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A geometrical approach to the theory of Jacobi forms

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Introduction

In this note we give a geometrical approach to the theory of Jacobi forms, which was originated in an analytic way by Eichler and Zagier (cf. [4]). There Jacobi forms are introduced as holomorphic functions on $\mathscr{H} \times \mathbb{C}$ ($\mathscr{H}$: upper half-plane) satisfying a certain transformation law with respect to the semi-direct product $\Gamma \ltimes \mathbb{Z}^2$ (Γ: subgroup of finite index in $SL_2(\mathbb{Z})$) and having distinguished Fourier expansions at the cusps.

We start in the first section by recalling some basic facts about the elliptic modular surface X_Γ associated to Γ following the article [9]. To simplify the exposition, we restrict ourselves to the cases, where Γ acts without fixed points.

In the second section, we characterize the space of Jacobi cusp forms of weight $k \in 2\mathbb{N}$, index $m \in \mathbb{N}$ with respect to Γ as the space of global sections of a distinguished subsheaf $\mathscr{F}_{k,m}$ of a certain line bundle $\mathcal{O}_{X_\Gamma}(D_{k,m})$ on X_Γ.

In the third section, we prove the ampleness of the line bundle $\mathcal{O}_{X_\Gamma}(D_{k,m})$ for k, m being large enough. By using the Riemann–Roch Theorem for the surface X_Γ, we then determine the dimension of $H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m}))$. By proving the vanishing of $H^1(X_\Gamma, \mathscr{F}_{k,m})$, we are finally in position to derive a formula for the dimension of the space of Jacobi cusp forms, which can be made more explicit under the extra hypothesis that the least common multiple of all the cusp widths divides m. Formulas for the dimension of the space of Jacobi cusp forms were already obtained for $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_0(p)$ (p a prime) by analytic methods in [4], [6] and in full generality, by using a trace formula for Jacobi–Hecke operators (cf. [11]), in the preprint [10]. Our final formula is identical with the one given by Skoruppa in [10].

1. Notations and Definitions

Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$ and $\mathscr{H}$ the upper half-plane. Assume that Γ acts without fixed points and $-1 \notin \Gamma$. Let Y_Γ be the modular curve associated to Γ, i.e. the compactification of $\Gamma\backslash\mathscr{H}$ by adding the cusps.

Denote by σ_Γ the number of cusps and by n_j the cusp width of the jth cusp

$$P_j \in Y_\Gamma,\ P_j = \Gamma M_j i\infty \text{ with } M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL_2(\mathbb{Z})\ (j = 1, \ldots, \sigma_\Gamma).$$

Assume furthermore that Y_Γ has no cusps of the second kind.

Denote by X_Γ the elliptic modular surface associated to Γ, by $\pi\colon X_\Gamma \to Y_\Gamma$ the natural projection and by $\sigma\colon Y_\Gamma \to X_\Gamma$ the zero section. From [5], §8 or [9], Part II, we know the following: Because of our assumption on Γ, the only singular fibres of π lie above the σ_Γ cusps of Y_Γ; these are n_j-gons, i.e.

$$\pi^{-1}(P_j) = \bigcup_{\nu=0}^{n_j-1} \Theta_{j,\nu},$$

where $\Theta_{j,\nu} \cong \mathbb{P}^1$ is embedded into X_Γ with self-intersection number -2, and otherwise

$$\Theta_{j,\nu} . \Theta_{j,\nu'} = \begin{cases} 1 & \nu' = \nu \pm 1 \\ 0 & |\nu - \nu'| \geqslant 2 \end{cases}$$

(here and below, ν, ν' have to be understood modulo n_j).

In terms of local coordinates the situation above the cusp P_j can be described as follows: $\Theta_{j,\nu} \subset X_\Gamma$ can be covered by two affine charts $W^0_{j,\nu}$, $W^1_{j,\nu} \subset X_\Gamma$, where the coordinates $u_{j,\nu}$, $v_{j,\nu}$ of $W^0_{j,\nu}$ can be chosen such that $\Theta_{j,\nu}|_{W^0_{j,\nu}}$ is given by the equation $v_{j,\nu} = 0$ ($u_{j,\nu}$-axis) and hence the coordinates of $W^1_{j,\nu}$ are $u^{-1}_{j,\nu}$, $u^2_{j,\nu} v_{j,\nu}$ (note that $\Theta_{j,\nu} . \Theta_{j,\nu} = -2$). Because $\Theta_{j,\nu+1}$ and $\Theta_{j,\nu}$ intersect transversally, we deduce from this the following relations

$$\begin{aligned} u_{j,\nu+1} &= v^{-1}_{j,\nu} \\ v_{j,\nu+1} &= u_{j,\nu} v^2_{j,\nu}. \end{aligned}$$

Furthermore we note (cf. [2], I.4 or [3], §1)

$$\begin{aligned} u_{j,\nu} v_{j,\nu} &= q_j \\ u^{\nu+1}_{j,\nu} v^{\nu}_{j,\nu} &= \zeta_j \end{aligned}$$

with $q_j = e^{2\pi i M_j^{-1}\tau/n_j}$, $\zeta_j = e^{2\pi i z/(-c_j\tau + a_j)}$ $(\tau \in \mathscr{H},\ z \in \mathbb{C})$.

By g_Γ resp. p_a we denote the genus of Y_Γ resp. the arithmetic genus of X_Γ. We have the formulae (cf. [5], §12 or [9], pp. 35–36)

$$g_\Gamma = \frac{\mu_\Gamma}{12} - \frac{\sigma_\Gamma}{2} + 1 \quad \text{resp.}$$

$$p_a = \frac{\mu_\Gamma}{12} - 1 \qquad \text{with } \mu_\Gamma = \frac{[SL_2(\mathbb{Z}):\Gamma]}{2}.$$

Finally, we denote by K_{Y_Γ} resp. K_{X_Γ} the canonical divisor of Y_Γ resp. X_Γ.

2. Geometrical characterization of Jacobi forms

Let $k \in 2\mathbb{N}$ and $m \in \mathbb{N}$. From [4], we recall the following

DEFINITION 2.1. A holomorphic function $f: \mathscr{H} \times \mathbb{C} \to \mathbb{C}$ is called *Jacobi cusp form of weight k, index m with respect to* Γ, if it satisfies the following properties:

(i) $$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k}\exp\left(2\pi i m\left[\lambda^2\tau+2\lambda z - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right]\right)$$
$$= f(\tau, z),$$
for all $\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}, [\lambda \quad \mu]\right) \in \Gamma \ltimes \mathbb{Z}^2$.

(ii) At the cusp P_j $(j = 1, \ldots, \sigma_\Gamma)$, f has a Fourier expansion of the form

$$f(\tau, z)(-c_j\tau + a_j)^k \exp\left(2\pi i m \frac{-c_j z^2}{-c_j\tau + a_j}\right) = \sum_{\substack{n>0, r\in\mathbb{Z} \\ 4mn - n_j r^2 > 0}} c_j(n, r) q_j^n \zeta_j^r.$$

The $\mathbb{C}$-span of these functions is denoted by $J_{k,m}^{\text{cusp}}(\Gamma)$.

We are now going to describe the elements of $J_{k,m}^{\text{cusp}}(\Gamma)$ as distinguished sections of a certain line bundle over X_Γ. We put $Y_\Gamma^0 := \Gamma\backslash\mathscr{H}$ and $X_\Gamma^0 := \Gamma \ltimes \mathbb{Z}^2 \backslash \mathscr{H} \times \mathbb{C}$. We make the following observation: It is immediately checked that

$$\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}, [\lambda \quad \mu]\right) \mapsto (c\tau + d)^{-k}$$

$$\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}, [\lambda \quad \mu]\right) \mapsto \exp\left(2\pi i m\left[\lambda^2\tau + 2\lambda z - \frac{c(z + \lambda\tau + \mu)^2}{c\tau + d}\right]\right)$$

define two 1-cocycles, whence two 1-cohomology classes γ_1, γ_2 resp. of $H^1(\Gamma \ltimes \mathbb{Z}^2, \mathbb{C}^*) \cong H^1(X_\Gamma^0, \mathcal{O}_{X_\Gamma^0}^*) \cong \text{Pic}(X_\Gamma^0)$ (cf. [7], Appendix to I.2). The classical theory of modular forms shows that the isomorphism class in $\mathrm{Pic}(X_\Gamma^0)$ corresponding to γ_1 can be represented by the line bundle

$$\mathcal{O}_{X_\Gamma}\left(\frac{k}{2}\pi^* K_{Y_\Gamma}\right)\Big|_{X_\Gamma^0}. \tag{1}$$

Now let $[\mathscr{L}]$ be the isomorphism class in $\mathrm{Pic}(X_\Gamma^0)$ corresponding to γ_2. By restricting $[\mathscr{L}]$ to an arbitrary fibre E_τ of $\Gamma\tau \in Y_\Gamma^0$, we obtain an isomorphism class of line bundles on the elliptic curve E_τ. The corresponding 1-cohomology class of $H^1(E_\tau, \mathcal{O}_{E_\tau}^*) \cong H^1(\Lambda, \mathbb{C}^*)$ (where Λ is the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$) is determined by the 1-cocycle

$$\lambda\tau + \mu \mapsto \exp(2\pi i m(\lambda^2\tau + 2\lambda z)) \qquad (\lambda, \mu \in \mathbb{Z}).$$

One checks that this 1-cocyle differs only by a 1-coboundary from the 1-cocyle defined by

$$u := \lambda\tau + \mu \mapsto \exp\left(\frac{\pi}{2} H(u, u) + \pi H(z, u)\right),$$

where

$$H(x, y) := \frac{4mi}{\bar{\tau} - \tau} x\bar{y} \qquad (x, y \in \mathbb{C}).$$

By the theorem of Appel–Humbert, we find that $[\mathscr{L}]|_{E_\tau} = L(H, 1)$ (cf. [7], p. 20) and therefore can be represented by $\mathcal{O}_{E_\tau}(2m \cdot O)$, where O denotes the origin of E_τ. Hence $[\mathscr{L}]$ can be represented by the line bundle

$$\mathcal{O}_{X_\Gamma}(2m\sigma(Y_\Gamma))|_{X_\Gamma^0}. \tag{2}$$

The problem now consists in finding appropriate extensions of the line bundles (1), (2) to the whole of X_Γ. To solve it, we determine the divisor of a suitably chosen (meromorphic) Jacobi form of weight k, index m with respect to Γ.

We first recall two elementary lemmas.

LEMMA 2.2. *The theta function*

$$\theta_{1,1}(\tau, z) := \sum_{n \in \mathbb{Z}} \exp\left(\pi i \tau \left[n + \frac{1}{2}\right]^2 + 2\pi i \left[z + \frac{1}{2}\right]\left[n + \frac{1}{2}\right]\right)$$

satisfies the functional equation

$$\theta_{1,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-1/2}\exp\left(\pi i\left[\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right]\right)$$
$$=\chi_1\theta_{1,1}(\tau,z)$$

for all $\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix},[\lambda\ \mu]\right)\in SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, *with* χ_1 *an* 8th *root of unity.*

Proof. Cf. e.g. [8], p. 36. □

LEMMA 2.3. *The eta function*

$$\eta(\tau):=e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$$

satisfies the functional equation

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}=\chi_2\eta(\tau)$$

for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, with χ_2 *a* 24th *root of unity.*

Proof. Cf. e.g. [1], p. 52. □

We put $\chi:=(\chi_1\chi_2^{-1})^{2m}$ and denote by $\mathcal{O}_{Y_\Gamma}(M_{k,\bar\chi})$ the line bundle associated to holomorphic modular forms of weight k and character $\bar\chi$ with respect to Γ; its degree is given by $k\mu/12$. For example, for $\chi=1$, we have

$$M_{k,1}=\frac{k}{2}K_{Y_\Gamma}+\sum_j\frac{k}{2}P_j.$$

PROPOSITION 2.4. *For any* $f\in J^{\mathrm{cusp}}_{k,m}(\Gamma)$, *we have*

$$\operatorname{div}f\sim\tilde{D}_{k,m}:=\pi^*M_{k,\bar\chi}+2m\sigma(Y_\Gamma)+\sum_{j,\nu}\left(-m\nu+\frac{mn_j}{6}\right)\Theta_{j,\nu},$$

where $\sim$ *means linear equivalence of divisors.*

Proof. Let g be any holomorphic modular form of weight k and character $\bar\chi$

with respect to Γ and put

$$G(\tau, z) := \left(\frac{\theta_{1,1}(\tau, z)}{\eta(\tau)}\right)^{2m} g(\tau).$$

By construction we have

$$\operatorname{div} f \sim \operatorname{dig} G \sim \pi^* M_{k,\bar{\chi}} + 2m \operatorname{div} \theta_{1,1} - 2m \operatorname{div} \eta.$$

To complete the proof, we have to compute $\operatorname{div} \theta_{1,1}$ and $\operatorname{div} \eta$. The only non-trivial part of this calculation is the determination of the multiplicities of $\theta_{1,1}$ on the components $\Theta_{j,\nu}$: From the Fourier expansion of $\theta_{1,1}$ we obtain

$$\begin{aligned}\theta_{1,1}(\tau, z)(-c_j\tau + a_j)^{1/2} \exp\left(\pi i \frac{-c_j z^2}{-c_j \tau + a_j}\right) &= \chi_1^{-1} \sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)} q_j^{n_j/2(n+1/2)^2} \zeta_j^{(n+1/2)} \\ &= \chi_1^{-1} \sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)} u_{j,\nu}^{n_j/2(n+1/2)^2+(\nu+1)(n+1/2)} v_{j,\nu}^{n_j/2(n+1/2)^2+\nu(n+1/2)},\end{aligned}$$

hence the multiplicity of $\theta_{1,1}$ on $\Theta_{j,\nu}$ equals

$$\min_{n\in\mathbb{Z}} \left(\frac{n_j}{2} n^2 + \left(\frac{n_j}{2} + \nu\right) n + \frac{n_j}{8} + \frac{\nu}{2}\right) = \left(-\frac{\nu}{2} + \frac{n_j}{8}\right).$$

Summing up, we get

$$\operatorname{div} f \sim \pi^* M_{k,\bar{\chi}} + 2m\sigma(Y_\Gamma) + 2m \sum_{j,\nu} \left(-\frac{\nu}{2} + \frac{n_j}{8} - \frac{n_j}{24}\right) \Theta_{j,\nu} = \tilde{D}_{k,m},$$

as claimed. □

THEOREM 2.5. *The map $f \mapsto \operatorname{div} f$ induces an injection*

$$\iota \colon J_{k,m}^{\mathrm{cusp}}(\Gamma) \hookrightarrow H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})),$$

where

$$D_{k,m} = \pi^* M_{k,\bar{\chi}} + 2m\sigma(Y_\Gamma) + \sum_{j,\nu} \left(-\left[\frac{n_j}{4m}\left\|\frac{2m\nu}{n_j}\right\|^2 + \frac{m\nu}{n_j}(n_j - \nu)\right] + \frac{mn_j}{6} - 1\right) \Theta_{j,\nu}$$

($\|x\|$ *meaning the distance of* x *to the nearest integer and* $[x]$ *the greatest integer* $\leq x$).

Proof. Let $f \in J_{k,m}^{\mathrm{cusp}}(\Gamma)$. By definition $\operatorname{div} f|_{X_\Gamma^0}$ is effective; therefore it remains to determine the multiplicities of f on the components $\Theta_{j,v}$: By Definiton 2.1(ii), we obtain

$$f(\tau, z)(-c_j\tau + a_j)^k \exp\left(2\pi i m \frac{-c_j z^2}{-c_j\tau + a_j}\right)$$
$$= \sum_{\substack{n>0, r\in\mathbb{Z} \\ 4mn - n_j r^2 > 0}} c_j(n, r) q_j^n \zeta_j^r = \sum_{\substack{n>0, r\in\mathbb{Z} \\ 4mn - n_j r^2 > 0}} c_j(n, r) u_{j,v}^{n+(v+1)r} v_{j,v}^{n+vr},$$

hence the multiplicity of f on $\Theta_{j,v}$ is bigger or equal to

$$\min_{\substack{n>0, r\in\mathbb{Z} \\ 4mn - n_j r^2 > 0}} (n + vr) = \min_{r\in\mathbb{Z}} \left(\left[\frac{n_j r^2}{4m}\right] + vr + 1\right)$$
$$= \left[\min_{r\in\mathbb{Z}} \left(\frac{n_j}{4m}\left(r + \frac{2mv}{n_j}\right)^2 - \frac{mv^2}{n_j} + 1\right)\right] = \left[\frac{n_j}{4m}\left\|\frac{2mv}{n_j}\right\|^2 - \frac{mv^2}{n_j}\right] + 1 =: \beta_{j,v}.$$

From Proposition 2.4 and the above computation we get

$$\operatorname{div} f \in \left\{D \in \operatorname{Div}(X_\Gamma) : D \sim \tilde{D}_{k,m},\ D \geqslant \sum_{j,v} \beta_{j,v}\Theta_{j,v}\right\}$$
$$\cong \{D \in \operatorname{Div}(X_\Gamma) : D \sim D_{k,m},\ D \geqslant 0\} \cong H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) - \{0\}/\mathbb{C}^*.$$

This completes the proof of Theorem 2.5. □

To describe the subspace $\operatorname{im} \iota \subseteq H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m}))$ corresponding to the space $J_{k,m}^{\mathrm{cusp}}(\Gamma)$, we have to introduce some definitions:

For $x \in \mathbb{R}$, denote by $\langle x \rangle$ the nearest integer of x. For $j = 1, \ldots, \sigma_\Gamma$ and $v = 0, \ldots, n_j - 1$ define integers $v^- \leqslant 0$, $v^+ \geqslant 0$ satisfying the properties

(i) the integers in the interval

$$\bigcup_{v=0}^{n_j-1} \left[v^- - \left\langle\frac{2mv}{n_j}\right\rangle, v^+ - \left\langle\frac{2mv}{n_j}\right\rangle\right]$$

form a complete set of representatives of the residue classes modulo $2m$,

(ii) the sum

$$\sum_{v=0}^{n_j-1} \sum_{\rho=v^- - \langle 2mv/n_j\rangle}^{v^+ - \langle 2mv/n_j\rangle} \left(\left[\frac{n_j\rho^2}{4m}\right] + v\rho - \beta_{j,v} + 1\right)$$

is minimal.

Finally, define the subsheaf $\mathscr{F}_{k,m}$ of $\mathcal{O}_{X_\Gamma}(D_{k,m})$ to consist of those sections s having the property that in their Taylor expansions

$$\sum_{k,l\geqslant 0} b_{j,\nu}(k,l)u_{j,\nu}^k v_{j,\nu}^l \tag{3}$$

on $W_{j,\nu}^0$, i.e. around the vertices $Q_{j,\nu} := \Theta_{j,\nu-1} \cap \Theta_{j,\nu}$ of the n_j-gons $\pi^{-1}(P_j)$, the coefficients $b_{j,\nu}(k,l)$ vanish for

$$l < \left[\frac{n_j\rho^2}{4m}\right] + \nu\rho - \beta_{j,\nu} + 1, \tag{4}$$

where $\rho := k - l + \beta_{j,\nu+1} - \beta_{j,\nu}$ is running from

$$\nu^- - \langle 2m\nu/n_j\rangle, \ldots, \nu^+ - \langle 2m\nu/n_j\rangle$$

$(j = 1, \ldots, \sigma_\Gamma;\ \nu = 0, \ldots, n_j - 1)$. We note that $\mathscr{F}_{k,m}$ is *not* an $\mathcal{O}_{X_\Gamma}$-submodule of $\mathcal{O}_{X_\Gamma}(D_{k,m})$ in general.

With these definitions we prove

THEOREM 2.6. *There is an isomorphism*

$$J_{k,m}^{\text{cusp}}(\Gamma) \cong H^0(X_\Gamma, \mathscr{F}_{k,m}).$$

Proof. Let $s \in H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m}))$ be a global section belonging to im ι. From the proof of Theorem 2.5 we know that the Fourier expansion at the cusp P_j of the Jacobi form corresponding to s is obtained by taking one of the Taylor expansions (3) of s (around the vertices $Q_{j,\nu}$), multiplying it by $u_{j,\nu}^{\beta_{j,\nu+1}} v_{j,\nu}^{\beta_{j,\nu}}$ and substituting $u_{j,\nu}$, $v_{j,\nu}$ by $q_j^{-\nu}\zeta_j$, $q_j^{\nu+1}\zeta_j^{-1}$ respectively; putting $\rho := k - l + \beta_{j,\nu+1} - \beta_{j,\nu}$, we find the Fourier expansion

$$\sum_{k,l\geqslant 0} b_{j,\nu}(k,l) q_j^{l+\beta_{j,\nu}-\nu\rho} \zeta_j^\rho.$$

The above analysis shows: $s \in \text{im}\ \iota$, if and only if the coefficients $b_{j,\nu}(k,l)$ in the Taylor expansions (3) vanish for

$$l + \beta_{j,\nu} - \nu\rho < \left[\frac{n_j\rho^2}{4m}\right] + 1 \tag{5}$$

$(j = 1, \ldots, \sigma_\Gamma;\ \nu = 0, \ldots, n_j - 1)$. The number of the conditions (5) can now be minimized as follows: By the argument of [4], p. 23, we know that (5) has to be fulfilled only for ρ running through a complete set of representatives of the residue classes modulo $2m$. Observing that the n_j Taylor expansions (3) around the vertices $Q_{j,0}, \dots, Q_{j,n_j-1}$ determine *the* Fourier expansion of the corresponding Jacobi form at the cusp P_j, we optimize the problem by distributing the conditions (5) in a minimal way among the n_j Taylor expansions (3). Recalling the definitions preceding Theorem 2.6, we see that the optimal result is achieved by imposing at each vertex $Q_{j,v}$ the conditions $b_{j,v}(k, l) = 0$ for

$$l < \left\lceil \frac{n_j \rho^2}{4m} \right\rceil + v\rho - \beta_{j,v} + 1,$$

where ρ is running from

$$v^- - \langle 2mv/n_j \rangle, \dots, v^+ - \langle 2mv/n_j \rangle,$$

i.e., $s \in \operatorname{im} \iota$, if and only if $s \in H^0(X_\Gamma, \mathscr{F}_{k,m})$. $\square$

REMARK 2.7. From inequality (4) and the estimate

$$l = k - \rho + \beta_{j,v+1} - \beta_{j,v} \geqslant -\rho + \beta_{j,v+1} - \beta_{j,v}$$

we note that the number C_j of conditions for each cusp P_j, which define the sheaf $\mathscr{F}_{k,m}$ as a subsheaf of $\mathcal{O}_{X_\Gamma}(D_{k,m})$, is given by the formula

$$C_j = \sum_{v=0}^{n_j-1} \sum_{\rho = v^- - \langle 2mv/n_j \rangle}^{v^+ - \langle 2mv/n_j \rangle} \left(\left\lceil \frac{n_j \rho^2}{4m} \right\rceil + v\rho - \beta_{j,v} + 1 - \max(0, -\rho + \beta_{j,v+1} - \beta_{j,v}) \right). \tag{6}$$

3. Dimension computations

To compute $\dim H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m}))$ and $\dim H^0(X_\Gamma, \mathscr{F}_{k,m})$, we need the following

LEMMA 3.1. *Define* $\alpha_{j,v}$ and $\gamma_{j,v}$ *by*

$$\alpha_{j,v} := -\left[\frac{n_j}{4m} \left\| \frac{2mv}{n_j} \right\|^2 + \frac{mv}{n_j}(n_j - v) \right] + \frac{mn_j}{6} - 1 \tag{7}$$

for $j = 1, \dots, \sigma_\Gamma$ *and* $v = 0, \dots, n_j - 1$, *extended to all* $v \in \mathbb{Z}$ *by periodicity in* v *with period* n_j, *and*

$$\gamma_{j,v} := \alpha_{j,v-1} - 2\alpha_{j,v} + \alpha_{j,v+1} + \begin{cases} 2m & v \equiv 0 \bmod n_j \\ 0 & \textit{otherwise} \end{cases}.$$

If $m \geqslant n_j$, then $\gamma_{j,v} > 0$ for all $v \in \mathbb{Z}$.

Proof. The case $m = n_j$ being easily treated by direct computation, we assume from now on $m > n_j$. With $0 \leqslant \rho_{j,\nu} < 1$, (7) equals

$$\alpha_{j,\nu} = -\frac{m\nu}{n_j}(n_j - \nu) + \frac{mn_j}{6} - 1 - \frac{n_j}{4m}\left\|\frac{2m\nu}{n_j}\right\|^2 + \rho_{j,\nu}. \tag{8}$$

Introducing the polynomial $P_j(\nu) := -m\nu(n_j - \nu)/n_j + mn_j/6 - 1$ and $\lambda_j := 2m/n_j$, we can rewrite (8) as

$$\alpha_{j,\nu} = P_j(\nu) - \frac{1}{2\lambda_j}\|\lambda_j\nu\|^2 + \rho_{j,\nu}.$$

Observing

$$P_j(\nu - 1) - 2P_j(\nu) + P_j(\nu + 1) = \begin{cases} \lambda_j - 2m & \nu \equiv 0 \bmod n_j \\ \lambda_j & \text{otherwise} \end{cases}$$

and $\rho_{j,\nu-1} - 2\rho_{j,\nu} + \rho_{j,\nu+1} > -2$, we derive the estimate

$$\gamma_{j,\nu} > \lambda_j - 2 - \frac{1}{2\lambda_j}(\|\lambda_j\nu - \lambda_j\|^2 - 2\|\lambda_j\nu\|^2 + \|\lambda_j\nu + \lambda_j\|^2).$$

Noting that for all real x and $-1/2 \leqslant \xi \leqslant +1/2$, we have

$$\|\xi - x\|^2 - 2\|\xi\|^2 + \|\xi + x\|^2 \leqslant (\xi - x)^2 - 2\xi^2 + (\xi + x)^2 = 2x^2,$$

which by periodicity also holds for all real x and ξ, we obtain (by taking $x = \lambda_j - 2$ and $\xi = \lambda_j\nu$)

$$\gamma_{j,\nu} > \lambda_j - 2 - \frac{(\lambda_j - 2)^2}{\lambda_j} = \frac{2(m - n_j)}{m} > 0.$$

This completes the proof of the lemma. □

PROPOSITION 3.2. *If* $k \geqslant m + 4$ *and* $m \geqslant n_j$ *(all* j*), the line bundle* $\mathcal{O}_{X_\Gamma}(D_{k,m} - K_{X_\Gamma})$ *is ample.*

Proof. We have (cf. Theorem 2.5 and (7))

$$D_{k,m} = \pi^* M_{k,\bar{\chi}} + 2m\sigma(Y_\Gamma) + \sum_{j,\nu} \alpha_{j,\nu}\Theta_{j,\nu}.$$

By the criterion of Nakai–Moishezon, we have to show that the intersection numbers of $D_{k,m} - K_{X_\Gamma}$ with all irreducible curves $C \subset X_\Gamma$ are positive. We distinguish the following three cases:

—$C = E$, a regular fibre of π, i.e. an elliptic curve
—$C = \Theta_{j,v}$ (all j, v)
—C is a horizontal divisor, i.e., $\pi(C) = Y_\Gamma$.

In the first case, we have

$$(D_{k,m} - K_{X_\Gamma}).E = 2m > 0.$$

In the second case, we have by Lemma 3.1

$$(D_{k,m} - K_{X_\Gamma}).\Theta_{j,v} = \gamma_{j,v} > 0.$$

Before treating the last case, we recall from [5], §12 that $K_{X_\Gamma} = \pi^*(K_{Y_\Gamma} - F)$ with $F \in \mathrm{Div}(Y_\Gamma)$, $\deg F = -p_a - 1$; with the adjunction formula we therefore obtain

$$\sigma(Y_\Gamma).\sigma(Y_\Gamma) = -K_{X_\Gamma}.\sigma(Y_\Gamma) + 2g_\Gamma - 2 = -p_a - 1 = -\frac{\mu_\Gamma}{12}.$$

For the last case we distinguish $C = \sigma(Y_\Gamma)$ and $C \neq \sigma(Y_\Gamma)$. In the first subcase, we have

$$\begin{aligned}(D_{k,m} - K_{X_\Gamma}).\sigma(Y_\Gamma) &= \deg M_{k,\bar{\chi}} - K_{X_\Gamma}.\sigma(Y_\Gamma) + 2m\sigma(Y_\Gamma).\sigma(Y_\Gamma) + \sum_j \alpha_{j,0} \\ &= \frac{k\mu_\Gamma}{12} - (2g_\Gamma - 2 + p_a + 1) - \frac{m\mu_\Gamma}{6} + \sum_j \left(\frac{mn_j}{6} - 1\right).\end{aligned}$$

The formulae for the genus g_Γ and the arithmetic genus p_a given in the first section and the relation $\mu_\Gamma = \Sigma_j n_j$ now imply

$$(D_{k,m} - K_{X_\Gamma}).\sigma(Y_\Gamma) = \frac{\mu_\Gamma}{12}(k - 3) > 0.$$

For the second subcase, we first note that $\Theta_{j,v}.C$ equals zero for all but one $v =: v_j$, where we have $\Theta_{j,v_j}.C = 1$. We now obtain the estimate

$$\begin{aligned}(D_{k,m} - K_{X_\Gamma}).C &= \deg M_{k,\bar{\chi}} - K_{X_\Gamma}.C + \sum_j \alpha_{j,v_j} \\ &= \frac{k\mu_\Gamma}{12} - (2g_\Gamma - 2 + p_a + 1) + \sum_j \left(- \left[\frac{n_j}{4m}\left\|\frac{2mv_j}{n_j}\right\|^2 + \frac{mv_j}{n_j}(n_j - v_j)\right] + \frac{mn_j}{6} - 1\right) \\ &\geqslant \frac{\mu_\Gamma}{12}(k-3) + \sum_j \left(-\frac{mv_j}{n_j}(n_j - v_j) + \frac{mn_j}{6} - \frac{1}{16}\right) \\ &> \frac{\mu_\Gamma}{12}(k-3) + \sum_j \left(-\frac{mn_j}{12} - \frac{n_j}{12}\right) = \frac{\mu_\Gamma}{12}(k - m - 4) \geqslant 0.\end{aligned}$$

This finishes the proof of the proposition. □

THEOREM 3.3. *If $k \geqslant m+4$ and $m \geqslant n_j$ (all j), we have the dimension formula*

$$\dim H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m}))$$
$$= \frac{\mu_\Gamma}{6}km + \frac{\mu_\Gamma}{6}m^2 - \left(\frac{\mu_\Gamma}{4} + \sigma_\Gamma\right)m + \frac{\mu_\Gamma}{12} + \sum_{j,\nu} \frac{\alpha_{j,\nu}}{2}(\alpha_{j,\nu-1} - 2\alpha_{j,\nu} + \alpha_{j,\nu+1}).$$

Proof. By Proposition 3.2 the line bundle $\mathcal{O}_{X_\Gamma}(D_{k,m} - K_{X_\Gamma})$ is ample and therefore we have by the Kodaira Vanishing Theorem

$$\dim H^1(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) = \dim H^2(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) = 0.$$

Now we obtain by the Riemann–Roch Theorem

$$\dim H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) = \chi(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) = \tfrac{1}{2}D_{k,m}\,.\,(D_{k,m} - K_{X_\Gamma}) + p_a + 1$$
$$= 2m \deg M_{k,\bar{\chi}} + 2m^2\sigma(Y_\Gamma)\,.\,\sigma(Y_\Gamma) + 2m\sum_j \alpha_{j,0} - mK_{X_\Gamma}\,.\,\sigma(Y_\Gamma)$$
$$+ \sum_{j,\nu}\frac{\alpha_{j,\nu}}{2}(\alpha_{j,\nu-1} - 2\alpha_{j,\nu} + \alpha_{j,\nu+1}) + p_a + 1 = \frac{\mu_\Gamma}{6}km + \frac{\mu_\Gamma}{6}m^2 - \left(\frac{\mu_\Gamma}{4} + \sigma_\Gamma\right)m$$
$$+ \frac{\mu_\Gamma}{12} + \sum_{j,\nu}\frac{\alpha_{j,\nu}}{2}(\alpha_{j,\nu-1} - 2\alpha_{j,\nu} + \alpha_{j,\nu+1}). \qquad \square$$

We denote by $\mathcal{S}_{k,m}$ the quotient sheaf $\mathcal{O}_{X_\Gamma}(D_{k,m})/\mathcal{F}_{k,m}$. It is a skyscraper sheaf supported on the vertices $Q_{j,\nu}$ of the n_j-gons $\pi^{-1}(P_j)$. By definition we have

$$\dim H^0(X_\Gamma, \mathcal{S}_{k,m}) = \sum_j C_j \tag{9}$$

with C_j given by (6). We are now able to prove

THEOREM 3.4. *If $k \gg 0$ and $m \geqslant n_j$ (all j), we have the dimension formula*

$$\dim J_{k,m}^{\mathrm{cusp}}(\Gamma) = \frac{\mu_\Gamma}{6}km + \frac{\mu_\Gamma}{6}m^2 - \left(\frac{\mu_\Gamma}{4} + \sigma_\Gamma\right)m + \frac{\mu_\Gamma}{12}$$
$$+ \sum_{j,\nu}\frac{\alpha_{j,\nu}}{2}(\alpha_{j,\nu-1} - 2\alpha_{j,\nu} + \alpha_{j,\nu+1}) - \sum_j C_j.$$

Proof. Consider the exact sequence of sheaves

$$0 \to \mathcal{F}_{k,m} \to \mathcal{O}_{X_\Gamma}(D_{k,m}) \to \mathcal{S}_{k,m} \to 0. \tag{10}$$

If we are able to show $H^1(X_\Gamma, \mathcal{F}_{k,m}) = \{0\}$, the long exact sequence in cohomology associated to (10) implies

$$\dim H^0(X_\Gamma, \mathscr{F}_{k,m}) = \dim H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) - \dim H^0(X_\Gamma, \mathscr{S}_{k,m}).$$

The claim then follows immediately from Theorems 2.6, 3.3 and formula (9).

We are left to show $H^1(X_\Gamma, \mathscr{F}_{k,m}) = \{0\}$. This will be done in two steps:

1st step. We introduce an auxiliary subsheaf $\mathscr{F}'_{k,m}$ of $\mathscr{F}_{k,m}$ as follows: With $\rho^{(\pm)} := v^{(\pm)} - \langle 2mv/n_j \rangle$ put

$$l_{j,v}^{(\pm)} := \left[\frac{n_j \rho^{(\pm)2}}{4m} \right] + v\rho^{(\pm)} - \beta_{j,v} + 1$$

$$k_{j,v}^{(\pm)} := l_{j,v}^{(\pm)} + \rho^{(\pm)} - \beta_{j,v+1} + \beta_{j,v}$$

$$h_j := \max_v \max(k_{j,v}^+, k_{j,v}^-, l_{j,v}^+, l_{j,v}^-).$$

Define the $\mathcal{O}_{X_\Gamma}$-submodule $\mathscr{F}'_{k,m}$ of $\mathcal{O}_{X_\Gamma}(D_{k,m})$ to consist of those sections s having the property that in their Taylor expansions (3) around the vertices $Q_{j,v}$ of the n_j-gons $\pi^{-1}(P_j)$, the coefficients $b_{j,v}(k, l)$ vanish for

$$0 \leqslant k,\ l \leqslant h_j$$

$(j = 1, \dots, \sigma_\Gamma;\ v = 0, \dots, n_j - 1)$. By construction $\mathscr{F}'_{k,m}$ is a subsheaf of $\mathscr{F}_{k,m}$ and the quotient $\mathcal{O}_{X_\Gamma}(D_{k,m})/\mathscr{F}'_{k,m}$ is again a skyscraper sheaf $\mathscr{S}'_{k,m}$ supported on the vertices $Q_{j,v}$. We have the following exact sequences of sheaves

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ 0 \arrow[r] & \mathscr{F}'_{k,m} \arrow[r] \arrow[d] & \mathcal{O}_{X_\Gamma}(D_{k,m}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{S}'_{k,m} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}_{k,m} \arrow[r] & \mathcal{O}_{X_\Gamma}(D_{k,m}) \arrow[r] & \mathscr{S}_{k,m} \arrow[r] \arrow[d] & 0. \\ & & & 0 & \end{tikzcd} \tag{11}$$

The first part of the long exact sequence in cohomology associated to (11) leads to the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) \arrow[r] \arrow[d, "\mathrm{id}"] & H^0(X_\Gamma, \mathscr{S}'_{k,m}) \arrow[r] \arrow[d] & H^1(X_\Gamma, \mathscr{F}'_{k,m}) \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & H^0(X_\Gamma, \mathcal{O}_{X_\Gamma}(D_{k,m})) \arrow[r] & H^0(X_\Gamma, \mathscr{S}_{k,m}) \arrow[r] \arrow[d] & H^1(X_\Gamma, \mathscr{F}_{k,m}) \arrow[r] & 0. \\ & & 0 & & \end{tikzcd} \tag{12}$$

From (12) we easily derive that $H^1(X_\Gamma, \mathscr{F}'_{k,m}) = \{0\}$ implies $H^1(X_\Gamma, \mathscr{F}_{k,m}) = \{0\}$.

2nd step. To prove the vanishing of $H^1(X_\Gamma, \mathscr{F}'_{k,m})$, we introduce the auxiliary line bundle $\mathscr{L}'_{k,m} := \mathcal{O}_{X_\Gamma}(D'_{k,m})$ with

$$D'_{k,m} := D_{k,m} - \sum_{j,v} h_j \Theta_{j,v} \in \mathrm{Div}(X_\Gamma).$$

From the exact sequence of $\mathcal{O}_{X_\Gamma}$-modules

$$0 \to \mathscr{L}'_{k,m} \to \mathscr{F}'_{k,m} \to \mathscr{Q}'_{k,m} := \mathscr{F}'_{k,m}/\mathscr{L}'_{k,m} \to 0,$$

we deduce that the vanishing of $H^1(X_\Gamma, \mathscr{L}'_{k,m})$ and $H^1(X_\Gamma, \mathscr{Q}'_{k,m})$ implies the triviality of $H^1(X_\Gamma, \mathscr{F}'_{k,m})$. The vanishing of $H^1(X_\Gamma, \mathscr{L}'_{k,m})$ follows from the ampleness of the line bundle $\mathcal{O}_{X_\Gamma}(D'_{k,m} - K_{X_\Gamma})$ for k being large enough, which is proved along the same lines as Proposition 3.2; a crude estimate shows

$$k \geqslant \left(72\frac{\sigma_\Gamma}{\mu_\Gamma} + 1\right)m + 204\frac{\sigma_\Gamma}{\mu_\Gamma} + 4.$$

To prove $H^1(X_\Gamma, \mathscr{Q}'_{k,m}) = \{0\}$, we note (after a short calculation) that the quotient $\mathscr{Q}'_{k,m} = \mathscr{F}'_{k,m}/\mathscr{L}'_{k,m}$ is isomorphic to

$$\bigoplus_{j,v} \mathcal{O}_{h_j\Theta_{j,v}}(h_j),$$

whence

$$H^1(X_\Gamma, \mathscr{Q}'_{k,m}) \cong \bigoplus_{j,v} H^1(h_j\Theta_{j,v}, \mathcal{O}_{h_j\Theta_{j,v}}(h_j)) = \{0\},$$

because $\Theta_{j,v} \cong \mathbb{P}^1$ has self-intersection number -2 and $h_j > 0$ (all j). This finishes the proof of the theorem. $\square$

From Theorem 3.4 we are able to deduce an explicit formula for $\dim J^{\mathrm{cusp}}_{k,m}(\Gamma)$ assuming $k \gg 0$ and $n_j \mid m$ (all j). To do this, we use a slight generalization of the lemma proved in [4], p. 124. Actually, we will prove a little more than we finally need. As in [4] we introduce the function

$$((x)) := \begin{cases} x - [x] - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

LEMMA 3.5. *For positive integers n, N, we have the formula*

$$\sum_{\kappa \bmod . N} \left(\left(\frac{n\kappa^2}{N}\right)\right) = -(n, N) \sum_{\substack{\Delta | N/(n,N), \Delta < 0 \\ N/(n,N)\Delta \text{ squarefree}}} \left(\frac{\Delta}{n/(n,N)}\right) H(\Delta), \tag{13}$$

where $H(\Delta)$ denotes the Hurwitz class number, i.e. the (weighted) number of $SL_2(\mathbb{Z})$ equivalence classes of all positive definite, integral, binary quadratic forms of discriminant Δ.

Proof. First suppose $(n, N) = 1$ and N is a prime. Then, as in [4], p. 124, we observe that the cases $N = 2$ and $N \equiv 1 \bmod. 4$ are trivial; in the case $N \equiv 3 \bmod. 4$, we obtain for the left-hand side of (13)

$$\sum_{x \bmod. N} \left(\left(\frac{x}{N}\right)\right) \sum_{\substack{\kappa \bmod. N \\ \kappa^2 \equiv n^{-1} x \bmod. N}} 1 = \sum_{x \bmod. N} \left(\left(\frac{x}{N}\right)\right)\left(1 + \left(\frac{n^{-1}x}{N}\right)\right)$$

$$= \sum_{x=1}^{N-1} \left(\frac{n}{N}\right)\left(\frac{x}{N} - \frac{1}{2}\right)\left(\frac{x}{N}\right) = \left(\frac{n}{-N}\right) \sum_{x=1}^{N-1} \frac{x}{N}\left(\frac{x}{N}\right) = -\left(\frac{-N}{n}\right) H(-N),$$

by the quadratic reciprocity law and Dirichlet's class number formula. The case of composite N and $(n, N) = 1$ is then treated following the sketch given in [4], p. 125. To prove (13) in the general case, we observe that the function $\left(\left(\frac{n\kappa^2}{N}\right)\right)$ is periodic in κ with period $N/(n, N)$; this gives

$$\sum_{\kappa \bmod. N} \left(\left(\frac{n\kappa^2}{N}\right)\right) = (n, N) \sum_{\kappa \bmod. N/(n,N)} \left(\left(\frac{n/(n, N) \cdot \kappa^2}{N/(n, N)}\right)\right).$$

Now formula (13) follows from the preceding case. □

LEMMA 3.6. *For m, n_j as above, the following formula holds*

$$(m, n_j) \sum_{\kappa \bmod. 2m/(m,n_j)} \left(\frac{n_j\kappa^2}{4m} - \left[\frac{n_j\kappa^2}{4m}\right] - \frac{1}{2}\right)$$

$$= -\frac{(4m, n_j)}{4} Q\left(\frac{4m}{(4m, n_j)}\right) - \frac{(4m, n_j)}{2} \sum_{\substack{\Delta | 4m/(4m,n_j), \Delta < 0 \\ 4m/(4m,n_j)\Delta \text{ squarefree}}} \left(\frac{\Delta}{n_j/(4m, n_j)}\right) H(\Delta),$$

where $Q(N)$ denotes the largest integer, whose square divides N.

Proof. Applying Lemma 3.5 with $n = n_j$ and $N = 4m$, we obtain

$$\sum_{\kappa \bmod. 4m} \left(\left(\frac{n_j\kappa^2}{4m}\right)\right) = -(4m, n_j) \sum_{\substack{\Delta | 4m/(4m,n_j), \Delta < 0 \\ 4m/(4m,n_j)\Delta \text{ squarefree}}} \left(\frac{\Delta}{n_j/(4m, n_j)}\right) H(\Delta). \tag{14}$$

Observing that the function $\left(\left(\frac{n_j\kappa^2}{4m}\right)\right)$ is periodic in κ with period $2m/(m, n_j)$, we can rewrite the left-hand side of (14) to

$$2(m, n_j) \sum_{\kappa \bmod . 2m/(m,n_j)} \left(\left(\frac{n_j\kappa^2}{4m}\right)\right).$$

A short calculation shows that this equals

$$2(m, n_j)\left(\sum_{\kappa \bmod . 2m/(m,n_j)} \left(\frac{n_j\kappa^2}{4m} - \left[\frac{n_j\kappa^2}{4m}\right] - \frac{1}{2}\right) + \frac{(4m, n_j)}{4(m, n_j)} Q\left(\frac{4m}{(4m, n_j)}\right)\right),$$

which completes the proof of the lemma. □

Finally, we need

LEMMA 3.7. *If $n_j \mid m$, we have*

$$C_j = n_j \sum_{\kappa = -m/n_j+1}^{m/n_j} \left[\frac{n_j\kappa^2}{4m}\right].$$

Proof. We have

$$\beta_{j,v} = -\frac{mv^2}{n_j} + 1$$

and, by symmetry,

$$v^- = -\frac{m}{n_j} + 1, \qquad v^+ = \frac{m}{n_j}$$

for $v = 0, \ldots, n_j - 1$. Therefore we have $\max(0, -\rho + \beta_{j,v+1} - \beta_{j,v}) = 0$ for ρ running from $v^- - 2mv/n_j, \ldots, v^+ - 2mv/n_j$, and we obtain

$$C_j = \sum_{v=0}^{n_j-1} \sum_{\rho = v^- - 2mv/n_j}^{v^+ - 2mv/n_j} \left(\left[\frac{n_j\rho^2}{4m}\right] + v\rho + \frac{mv^2}{n_j}\right) = n_j \sum_{\kappa = -m/n_j+1}^{m/n_j} \left[\frac{n_j\kappa^2}{4m}\right]. \qquad \square$$

We now prove

THEOREM 3.8. *If $k \gg 0$ and $n_j \mid m$ (all j), we have the dimension formula*

$$\dim J_{k,m}^{\mathrm{cusp}}(\Gamma) = \frac{\mu_\Gamma}{6} km - \frac{\mu_\Gamma}{4} m - \sum_j \left(\frac{n_j}{4} Q\left(\frac{4m}{n_j}\right) + \frac{n_j}{2} \sum_{\substack{\Delta \mid 4m/n_j, \Delta < 0 \\ 4m/n_j \Delta \,\mathrm{squarefree}}} H(\Delta)\right).$$

Proof. Because $n_j \mid m$ (all j), we have

$$\alpha_{j,\nu} = \frac{m\nu}{n_j}(\nu - n_j) + \frac{mn_j}{6} - 1,$$

whence

$$\sum_{j,\nu} \frac{\alpha_{j,\nu}}{2}(\alpha_{j,\nu-1} - 2\alpha_{j,\nu} + \alpha_{j,\nu+1}) = \frac{m^2}{6}\sum_j \left(\frac{1}{n_j} - n_j\right).$$

By Theorem 3.4 and Lemma 3.7, we then obtain

$$\dim J_{k,m}^{\mathrm{cusp}}(\Gamma) = \frac{\mu_\Gamma}{6} km - \frac{\mu_\Gamma}{4} m + \sum_j \left(\frac{m^2}{6n_j} - m + \frac{n_j}{12} - n_j \sum_{\kappa = -m/n_j + 1}^{m/n_j} \left[\frac{n_j \kappa^2}{4m}\right]\right). \tag{15}$$

Using Lemma 3.6, we have

$$\begin{aligned} &\frac{m^2}{6n_j} - m + \frac{n_j}{12} - n_j \sum_{\kappa = -m/n_j + 1}^{m/n_j} \left[\frac{n_j \kappa^2}{4m}\right] \\ &\quad = -\frac{n_j}{4} Q\left(\frac{4m}{n_j}\right) - \frac{n_j}{2} \sum_{\substack{\Delta \mid 4m/n_j, \Delta < 0 \\ 4m/n_j \Delta \,\mathrm{squarefree}}} \left(\frac{\Delta}{1}\right) H(\Delta). \end{aligned} \tag{16}$$

By substituting (16) into (15), we derive the desired formula. □

REMARK 3.9. The formula for $\dim J_{k,m}^{\mathrm{cusp}}(\Gamma)$ proved in Theorem 3.8 is identical with the one given by Skoruppa in [10], derived by means of a trace formula (cf. [11]).

4. Concluding remarks

1. A particularly simple situation is the so-called geometric case, where $\Gamma = \Gamma(m)$ is the full congruence subgroup of level $m \geqslant 3$. Then $n_j = m$ for all j, and Lemma 3.7 shows $C_j = 0$, i.e.,

$$\mathscr{F}_{k,m} = \mathcal{O}_{X_{\Gamma(m)}}(D_{k,m})$$

and therefore

$$J_{k,m}^{\mathrm{cusp}}(\Gamma(m)) \cong H^0(X_{\Gamma(m)}, \mathcal{O}_{X_{\Gamma(m)}}(D_{k,m})).$$

2. Because we were only able to find explicit formulas for the numbers $\alpha_{j,\nu}$ and C_j in the case $n_j \mid m$ (all j), it is still an open problem to derive Skoruppa's formula for dim $J_{k,m}^{\text{cusp}}(\Gamma)$ of [10] in the more general case $m \geqslant n_j$ by geometrical methods, using Theorem 3.4 and Lemma 3.6.

3. If we allow Y_Γ to have elliptic fixed points and cusps of the second kind (still assuming $-1 \notin \Gamma$), we have in addition to take into account the following new types of singular fibres (cf. [5], §6)

$$II,\ II^*;\ III,\ III^*;\ IV,\ IV^*;\ I_{n_j}^*.$$

Using the corresponding local coordinates, which are explicitly given in [5], §8, we easily find the appropriate modifications of Theorems 2.5 and 2.6. The dimension computations are more delicate; one needs analogues of Lemma 3.1 and Proposition 3.2. We have not touched this problem.

If we allow in addition $-1 \in \Gamma$, the surface X_Γ will not be elliptic any more; the fibres of π become rational curves. Respecting the changed geometrical nature of X_Γ, we are able to find the analogues of Theorems 2.5 and 2.6. Again, the dimension computations are more difficult to handle; we have done this in the simple case $\Gamma = SL_2(\mathbb{Z})$, where we rediscover the formula for dim $J_{k,m}^{\text{cusp}}(\Gamma)$ given in [4].

4. We finish this series of remarks by mentioning the following problem: The Jacobi–Hecke operators, introduced in [4], can be interpreted as being induced by certain correspondences of the elliptic surface X_Γ on the space of sections $H^0(X_\Gamma, \mathscr{F}_{k,m})$. Although the sheaf $\mathscr{F}_{k,m}$ is not an $\mathcal{O}_{X_\Gamma}$-module, we wonder if the trace of the Jacobi–Hecke operators can be computed by means of the Lefschetz–Verdier trace formula (cf. [12], p. 133).

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