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The cohomological dimension of the quotient field of the two dimensional complete local domain


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Let $A$ be a complete Noetherian local domain with separably closed residue field $k$ and with quotient field $K$. For a prime number $p$ which is different from $\text{char}(K)$, let $\text{cd}_p(K)$ be the $p$-cohomological dimension of the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ (cf. [13], [14], here $K^{\text{sep}}$ denotes the separable closure of $K$). In this paper, we determine $\text{cd}_p(K)$ in the case $\dim(A) = 2$, $\text{char}(k) = p$ and $\text{char}(K) = 0$.

In general, if $\text{char}(k) \neq p$, a standard conjecture (Artin [2]) is that

$$\text{cd}_p(K) = \dim(A). \tag{0.1}$$

In the more delicate case where $\text{char}(k) = p$ and $\text{char}(K) = 0$, Artin suggests in [2] that the rank of the absolute differential module $\Omega^1_k = \Omega^1_{k/\mathbb{Z}}$ should be involved in $\text{cd}_p(K)$. The precise form of his conjecture in this case should be

$$\text{cd}_p(K) = \dim(A) + \dim_k(\Omega^1_k). \tag{0.2}$$

The aim of this paper is to prove (0.2) in the case $\dim(A) = 2$.

**THEOREM.** Let $A$ be a complete Noetherian two dimensional local domain of mixed characteristic $(0, p)$ with a separably closed residue field $k$, and $K$ be the quotient field of $A$. Then,

$$\text{cd}_p(K) = \dim_k(\Omega^1_k) + 2.$$
Notation

A ring means a commutative ring with a unit.

For a local ring $A$,

- $\hat{A}$: the completion of $A$ by the maximal ideal $m_A$,
- $A_{\mathfrak{p}}$: the localization of $A$ at a prime ideal $\mathfrak{p}$,
- $\kappa(\mathfrak{p})$: the residue field of $A_{\mathfrak{p}}$,
- $\Omega_A^i$: the $i$th exterior power over $A$ of the absolute differential module $\Omega_A^1$.

For a field $k$,

- $K_i(k)$: the $i$th Milnor's $K$-group of $k$ ([11]).

For an abelian group $M$ and a family $S_\lambda (\lambda \in \Lambda)$ of elements of $M$, $\langle S_\lambda; \lambda \in \Lambda \rangle$ is the subgroup of $M$ generated by $S_\lambda$ for $\lambda \in \Lambda$.

Proof of theorem. Throughout this paper, let $A$ be a complete Noetherian two dimensional local domain with a separably closed residue field $k$ of characteristic $p > 0$ and with the quotient field $K$ of characteristic 0. Without loss of generality, we assume that if $p \not= 2$ (resp. $p = 2$) $K$ contains a primitive $p$th (resp. 4th) root of unity ([15]).

First of all, we have

PROPOSITION 1.

\[ \text{cd}_p(K) \geq \dim_k(\Omega_A^1) + 2. \]

Proof. Let $\mathfrak{p}$ be a prime ideal of $A$ such that $\text{ht}(\mathfrak{p}) = 1$ and $\text{char}(\kappa(\mathfrak{p})) = 0$, and let $K_{\mathfrak{p}}$ be the quotient field of the henselization of the local ring of $A$ at $\mathfrak{p}$. Then we have

\[ \text{cd}_p(K) \succeq \text{cd}_p(K_{\mathfrak{p}}) \]
\[ = \text{cd}_p(\kappa(\mathfrak{p})) + 1 \]
\[ = \dim_k(\Omega_A^1) + 2. \]

Hence it remains to prove that $\text{cd}_p(K) \leq \dim_k(\Omega_A^1) + 2$. In the rest of this paper, we assume $\dim_k(\Omega_A^1) < \infty$. Let $r = \dim_k(\Omega_A^1)$ (so $[k:k^p] = p^r$).

We fix an algebraic closer $\overline{K}$ of $K$, and $r$ elements $b_1, b_2, \ldots, b_r$ of $A$ such that the residue classes $\overline{b_i}$ of $b_i$ ($1 \leq i \leq r$) form a $p$-basis of $k$. Then we can pick up elements $\{b_{i,j}; \ 1 \leq i \leq r, j = 0, 1, 2, \ldots \}$ of $\overline{K}$ which satisfy the following conditions:
For integers \( n \) (\( n = 0, 1, 2, \ldots, \infty \)), we define extensions \( A^{(n)} \) (resp. \( K^{(n)} \)) of \( A \) (resp. \( K \)) by

\[
A^{(n)} = A[b_{i,n}; 1 \leq i \leq r]
\]

\[
A^{(\infty)} = \bigcup_{n=0}^{\infty} A^{(n)}
\]

\[
K^{(n)} = K(b_{i,n}; 1 \leq i \leq r)
\]

\[
K^{(\infty)} = \bigcup_{n=0}^{\infty} K^{(n)}.
\]

**PROPOSITION 2.**

\[\text{cd}_p(K) \leq r + 3.\]

**Proof.** In Lemma 1 below, we shall prove that \( A^{(\infty)} \) is an excellent henselian local domain. Since the residue field of \( A^{(\infty)} \) is algebraically closed, \( \text{cd}_p(K^{(\infty)}) = 2 \) (cf. [12] §5 Th. the excellence of \( A^{(\infty)} \) is needed here). Let \( \zeta_p = \) be a subgroup of \( K^* \) which consists of all roots of unity of \( p \)-primary orders. Then \( \text{cd}_p(K^{(\infty)}(\zeta_p)) \leq 2 \) ([14]).

On the other hand, the field \( K^{(\infty)}(\zeta_p) \) is a Galois extension of \( K \) and the Galois group of \( K^{(\infty)}(\zeta_p)/K \) is isomorphic to \( \mathbb{Z}_{p}^{r+1} \) (\( \mathbb{Z}_p \) is the ring of \( p \)-adic integers). Then we have inequalities

\[
\text{cd}_p(K) \leq \text{cd}_p(\mathbb{Z}_{p}^{r+1}) + \text{cd}_p(K^{(\infty)}(\zeta_p))
\]

\[
\leq r + 3. \quad ([14])
\]

**LEMMA 1.** \( A^{(\infty)} \) is an excellent henselian two dimensional local ring.

**Proof.** By [9], \( A \) is finite over \( R[[X]] \) where \( R \) is a complete discrete valuation ring with mixed characteristic containing \( b_1, b_2, \ldots, b_r \), whose residue field is the same as that of \( A \), and \( X \) is a variable. So we may assume that \( A = R[[X]], b_1, b_2, \ldots, b_r \in R \). We define rings \( R^{(n)} = R[b_{i,n}; 1 \leq i \leq r] \) for integers \( n \geq 0 \) and \( R^{(\infty)} = \bigcup_{n=0}^{\infty} R^{(n)} \), and fix a prime element \( \pi \) of \( R \).

First, we will prove that \( A^{(\infty)} \) is Noetherian.

It is enough to show that every prime ideal \( \mathcal{P} \) of \( A^{(\infty)} \) is finitely generated ([9]). Since \( A^{(\infty)} \) is a two dimensional ring, every prime ideal \( \neq (0) \) is either maximal or of height one. Assume \( \mathcal{M} \) is a maximal ideal of \( A^{(\infty)} \). Then \( \mathcal{M} \cap R^{(n)}[[X]] \) is a
maximal ideal of $R^{(n)}[[X]]$ for all integers $n \geq 0$. Hence $\mathcal{M} \cap R^{(n)}[[X]] = (\pi, X)$ for all $n \geq 0$, and this implies $\mathcal{M} = (\pi, X)$. On the other hand, if $\mathcal{P}$ is a prime ideal of $A^{(\infty)}$ of height one, $\mathcal{P} \cap R^{(n)}[[X]] = (\pi)$ or $(X^m + a_1X^{m-1} + \cdots + a_m)$ ($m$ is a positive integer and $a_i$ are elements of the maximal ideal of $R^{(n)}$ for $1 \leq i \leq r$) for all integers $n \geq 0$. When $\mathcal{P} \cap R[[X]] = (\pi)$, $\mathcal{P} \cap R^{(n)}[[X]] = (\pi)$ also. This implies $\mathcal{P} = (\pi)$. When $\mathcal{P} \cap R[[X]] \neq (\pi)$, the degree $m$ of the above polynomial becomes stable for sufficiently large integers $n$. So $\mathcal{P}$ is generated by an element which generates $\mathcal{P} \cap R^{(n)}[[X]]$ for integers $n > 0$. Thus $A^{(\infty)}$ is Noetherian.

Secondly, recall that, a Noetherian local ring $S$ is excellent, if and only if, $S$ is a $G$-ring and universally catenary (cf. [9] Ch. 13, 34).

It is easily deduced that $A^{(\infty)}$ is universally catenary from the fact that $A^{(\infty)}$ is the union of subrings which are finite over the excellent ring $A$. And $A^{(\infty)}$ is a $G$-ring when $\hat{A}^{(\infty)} \otimes_{A^{(\infty)}} L$ is regular for any prime ideal $\mathcal{P}$ of $A^{(\infty)}$ and any finite extension $L$ of $K(\mathcal{P})$. The regularity is easy and we omit the proof.

We have shown that $\text{cd}_p(K) = r + 2$ or $r + 3$. To prove that $\text{cd}_p(K) = r + 2$, it is sufficient to show that the Galois cohomology groups $H^{r+3}(L, \mathbb{Z}/p\mathbb{Z})$ vanish for all finite extension fields $L$ over $K$.

**Lemma 2.** The cohomology symbol map (cf. [6] II)

$$h^{r+3}_K; K_{r+3}(K)/p \rightarrow H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

is surjective.

**Proof.** In the first place, we consider the fact ($\ast$).

($\ast$) Let $k$ be a field, $S$ a Galois extension of $k$ of infinite degree, $p$ a prime number which is invertible in $k$, and $q \geq 0$ an integer. Suppose that $\text{cd}_p(\text{Gal}(S/k)) \leq q$ and $\text{cd}_p(S) \leq 2$, and that for any open subgroup $J$ of $\text{Gal}(S/k)$, the cup product

$$\otimes^g H^1(J, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^g(J, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Then, $h^{q+2}_K$ is surjective.

Using the fact that $h^2_k$ is surjective for any field $k$ ([10]), the arguments in the proof of Proposition 3 of [6] II, §1.3 can be used to prove ($\ast$) (we replace $\text{cd}_p(S) \leq 1$ (resp. $h^{q+1}_k$) by $\text{cd}_p(S) \leq 2$ (resp. $h^{q+2}_k$)).

We apply ($\ast$) to $k = K$, $S = K^{(\infty)}(\zeta_{p^\infty})$ and $q = r + 1$. From the proof of Proposition 2, $\text{cd}_p(K^{(\infty)}(\zeta_{p^\infty})) \leq 2$, $\text{cd}_p(\text{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K)) = r + 1$ and $\text{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K) \cong \mathbb{Z}_p^{r+1}$. Any open subgroup of $\mathbb{Z}_p^{r+1}$ is isomorphic to $\mathbb{Z}_p^{r+1}$ and the cup product

$$\otimes^{r+1} H^1(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{r+1}(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z})$$
is surjective. Hence the assumption of (*) is satisfied. This lemma is proved.

We consider the condition.

(**) A is regular and there exist two elements u and v of A which generate the maximal ideal of A, such that p is invertible in A[1/uv].

For w = u or v, let \( \mathfrak{f}_w = (w) \), \( \mathcal{A}_w = A/\mathfrak{f}_w \) and

\[
f_w = \begin{cases} 
(p/p - 1)\text{ord}_w(p) & \text{if char } \kappa(\mathfrak{f}_w) = p \\
0 & \text{if char } \kappa(\mathfrak{f}_w) = 0.
\end{cases}
\]

We distinguish two cases;

Case (I) \( \text{char}(\kappa(\mathfrak{f}_u)) = \text{char}(\kappa(\mathfrak{f}_v)) = p \)

Case (II) \( \text{char}(\kappa(\mathfrak{f}_u)) = 0, \text{char}(\kappa(\mathfrak{f}_v)) = p \).

**Lemma 3.** Assume (**). Let

\[
\Delta = \left< \left\{ a, b_1, \ldots, b_{r+2}; \ b_s \in A[1/uv]^* \text{ for } 1 \leq s \leq r + 2 \text{ and } \right. \right.
\]
\[
a \in 1 + uvA \text{ in the case (I)} \]
\[
\left( \text{resp. } a \in 1 + vA \text{ in the case (II)} \right) \]
\[
\left. \right> \text{ in } K_{r+3}(K)/p.
\]

Then \( \Delta = 0 \).

*Proof.* For integers \( i \geq 0 \) and \( j > 0 \), we define

\[
\Delta_{i,j} = \left< \left\{ 1 + au^j v^j, b_1, \ldots, b_r, c, d; \ a \in A, b_s \in A^* \text{ for } 1 \leq s \leq r \text{ and } c, d \in A[1/uv]^* \right. \right.
\]
\[
\left. \left. \right> \text{ in } K_{r+3}(K)/p.
\]

In the case (I) (resp. (II)), we deduce \( \Delta = 0 \) from the following facts,

1. \( \Delta = \Delta_{1,1} \) (resp. \( \Delta = \Delta_{0,1} \))
2. if \( 0 \leq i \leq f_u, 0 < j < f_v \) and either \( p \not| i \) or \( p \not| j \), we have \( \Delta_{i,j} = \Delta_{i,j+1} \)
3. \( \Delta_{f_u, f_v} = 0 \).

*Proof of (1).* This is easy and we omit the proof.

*Proof of (2).* We can define the homomorphism

\[
\chi_l; \quad \Omega_{A_v}^{+2} \to \Delta_{i,j}/\Delta_{i,j+1}
\]

by

\[
\frac{d \bar{a}_i}{\bar{a}_1} \wedge \cdots \wedge \frac{d \bar{a}_{r+2}}{\bar{a}_{r+1}} \mapsto \{ 1 + au^j v^j, a_1, \ldots, a_{r+2} \}.
\]
If $p \nmid j$, by the following calculation in $K_2(K)$, the map $\chi_1$ is surjective.

$$j\{1 + au^j v, v\} = \{1 + au^j v, -au^j\}$$

$$\{1 + au^j v, 1 + bv\} = \{1 + au^j v, 1 - abu^j v + 1\} - \{1 + bv, 1 - abu^j v + 1\}$$

for $a \in A^*$, $b \in A$ and $h \geq i$.

In the case $p | j$ and $p | i$, we can define the homomorphism

$$\chi_2: \Omega^{r+1}_{A_i} \to \Delta_{i,j}/\Delta_{i,j+1}$$

by

$$\frac{d\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{d\bar{a}_{r+1}}{\bar{a}_{r+1}} \mapsto \{1 + au^j v, a_1, \ldots, a_{r+1}, v\}.$$  

Then every element of $\Delta_{i,j}/\Delta_{i,j+1}$ is a sum of elements of the images of $\chi_1$ and $\chi_2$.

On the other hand, the equalities $\text{char}(K(\mu_p)) = p$ and $[\kappa(\mu_p): K(\mu_p)^{p^r}] = p^{r+1}$ imply $\Omega^{r+2}_{A_i} = 0$.

$$(\text{[7] II}). \text{ Then we have } \Delta_{i,j} = \Delta_{i,j+1}.$$  

Proof of (3). Let $c$ be the element of $A^*$ such that $p = cu^{(p-1)}/pv^{(p-1)}/p.$ We can take a solution $x \in A$ of the equation $X^p + cX - a = 0$ for any $a \in A$. Then,

$$1 + au^j v + c = (1 + xu^{(p-1)}pv^{(p-1)p})^p.$$  

Thus the proof of Lemma 3 is complete.

**LEMMA 4.** Assume (**). Let

$$\Delta' = \langle\{a, b_1, \ldots, b_{r+2}\}; a \in 1 + m_A \text{ and } b_i \in A[1/uv]^* \text{ for } 1 \leq j \leq r + 2\rangle$$

in $K_{r+3}(K)/p$.

where $m_A$ is the maximal ideal of $A$.

Then $\Delta' = 0$.

Proof. As is easily seen, $1 + m_A$ is generated by elements of the form $1 - aw$ ($a \in A^*$) for $w = u$ or $v$. From this, we see that $\Delta'$ is generated by elements of the forms $\{1 - aw, b_1, \ldots, b_r, c, d\}$ with $a, b_1, \ldots, b_r$ and $c \in A^*$, $d \in A[1/uv]^*$ and $w = u$ or $v$ such that $b_1, \ldots, b_{r-1}$ and $b_r$ form a $p$-basis of $k$.  


Hence it suffices to prove
\[ \{ 1 - au, b_1, \ldots, b_r, c, d \} \in pK_{r+3}(K) \]
for \( a, b_1, \ldots, b_r, c \) and \( d \) as above. Let
\[
B = A[\{(au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}\}] \subseteq \bar{K}
\]
and \( L \) be the quotient field of \( B \). Since \( B/vB = (A/vA)^{1/p} \), there exist elements \( c' \in B^* \) and \( c'' \in B \) such that \( c = (c')^p(1 + c''v) \). We apply Lemma 3 to \( B \),
\[ \{ 1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, 1 + c''v, d \} \in pK_{r+3}(L). \]
With \( N_{L/K} \) denoting the norm map, we have
\[ \{ 1 - au, b_1, \ldots, b_r, c, d \} = N_{L/K}(\{ 1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, c, d \}) \]
\[ \in N_{L/K}(pK_{r+3}(L)) \subseteq pK_{r+3}(K). \]

**LEMMA 5.** Assume (**). Let
\[ \Delta'' = \langle \{ a_1, a_2, \ldots, a_{r+3} \}; a_j \in A[1/uv]^* \text{ for } 1 \leq j \leq r + 3 \rangle \]
in \( K_{r+3}(K)/p \).

Then \( \Delta'' = 0 \).

**Proof.**
\[ \Delta'' = \langle \{ a_1, \ldots, a_{r+1}, b, c \}; a_j \in A^* \text{ for } 1 \leq j \leq r + 1, b, c \in A[1/uv]^* \rangle \]
Hence it suffices to prove
\[ \{ a_1, \ldots, a_{r+1}, b, c \} \in pK_{r+3}(K) \]
for \( a_1, \ldots, a_{r+1}, b \) and \( c \) as above. Since \( \tilde{a}_i = a_i \mod m_A (1 \leq i \leq r + 1) \) cannot be \( p \)-independent, there exists \( s \) such that \( 1 \leq s \leq r \) and \( \tilde{a}_{s+1} \in k^p(a_1, \ldots, a_s) \). Let
\[
B = A[\{ a_1^{1/p}, \ldots, a_s^{1/p} \}] \subseteq \bar{K}
\]
and \( L \) be the quotient field of \( B \). Since the residue field of \( B \) contains \( \tilde{a}_{s+1}^{1/p} \), there exist elements \( d' \in B^* \) and \( d'' \in m_B \) (\( m_B \) is the maximal ideal of \( B \)) such that
\[ a_{s+1} = (a')^p (1 + a'). \] By applying Lemma 4 to \( B \), we have

\[
\{a_1, \ldots, a_{r+1}, b, c\} = N_{L/K}(\{a_1^{1/p}, \ldots, a_s^{1/p}, a_{s+1}, \ldots, a_{r+1}, b, c\})
\in N_{L/K}(pK_{r+3}(L)) \subseteq pK_{r+3}(K).
\]

We follow the method of [12] §5.

**Lemma 6.** Let \( \mathfrak{x} \to \text{Spec}(A) \) be a proper birational morphism with regular such that \( Y = \mathfrak{x} \otimes_A A/\mathfrak{m}_A \) is a reduced divisor with normal crossing on \( \mathfrak{x} \) ([1], [5]). Then,

\[
H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z})
\]

where \( Y_0 \) denotes the set of closed points of \( Y \) and for each \( x \in Y_0 \), \( K_x \) denotes the quotient field of the henselization of \( \mathcal{O}_{K_X} \).

**Proof.** Let \( \lambda : \text{Spec}(K) \to \mathfrak{x} \) be the inclusion map and put \( \mathfrak{y} = R\lambda^*(\mathbb{Z}/p\mathbb{Z}) \). By the proper base change theorem, we have

\[
H^q_\mathfrak{y}(Y, i^*\mathfrak{y}) = H^q(K, \mathbb{Z}/p\mathbb{Z})
\]

where \( i : Y \to \mathfrak{x} \) is the inclusion map. From this, we obtain an exact sequence

\[
\to \bigoplus_{x \in Y_0} H^q_\mathfrak{y}(Y, i^*\mathfrak{y}) \to H^q(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{x \in Y_0} H^q(K_x, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{x \in Y_0} H^{q+1}_x(Y, i^*\mathfrak{y}) \to
\]

where \( \eta \) ranges over all generic points of \( Y \) and \( K_\eta \) denotes the quotient field of the henselization of \( \mathcal{O}_{X,\eta} \). For each \( x \in Y_0 \), we have an exact sequence

\[
\to H^q_\mathfrak{y}(Y, i^*\mathfrak{y}) \to H^q(K_x, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_v H^q(K_v, \mathbb{Z}/p\mathbb{Z}) \to H^{q+1}_x(Y, i^*\mathfrak{y}) \to
\]

where \( v \) ranges over all generic points of the henselization of \( \text{Spec}(\mathcal{O}_{Y,x}) \), and \( K_v \) denotes the quotient field of the henselization of the discrete valuation ring of \( K_x \) corresponding to \( v \).

Since \( \text{cd}_p(K_\eta) = \text{cd}_p(K_v) = r + 2 \) ([8]), we have

\[
H^{r+3}(K_\eta, \mathbb{Z}/p\mathbb{Z}) = H^{r+3}(K_v, \mathbb{Z}/p\mathbb{Z}) = 0.
\]
On the other hand, from the classical approximation theorem for a finite family of discrete valuation on $K^*$ and $K_{x}^*$, the maps

$$K_{r+2}(K) \to \bigoplus_{\eta} K_{r+2}(K_{\eta})/p \quad \text{and} \quad K_{r+2}(K_{x}) \to \bigoplus_{v} K_{r+2}(K_{v})/p$$

are subjective. By [3] §5, the cohomological symbol maps

$$K_{r+2}(K_{\eta})/p \to H^{r+2}(K_{\eta}, \mathbb{Z}/p\mathbb{Z})$$

and

$$K_{r+2}(K_{v})/p \to H^{r+2}(K_{v}, \mathbb{Z}/p\mathbb{Z})$$

are subjective. Hence the maps

$$H^{r+2}(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\eta} H^{r+2}(K_{\eta}, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+2}(K_{x}, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v} H^{r+2}(K_{v}, \mathbb{Z}/p\mathbb{Z})$$

are also subjective.

Putting these things together,

$$\bigoplus_{x \in \gamma_{0}} H^{r+3}_{x}(Y, i^{*}\mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+3}_{x}(Y, i^{*}\mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K_{x}, \mathbb{Z}/p\mathbb{Z}).$$

These isomorphisms induce the isomorphism of Lemma 6.

**PROPOSITION 3.** For $r = \text{dom}(\Omega^{k}_{x})$,

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

**Proof.** By Lemma 2 and 6, we have

$$\bigoplus_{x \in \gamma_{0}} K_{r+3}(K_{x})/p \to \bigoplus_{x \in \gamma_{0}} H^{r+3}(K_{x}, \mathbb{Z}/p\mathbb{Z}) \quad \uparrow \quad i^{\dagger}$$

$$K_{r+3}(K)/p \to H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}).$$
For any family of fixed elements $a_1, a_2, \ldots, a_{r+3} \in K^*$, we can take $\mathfrak{x}$ such that the union $Z$ of $Y$ with the supports of the divisor of $a_1, \ldots, a_{r+2}$ and $a_{r+3}$ on $\mathfrak{x}$ is normally crossing divisor ([5]). Then, by Lemma 5,

$$\{a_1, a_2, \ldots, a_{r+3}\} \in pK_{r+3}(K_\mathfrak{x})$$

for any $x \in Y_0$. This shows that

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$  

We are now in the position to complete the proof of our theorem. By Proposition 1 and 2, $\text{cd}_p(K) = r + 2$ or $r + 3$. For any finite extension field $K'$ over $K$,

$$H^{r+3}(K', \mathbb{Z}/p\mathbb{Z}) = 0$$

by Proposition 3. Hence $\text{cd}_p(K) = r + 2$ ([14]).

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References

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