with $b$, $b'$, and $s \in \mathcal{O}$. (When $S_h \subseteq J_{m,i}$, this is consistent with our parametrization of $(J_{m,i})^\wedge$).

Recall that $\phi$ is a fixed additive character of $F$ with conductor the ring of integers $\mathcal{O}$. Since the residual characteristic of $F$ is 2, we may choose $\rho \in \mathcal{O}^\times$ with the property that for $x \in \mathcal{O}$, we have $\phi(\pi^{-1}x^2) = \phi(\pi^{-1}\rho x)$.

LEMMA 3.8. *Let $k = \min\{\mathrm{val}(2), \mathrm{val}(r \pm s)\}$. Let $j$ be the smallest integer such that $\pi$ restricted to $S_j$ contains a character $\phi$ which is trivial on $S_j$. There are various cases that can occur.*

(a) $j \geqslant 2k + 1$: *$\varphi$ may be extended to $S_{j-2}/S_j$. This situation may always be transformed into one of the following conditions.*
   (i) *If $s \in \mathcal{O}^\times$ and $b, b' \in \mathcal{P}$, then $\varphi$ is principal.*
   (ii) *If $s \in \mathcal{P}$, $b \in \mathcal{O}^\times$, and $b' \in \mathcal{P} - \mathcal{P}^2$, then $\pi$ is irreducibly induced from $J$.*
   (iii) *If*

$$\begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

*is irreducible* mod $\mathcal{P}$, *$\pi$ is irreducibly induced from $J$.*

(b) $j = 2k$ *and* $k = \mathrm{val}(r + s) = \mathrm{val}(r - s)$: *In this case, let*

$$\tilde{s} = s + \frac{\rho^2\pi^{2k}}{(r + s)^2}$$

*If*

$$\begin{pmatrix} 0 & b \\ b' & \tilde{s} \end{pmatrix}$$

*is irreducible* mod $\mathcal{P}$, *then $\pi$ is irreducibly induced from $J$. Otherwise $\varphi$ is principal and $\pi$ is non-supercuspidal.*

(c) $j < 2k$ *or* $j = 2k = \mathrm{val}(2) < \mathrm{val}(r \pm s)$: *Here $(\pi, V)$ is never supercuspidal.*

*Proof.* In this proof we assume for convenience that $n - 1 = 2m$ and that $i = 0$. The other case is handled in a nearly identical fashion.

Since the illustration of principal characters in case (i) of (a) uses essentially the same arguments as in Section 4. of [KM], the proof is left to the reader.

Consider condition (ii) of case (a). Assume that some $x \in G$ intertwines $\mathrm{Ind}_{S_{j-2}}^{G} \varphi$. We need to show that $x \in J$. Let $t = t_1 t_2$ so that mod $Z(G)$ we may take $t$ to be the translation

$$t_{\alpha+2\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let $\mathscr{D} = \{w_{\alpha+2\beta}t^{\nu} \mid \nu \geqslant 0\} \cup \{t^{\nu} \mid \nu > 0\}$. It is enough to consider elements $g \in J\mathscr{D}J$ since intertwining by elements of other double cosets of $J$ would produce vectors fixed by subgroups already considered. We can write

$$J\mathscr{D}J = J_0\{I, z_J\}\mathscr{D}\{I, z_J\}J_0$$

where $J_0 = J_{0,0}$. The arguments for each piece of this double coset are essentially the same. We give the proof for $g \in J_0 t^{\nu} J_0$.

Let $g = j_1 t^{\nu} j_2 = n_1 j t^{\nu} n_2$ where

$$j \in J_0 \cap t^{\nu} J_0 t^{-\nu}$$

$$n_1 \in J_0/(J_0 \cap t^{\nu} J_0 t^{-\nu})$$

$$n_2 \in (J_0 \cap t^{-\nu} J_0 t^{\nu})\backslash J_0.$$

This means that

$$n_1 \in [w_{\beta} u_{-\beta}(\mathscr{P}/\mathscr{P}^{\nu}) \cup u_{-\beta}(\mathcal{O}/\mathscr{P}^{\nu})] u_{-(\alpha+2\beta)}(\mathscr{P}/\mathscr{P}^{1+2\nu}) u_{-(\alpha+\beta)}(\mathscr{P}/\mathscr{P}^{1+\nu})$$

and

$$n_2 \in u_{\alpha+2\beta}(\mathscr{P}/\mathscr{P}^{2\nu}) u_{\alpha+\beta}(\mathscr{P}/\mathscr{P}^{\nu})[u_{\beta}(\mathscr{P}/\mathscr{P}^{\nu}) w_{\beta} \cup u_{\beta}(\mathcal{O}/\mathscr{P}^{\nu})]$$

We claim that neither $n_1$ nor $n_2$ may have $w_{\beta}$ as a factor. In general, if $g$ intertwines $\varphi$, then $g\varphi = \varphi$ on $x\tilde{S}_{j-2}x^{-1} \cap \tilde{S}_{j-2}$. Therefore $t^{\nu} n_2 \varphi = j^{-1} n_1^{-1} \varphi$ on $J_{m,0} \cap t^{\nu} J_{m,0} t^{-\nu}$. If $n_2$ has $w_{\beta}$ as a factor, it is easy to see that $t^{\nu} n_2 \varphi$ has level $n + 2\nu$ on $u_{-(\alpha+2\beta)}$. This is a contradiction since $j^{-1} n_1^{-1} \varphi$ is trivial on $J_{n-1,0}$. Similar reasoning shows that $n_1$ cannot have $w_{\beta}$ as a factor either.

Since $j$ may be factored, we may write $x = n^- t^{\nu} n^+$ where $n^+$ and $n^-$ are respectively in the intersections of $J_0$ with the positive and negative root Borel subgroups. Therefore, on $J_{m,0} \cap t^{\nu} J_{m,0} t^{-\nu}$ we have

$$t^{\nu} n^+ \varphi = n^- \varphi$$

The left hand side of this equation is trivial on the $\beta$ and $\alpha + \beta$ root groups of $J_{m,0}$. We now write

$$n^- \varphi = u_{-\beta}(x) u_{-(\alpha+\beta)}(y) \varphi_1$$

where

$$\varphi_1 = u_{-\alpha}(p) u_{-(\alpha+2\beta)}(q) \varphi$$

One can verify that $\varphi_1$ is also trivial on the $\beta$ and $\alpha + \beta$ root groups of $J_{m,0}$. It follows that $x$ and $y$ are in $\mathscr{P}^{m-k+1}$.

The next thing to see is that $n^- \varphi_1$ still has precisely level $j - 1$ on $u_{\alpha+2\beta}$. This is true because $j \geqslant 2k + 1$ forces $[u_{\alpha+2\beta}(z), u_{-\beta}(x)] \in \ker \varphi_1$ whenever $z \in \mathscr{P}^{j-2}$ and $x \in \mathscr{P}^{m-k+1}$. A similar argument applies to the effect of $u_{-(\alpha+\beta)}(y)$ on $\varphi_1$.

We conclude that $n^- \varphi$ produces a ramified cuspidal character of $S_{j-2}$ of the same level as that produced by $\varphi$. A similar check shows the same to be true for $n^+ \varphi$. It is evident that $t^v$ cannot send one to the other. Similar arguments for the other kinds of double cosets of $J$ show that the only elements that can intertwine $\varphi$ are in $J$ itself. Thus $(\pi, V)$ is irreducibly induced from $J$. The proof for case (iii) of (a) is quite similar.

We now consider case (b). We may extend $\varphi$ to $\tilde{S}_{2k-2} u_{-(\alpha+\beta)}(\mathscr{P}^m)$ so that $\varphi$ is trivial on $u_{-(\alpha+\beta)}(\mathscr{P}^m)$. This is a key point in the entire problem since this extension exists precisely because $F$ has residual characteristic 2.

For $p \in \mathscr{P}^m$ and $q \in \mathscr{P}^{2k-1}$ we may write

$$\varphi(u_{-(\alpha+\beta)}(p) u_{-(\alpha+2\beta)}(q)) = \phi(\pi^{-n} ap + \pi^{-2k} bq)$$

where $a$ and $b$ are integers.

Conjugation of $\varphi$ by $u_{\alpha+\beta}(y) u_{\alpha+2\beta}(z)$ changes $\varphi$ only on $u_{-(\alpha+\beta)}$ and $u_{-(\alpha+2\beta)}$. This action changes the coefficient of $p$ in the formula above by $\pi^{-n}(r + s)y + \pi^{-(m+1)} \rho z^{1/2}$. The $q$-coefficient is changed by the amount $\pi^{-n+1} y^2 + \pi^{-2k}(b'z^2 + sz)$.

$\varphi$ is a principal character when there exist elements $y$ and $z$ to make the $q$-coefficient zero without modifying the $p$-coefficient. This requires that

$$y^2 = \frac{\pi^{2m} \rho^2 z}{(r + s)^2}.$$

Thus the modification to the $q$-coefficient is

$$\pi^{-2k} \left[ \left( s + \frac{\pi^{2k} \rho^2}{(r + s)^2} \right) z + b'z^2 \right].$$

Thus the new $q$-coefficient will be

$$\pi^{-2k}(b + \tilde{s}z + b'z^2).$$

This leads immediately to the condition of case (b). When that condition fails, an argument similar to the one given in case (a) shows that $(\pi, V)$ is irreducibly induced from $J$.

We now consider case (c). Let

$$\gamma = \begin{cases} \beta & \text{for } n-1 \text{ odd;} \\ \alpha+\beta & \text{for } n-1 \text{ even.} \end{cases}$$

Then $\varphi$ may be extended to $\tilde{S}_j u_{-\gamma}(\mathscr{P}^m)$ as before. As in the above cases, we give the proof for when $n-1$ is even and $\gamma = \alpha + \beta$.

We may assume that the extended $\varphi$ is trivial on $u_{-(\alpha+\beta)}(\mathscr{P}^m)$. Note that conjugating $u_{-(a+b)}(y)$ by $u_{\alpha+2\beta}(z)$ gives

$$u_{-(\alpha+\beta)}(y)u_\beta(zy)u_{-\alpha}(zy^2).$$

When $y \in \mathscr{P}^m$ and $z \in \mathscr{O}$, we have $\varphi(u_\beta(zy)) = 1$. Also (since $F$ has residual characteristic 2) we find that the map $y \mapsto \varphi(u_{-\alpha}(zy^2))$ is an additive character of $\mathscr{P}^m/\mathscr{P}^{m+1}$. It follows that $z \in \mathscr{O}$ may be chosen so that $\varphi_1 = \mathrm{Ad}(u_{\alpha+2\beta}(z))(\varphi)$ is a character extending the original $\varphi$ which is trivial on $u_{-(\alpha+\beta)}(\mathscr{P}^m)$.

Let

$$\tilde{t} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

We will show that $\varphi_1$ is a principal character. Let $\varphi_2$ be any extension to $\tilde{S}_j u_{-(\alpha+\beta)}(\mathscr{P}_m)$ of $\mathrm{Ad}(\tilde{t})(\varphi_1)$. Let $P$ be the positive root Borel subgroup of $S_{\alpha+2\beta}$ and set $P_j = P \cap K_j$. Then $\varphi_2$ is trivial on $\tilde{S}_{2k}P_j$. We may choose $y \in \mathscr{P}^{m-k}$ such that $\mathrm{Ad}(u_{\alpha+\beta}(y))(\varphi_2)$ is a character of $\tilde{S}_j$ satisfying the initial conditions of the lemma. This demonstrates that $\varphi$ is a principal character.

THEOREM 3.9. *All supercuspidal representations occurring in the nilpotent case are irreducibly induced from some open compact-by-center subgroup of G.*

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