COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 80, nº 1 (1991), p. 75-84
<http://www.numdam.org/item?id=CM_1991__80_1_75_0>

Compositio Mathematica **80**: 75–84, 1991.

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Received 8 March 1990; accepted 18 December 1990

1. Introduction

The object of this paper is to define an adelic capacity theory for varieties over global fields. A theory of this kind was developed for P^1 by Cantor [1], and Cantor's work was generalized to all curves by Rumely [6]. The approach we will take here for arbitrary varieties is very different and was suggested by recent work of Gillet and Soulé [4]. We will begin by discussing this approach in the classical case of a compact subset of **C**.

Let E be a non-empty compact subset of **C** which is stable under complex conjugation. The capacity $\gamma(E)$ of E may be defined in several ways (cf. [3], [6, p. 7]). One is the transfinite diameter of E:

$$\gamma(E) = \lim_{n\to\infty} \max_{z_1,\ldots,z_n\in E} \prod_{i\neq j} |z_i - z_j|^{1/n(n-1)} \tag{1.1}$$

where the maximum is over all choices of n points of E. Another is the Chebyshev constant of E:

$$\gamma(E) = \lim_{n\to\infty} \min_{\substack{p(z)\in \mathbf{C}[z]\text{monic,}\\ \deg(p(z))=n}} \sup\{|p(z)|^{1/n} : z\in E\} \tag{1.2}$$

where the minimum is over all monic polynomials of degree n.

The following theorem of Fekete and Szegö connects $\gamma(E)$ to arithmetic.

THEOREM 1.1 (Fekete–Szegö [6]):

(A) *If* $\gamma(E) < 1$, *there is an open set* U *containing* E *such that there are only finitely many algebraic integers* α *which have all of their conjugates in* U, *and*
(B) *if* $\gamma(E) > 1$, *then every open* U *containing* E *contains infinitely many such* α.

In Section 3 we prove the following new definition for $\gamma(E)$.

THEOREM 1.2. *For* $0 \leqslant n \in \mathbf{Z}$ *and* $a = (a_0, \ldots, a_n) \in \mathbf{R}^{n+1}$ *let* $f_a(z)$ *be the polynomial* $a_0 + a_1 z + \cdots + a_n z^n$. *Let* $F_n(E)$ *be the set of* $a \in \mathbf{R}^{n+1}$ *such that*

* Partially supported by NSF grant DMS8814768 and NSA grant MDA90490H4033.

$|f_a(z)| < 1$ *for all* $z \in E$. *Let* ψ_n *be the usual Euclidean Haar measure on* $\mathbf{R}^{n+1}$. *Then*

$$\lim_{n \to \infty} n^{-2} 2 \log \psi_n(F_n(E)) = -\log \gamma(E). \tag{1.3}$$

Using Theorem 1.2, one may prove part A of Theorem 1.1 in the following way. The set $F_n(E)$ is a convex subset of $\mathbf{R}^{n+1}$ which is symmetric about the origin, and $F_n(E)$ is open because E is compact. If $\gamma(E) < 1$, then (1.3) shows that $F_n(E)$ has large volume for large n. Minkowski's lemma implies that for large n, $F_n(E)$ contains a non-zero integral vector $a \in \mathbf{Z}^{n+1}$. Then $U = \{z \in \mathbf{C}: |f_a(z)| < 1\}$ is an open set containing E. Furthermore, if α is an algebraic integer which has all of its conjugates in U, then $f_a(\alpha)$ is an algebraic integer having all of its conjugates inside the unit disk. Hence $f_a(\alpha) = 0$, so α is one of the finitely many roots of $f_a(z)$.

The virtue of this approach to capacity theory is that unlike other ones, it generalizes readily to study sets of Galois conjugate algebraic points on varieties over a global field K. We will consider v-adic constraints on the locations of such points as v ranges over all of the places of K. Our main result is a generalization of part A of Theorem 1.1. To proceed further we must introduce some notation.

Let X be a regular projective variety of dimension d over Spec(K). Let K_v be the completion of K at the place v. Define $\mathbf{C}_v$ to be the completion of an algebraic closure of K_v. For each v let E_v be a subset of $X(\mathbf{C}_v)$, and let $\mathbf{E} = \Pi_v E_v$. By a thickening of $\mathbf{E}$ we will mean the product $\mathbf{U} = \Pi_v U_v$ of open neighborhoods U_v of E_v in $X(\mathbf{C}_v)$. Let $\bar{K}$ be an algebraic closure of K. We will say that an embedding $\sigma: \bar{K} \to \mathbf{C}_v$ is over v if σ extends the natural inclusion of K into K_v. Define $X(\bar{K}; \mathbf{E})$ to be the set of all $P \in X(\bar{K})$ such that $\sigma(P) \in E_v$ for all v and all embeddings $\sigma: \bar{K} \to \mathbf{C}_v$ over v. Let $\bar{X} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$.

Let Div(X) be the group of Weil divisors on X, and let $\mathrm{Div}_{\mathbf{R}}(X) = \mathrm{Div}(X) \otimes_{\mathbf{Z}} \mathbf{R}$. We may write an element D of $\mathrm{Div}_{\mathbf{R}}(X)$ in a unique way as a real linear combination $\Sigma r_i D_i$ of irreducible Weil divisors D_i of X. Let $[r]$ be the greatest integer less than or equal to the real number r, and define $[D] = \Sigma [r_i] D_i$. For $n \in \mathbf{Z}$ define $H^0(L^{\otimes n}) = H^0(X, O_X([nD]))$, considered as a finite dimensional vector space over K inside the function field $K(X)$ of X. Thus the non-zero functions in $H^0(L^{\otimes n})$ are those $f \in K(X)$ such that $\mathrm{div}(f) + nD$ is a non-negative real combination of irreducible Weil divisors on X. For each place v of K let $H^0(L^{\otimes n})_v = H^0(L^{\otimes n}) \otimes_K K_v$. Let $\mathbf{A}_K$ be the adele ring of K. Define $H^0(L^{\otimes n})_{\mathbf{A}} = H^0(L^{\otimes n}) \otimes_K \mathbf{A}_K$, which we will view as a subgroup of the direct product $\Pi_v H^0(L^{\otimes n})_v$. Let $|\ |_v$ be the continuous extension to $\mathbf{C}_v$ of the normalized absolute value on K_v. For $f_v \in H^0(L^{\otimes n})_v$ define $\sup(f_v, E_v) = \sup\{|f_v(z)|_v : z \in E_v\}$, where the supremum of the empty set of real numbers is defined to be $-\infty$.

For each place v of K, let $F_n(E_v)$ be the subset of $f_v \in H^0(L^{\otimes n})_v$ for which the following is true:

$$\sup(f_v, E_v) \leqslant 1 \text{ (resp. } \sup(f_v, E_v) < 1) \text{ if } v \text{ is non-archimedean (resp. if } v \text{ is archimedean).} \tag{1.4}$$

Define $\mathbf{F}_n = \mathbf{F}_n(\mathbf{E})$ for the adelic set $\mathbf{E} = \Pi_v E_v$ to be $\mathbf{F}_n = H^0(L^{\otimes n})_{\mathbf{A}} \cap \Pi_v F_n(E_v)$.

DEFINITION 1.3. The adelic set $\mathbf{E}$ has a capacity relative to D if for all sufficiently large n, $\mathbf{F}_n$ is an open subset of $H^0(L^{\otimes n})_{\mathbf{A}}$.

We will assume in what follows that $\mathbf{E}$ has a capacity relative to D. In Section 4 we discuss necessary and sufficient conditions for this to be so. A sufficient condition is that there is a very ample divisor D' whose support contains that of D such that E_v has positive v-adic distance from D' for all v and distance at least 1 for almost all non-archimedean v. In this case the points of $X(\bar{K}; \mathbf{E})$ lie in $X - D'$ and have affine coordinates with bounded denominators and bounded archimedean absolute values.

The natural embedding $H^0(L^{\otimes n}) \to H^0(L^{\otimes n})_{\mathbf{A}}$ is discrete with respect to the topology of $H^0(L^{\otimes n})_{\mathbf{A}}$. Thus the choice of a Haar measure ψ on $H^0(L^{\otimes n})_{\mathbf{A}}$ induces a Haar measure on $H^0(L^{\otimes n})_{\mathbf{A}}/H^0(L^{\otimes n})$, which will also be denoted by ψ. If $\mathbf{F}_n$ is open and $\psi(\mathbf{F}_n)$ is finite define

$$\lambda_n(\mathbf{E}, D) = \psi(\mathbf{F}_n)/\psi(H^0(L^{\otimes n})_{\mathbf{A}}/H^0(L^{\otimes n})). \tag{1.5}$$

Since ψ is unique up to a positive real scalar, $\lambda_n(\mathbf{E}, D)$ does not depend on the choice of ψ. If $\mathbf{F}_n$ is not open or $\psi(\mathbf{F}_n)$ is infinite, define $\lambda_n(\mathbf{E}, D) = +\infty$.

DEFINITION 1.4. Suppose $\mathbf{E}$ has a capacity with respect to D. The inner sectional capacity of $\mathbf{E}$ with respect to D is

$$S\gamma(\mathbf{E}, D)^- = \exp\left\{\liminf_{n \to \infty} - n^{-(d+1)}(d+1)! \log \lambda_n(\mathbf{E}, D)\right\}. \tag{1.6}$$

The outer sectional capacity of $\mathbf{E}$ with respect to D is

$$S\gamma(\mathbf{E}, D)^+ = \exp\left\{\limsup_{n \to \infty} - n^{-(d+1)}(d+1)! \log \lambda_n(\mathbf{E}, D)\right\}. \tag{1.7}$$

If $S\gamma(\mathbf{E}, D)^- = S\gamma(\mathbf{E}, D)^+$, then the sectional capacity $S\gamma(\mathbf{E}, D)$ of $\mathbf{E}$ is $S\gamma(\mathbf{E}, D)^-$; otherwise $S\gamma(\mathbf{E}, D)$ is undefined.

We can now generalize part A of Theorem 1.1 to varieties.

THEOREM 1.5. *If $S\gamma(\mathbf{E}, D)^- < 1$ then there is a thickening $\mathbf{U}$ of $\mathbf{E}$ such that $X(\bar{K}; \mathbf{U})$ is not Zariski dense in $\bar{X}$.*

This theorem is proved in Section 2 using a Minkowski argument (Lemma 2.3) to construct for all ideles α of K and arbitrarily large n a non-zero element f of $\alpha\mathbf{F}_n(\mathbf{E}) \cap H^0(L^{\otimes n})$. Lemma 2.3 takes the place of the explicit construction of such f which is used in the Cantor–Rumely capacity theory for X of dimension 1 (cf. [6, Theorem 6.2.1]).

We now state two results which are proved in Section 3 by modifying the arguments showing Theorem 1.2.

Let $K = \mathbf{Q}$ and let X be the product $(P^1)^d$ of d copies of P^1. Let $\mathrm{pr}_k\colon X \to P^1$ be projection onto the kth factor of X. Let p_∞ be the point at infinity of $P^1(\mathbf{Q})$. Let $v(\infty)$ be the infinite place of $\mathbf{Q}$. Suppose that $E_1, \ldots, E_d$ are non-empty compact subsets of $\mathbf{C} = A^1(\mathbf{C}_{v(\infty)})$ which are stable under complex conjugation. Define $E_{v(\infty)}$ to be the subset $E_1 \times \cdots \times E_d$ of $X(\mathbf{C}_{v(\infty)})$. For each non-archimedean place v of $\mathbf{Q}$ let $\mathbf{O}_v = \{z \in \mathbf{C}_v : |z|_v \leqslant 1\}$ and let $E_v = (\mathbf{O}_v)^d \subseteq (A^1(\mathbf{C}_v))^d$. Define $\mathbf{E} = \Pi_v E_v$. Theorem 1.2 is equivalent to the special case $d = r_1 = 1$ of the following result.

THEOREM 1.6. *Suppose* $D = \Sigma_{k=1}^d r_k \,\mathrm{pr}_k^*(p_\infty)$ *for some positive real numbers* r_k. *If* $\gamma(E_k) = 0$ *for some* k *then* $S\gamma(\mathbf{E}, D) = 0$. *Otherwise*

$$\log S\gamma(\mathbf{E}, D) = \frac{(d+1)!}{2} r_1 \cdots r_d \sum_{k=1}^{d} r_k \log \gamma(E_k).$$

Now suppose $K = \mathbf{Q}$ and $X = P^1$. Let z be an affine coordinate for A^1 and let p_∞ and p_0 be the points $z = 0$ and $z = \infty$ of $P^1(\mathbf{Q})$. For v a finite place of $\mathbf{Q}$, let $E_v = \mathbf{O}_v^* = \{z \in \mathbf{C}_v : |z|_v = 1\} \subseteq A^1(\mathbf{C}_v)$. Let r be a positive real number, and let $E(r)$ be the subset $\{z \in \mathbf{C} : |z|_v = r\}$ of $A^1(\mathbf{C}_{v(\infty)}) = \mathbf{C}$. Define $\mathbf{E}(r) = \Pi_{v\,\mathrm{finite}} E_v \times E(r)$.

THEOREM 1.7. *Let* a, b *and* r *be positive real numbers and let* $D = ap_0 + bp_\infty$. *Then* $\log S\gamma(\mathbf{E}(r), D) = (b^2 - a^2)\log(r)$.

Theorem 1.7 shows in particular that there can be linear equivalent effective Weil divisors D_1 and D_2 such that for suitable $\mathbf{E}$, $S\gamma(\mathbf{E}, D_1) < 1 < S\gamma(\mathbf{E}, D_2)$.

We will end this introduction with some questions and remarks.

REMARK 1.8. It is not difficult to show that if $S\gamma(\mathbf{E}, rD)$ is well-defined for all rational $r > 0$ then $\log S\gamma(\mathbf{E}, rD) = r^{d+1} \log S\gamma(\mathbf{E}, D)$ for all such r. We ask when the following refinement of this statement is true. Suppose that $D = r_1 D_1 + \cdots + r_t D_t$ for some irreducible Weil divisors $D_1, \ldots, D_t$ and some positive real numbers r_i. Suppose that for all $0 < a_1, \ldots, a_t \in \mathbf{Z}$ such that $\sup_{i \neq j} |a_i/a_j - r_i/r_j|$ is small, $a_1 D_1 + \cdots + a_t D_t$ is ample. Suppose further that $S\gamma(\mathbf{E}, D')$ is well-defined for each $D' = r'_1 D_1 + \cdots + r'_t D_t$ such that the vector $r' = (r'_1, \ldots, r'_t)$ is sufficiently close to $r = (r_1, \ldots, r_t)$. Is $\log S\gamma(\mathbf{E}, D')$ then given by a homogeneous form of degree $(d+1)$ in the $r'_1, \ldots, r'_t$ for r' close to r? This is so in the cases treated in Theorems 1.6 and 1.7 and in the function field cases treated in [2].

REMARK 1.9. When $d = 1$ a homogeneous form of degree $d+1 = 2$ does arise naturally in the capacity theory of Cantor and Rumely. Suppose $x_1, \ldots, x_w$ are distinct points of $X(\bar{K})$, and that the set $\mathscr{X} = \{x_1, \ldots, x_w\}$ is stable under

$\mathrm{Aut}(\bar{K}/K)$. If $\mathbf{E}$ and $\mathscr{X}$ satisfy suitable hypotheses, Cantor and Rumely define in [1, 6] a $w \times w$ real symmetric matrix $\Gamma(\mathbf{E}, \mathscr{X})$ such that their capacity $\gamma(\mathbf{E}, \mathscr{X})$ is given by $\exp(-\mathrm{val}(\Gamma(\mathbf{E}, \mathscr{X})))$, where $\mathrm{val}(\Gamma(\mathbf{E}, \mathscr{X}))$ is the value of $\Gamma(\mathbf{E}, \mathscr{X})$ as a matrix game. In a preprint of this paper we asked whether the function $\log S\gamma(\mathbf{E}, r_1(x_1) + \cdots r_w(x_w))$ of the positive real variables $r_1, \ldots, r_w$ is equal to the quadratic form defined by the matrix $-\Gamma(\mathbf{E}, \mathscr{X})$. This is so in Theorem 1.7 and when $d = 1$ in Theorem 1.6. Rumely has proved it is so in general:

THEOREM 1.10 (Rumely [7]). *Suppose that $\Gamma(\mathbf{E}, \mathscr{X})$ is well defined and that $D = r_1(x_1) + + r_w(x_w)$ for some positive real numbers $r_1, \ldots, r_w$. Suppose furthermore that D is stable under* $\mathrm{Gal}(\bar{K}/K)$, *in the sense that $r_i = r_j$ if $x_i = \sigma x_j$ for some $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Then*

$$\log S\gamma(\mathbf{E}, D) = -(r_1, \ldots, r_w)\Gamma(\mathbf{E}, \mathscr{X})(r_1, \ldots, r_w)^{\mathrm{tr}}.$$

Rumely also proves for curves X various functorial properties of $S\gamma(\mathbf{E}, D)$ with respect to pullbacks and base extensions; see [7] for details.

REMARK 1.11. The definitions of $S\gamma(\mathbf{E}, D)^-$, $S\gamma(\mathbf{E}, D)^+$ and $S\gamma(\mathbf{E}, D)$ depend only on X, $\mathbf{E}$ and D, and not on the choice of an integral model for X or of metrics on $X(\mathbf{C}_v)$ for archimedean v. However, to compute $S\gamma(\mathbf{E}, D)^-$ and $S\gamma(\mathbf{E}, D)^+$ it may be useful to make such choices and to utilize intersection theory. If K is a global function field, the use of ordinary intersection theory on varieties of dimension $d+1$ over the prime field is discussed in [2]. If K is a number field, the work of Gillet and Soulé in [4] shows that for suitable X and D there is a natural ψ for which

$$-\log \psi(H^0(L^{\otimes n})_{\mathbf{A}}/H^0(L^{\otimes n})) = n^{d+1}\mathbf{L}^{d+1}/(d+1)! + O(n^d \log(n))$$

as $n \to \infty$, where $\mathbf{L}^{d+1}$ is an arithmetic intersection number. In view of Definition 1.4, the new analytic problem one faces in studying $S\gamma(\mathbf{E}, D)$ is to determine the asymptotic behavior of $\psi(\mathbf{F}_n)$ as $n \to \infty$.

2. Proof of Theorem 1.5

LEMMA 2.1. *Suppose $\mathbf{E}$ has a capacity relative to D in the sense of Definition* 1.3 *and that $\alpha = (\alpha_v)_v$ is an idele of K. Then*

$$\log \psi(\alpha\mathbf{F}_n) = \log \psi(\mathbf{F}_n) + O(n^d)$$

as $n \to \infty$.

Proof. For large n, $\psi(\alpha\mathbf{F}_n) = |\alpha|^{r(n)}\psi(\mathbf{F}_n)$ where $|\alpha| = \Pi_v |\alpha_v|_v$ is the norm of α as an idele and $r(n) = \dim_K H^0(L^{\otimes n})$. By dimension theory, $r(n) = O(n^d)$, so Lemma 2.1 holds.

REMARK 2.2. Lemma 2.1 shows that the values of $S\gamma(\mathbf{E}, D)^-$, $S\gamma(\mathbf{E}, D)^+$ and $S\gamma(\mathbf{E}, D)$ do not change if in their definition we replace $\mathbf{F}_n$ by $\alpha\mathbf{F}_n$.

LEMMA 2.3. *Suppose* $S\gamma(\mathbf{E}, D)^- < 1$. *Then for all ideles* α *of* K *there are infinitely many* $n > 0$ *for which* $\alpha\mathbf{F}_n \cap H^0(L^{\otimes n})$ *contains a non-zero function* f.

Proof. Let $c = -\log(S\gamma(\mathbf{E}, D)^-)/2 > 0$. By the definition of $S\gamma(\mathbf{E}, D)^-$, there are arbitrarily large integers n for which $\mathbf{F}_n$ is open and

$$cn^{(d+1)}/(d+1)! < \log \lambda_n(\mathbf{E}, D). \tag{2.1}$$

Let $\beta = (\beta_v)_v$ be the idele of K with component $\beta_v = 1/2$ (resp. $\beta_v = 1$) if v is archimedean (resp. non-archimedean). By Lemma 2.1, $\log \psi(\alpha\beta\mathbf{F}_n) = \log \psi(\mathbf{F}_n) + O(n^d)$. Hence (2.1) and (1.5) imply that for infinitely many positive integers n,

$$\psi(\alpha\beta\mathbf{F}_n) > \psi(H^0(L^{\otimes n})_{\mathbf{A}}/H^0(L^{\otimes n})).$$

For such n the projection of $\alpha\beta\mathbf{F}_n \subseteq H^0(L^{\otimes n})_{\mathbf{A}}$ into $H^0(L^{\otimes n})_{\mathbf{A}}/H^0(L^{\otimes n})$ cannot be an isometry. Hence there are distinct $f_1, f_2 \in \alpha\beta\mathbf{F}_n$ such that $f = f_1 - f_2$ lies in $H^0(L^{\otimes n})$. Then f is a non-zero element of $H^0(L^{\otimes n})$, and f is in $\alpha\mathbf{F}_n$ because f_1 and f_2 are in $\alpha\beta\mathbf{F}_n$.

Proof of Theorem 1.5. Suppose $S\gamma(\mathbf{E}, D)^- < 1$. We are to show that there is a thickening $\mathbf{U}$ of $\mathbf{E}$ such that $X(\bar{K}; \mathbf{U})$ is not Zariski dense in $\bar{X} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$. Let $\alpha = (\alpha_v)_v$ be an idele of K such that $|\alpha| = \Pi_v |\alpha_v|_v < 1$ and $|\alpha_v|_v \leqslant 1$ for all v. Let n be a positive integer for which there is a non-zero function $f \in \alpha\mathbf{F}_n(\mathbf{E}) \cap H^0(L^{\otimes n})$ as in Lemma 2.3. Define U_v to be the subset of $z \in X(\mathbf{C}_v)$ for which $|f(z)|_v \leqslant |\alpha_v|_v$ (resp. $|f(z)|_v < |\alpha_v|_v$) if v is non-archimedean (resp. if v is archimedean). By the definition of $\mathbf{F}_n(\mathbf{E})$ in (1.4), $\mathbf{U} = \Pi_v U_v$ is a thickening of $\mathbf{E}$. To prove Theorem 1.5, it will suffice to prove that $f(P) = 0$ if $P \in X(\bar{K}; \mathbf{U})$, since then $X(\bar{K}; \mathbf{U})$ will lie in the hypersurface in $\bar{X}$ defined by the vanishing of f. By the product formula, it will suffice to show that for all places v of K and all embeddings $\sigma: \bar{K} \to \mathbf{C}_v$ over v one has $|\sigma(f(P))|_v \leqslant 1$, with $|\sigma(f(P))|_v < 1$ for at least one such v and σ. Now f is in $K(X)$, so $\sigma(f(P)) = f(\sigma(P))$. By the definition of $X(\bar{K}; \mathbf{U})$, $P \in X(\bar{K}; \mathbf{U})$ implies $\sigma(P) \in U_v$, so $|\sigma(f(P))|_v = |f(\sigma(P))|_v \leqslant |\alpha_v|_v \leqslant 1$. Since $|\alpha| = \Pi_v |\alpha_v|_v < 1$, we must have $|\sigma(f(P))|_v < 1$ for at least one v and σ, so Theorem 1.5 is proved.

3. Proof of Theorems 1.2, 1.6 and 1.7

We will give the proof of Theorem 1.2 and then sketch how the argument can be modified to show Theorems 1.6 and 1.7.

For $a = (a_0, \ldots, a_n) \in \mathbf{C}^{n+1}$ define $f_a(z) = a_0 + \cdots + a_n z^n$. Let $F_n(E)$ (resp. $F_n(E)_{\mathbf{C}}$) be the set of $a \in \mathbf{R}^{n+1}$ (resp. $a \in \mathbf{C}^{n+1}$) such that $|f_a(z)| < 1$ for all $z \in E$. Since the compact subset $E \subseteq \mathbf{C}$ in Theorem 1.2 is assumed to be stable under complex conjugation, one finds

$$2^{-1}(F_n(E) + \sqrt{-1}\,F_n(E)) \subseteq F_n(E)_{\mathbf{C}} \subseteq F_n(E) + \sqrt{-1}\,F_n(E).$$

Let ψ_n (resp. μ_n) be the usual Euclidean Haar measure on $\mathbf{R}^{n+1}$ (resp. $\mathbf{C}^{n+1}$). Then

$$2^{-2(n+1)}\psi_n(F_n(E))^2 \leqslant \mu_n(F_n(E)_{\mathbf{C}}) \leqslant \psi_n(F_n(E))^2.$$

Hence to prove Theorem 1.2, it will suffice to show

$$\lim_{n\to\infty} n^{-2} \log \mu_n(F_n(E)_{\mathbf{C}}) = -\log \gamma(E). \tag{3.1}$$

Suppose now that $z_0, \ldots, z_n \in E$ are chosen to maximize

$$\prod_{0 \leqslant i \neq j \leqslant n} |z_i - z_j|^{1/n(n+1)}.$$

For $a \in F_n(E)_{\mathbf{C}}$, $f_a(z_i)$ lies inside the unit disc. Hence if M is the $(n+1)\times(n+1)$ matrix (z_i^j), Ma^{tr} lies in the product of $n+1$ copies of the unit disc. Thus $|\det(M)|^2\mu_n(F_n(E)_{\mathbf{C}}) \leqslant \pi^{n+1}$. When one uses the Vandermonde determinant formula for $\det(M)$ and the definition of $\gamma(E)$ in (1.1), one finds

$$\limsup_{n\to\infty} n^{-2} \log \mu_n(F_n(E)_{\mathbf{C}}) \leqslant -\log \gamma(E). \tag{3.2}$$

We now find a lower bound for $\mu_n(F_n(E)_{\mathbf{C}})$. For each $m \geqslant 1$ let $p_m(z) \in \mathbf{C}[z]$ be a monic complex polynomial of degree m which has minimal supremum over E over the set of all such polynomials. Define $P_m(E) = \sup\{|p_m(z)|^{1/m} : z \in E\}$. Let $t_m = P_m(E)^{-m}$ if $P_m(E) \neq 0$, and let $t_m = m^m$ otherwise. Define $g_m(z) = t_m p_m(z)$; then $|g_m(z)| \leqslant 1$ if $z \in E$. Because E is compact, the set $F_n(E)_{\mathbf{C}}$ contains the convex set $J(n)$ of coefficient vectors associated to the linear combinations $b_0 g_0(z) + \cdots + b_n g_n(z)$ for which $b_i \in \mathbf{C}$ and $\Sigma_i |b_i| < 1$. One can readily compute the volume $\mu_n(J(n))$ of $J(n)$ in terms of the t_m using the fact that $g_m(z)$ has leading coefficient $t_m z^m$. Furthermore, $\mu_n(J(n)) \leqslant \mu_n(F_n(E)_{\mathbf{C}})$. Combining this with the definition of $\gamma(E)$ in equation (1.2), we see after some simple calculations that

$$\liminf_{n\to\infty} n^{-2} \log \mu_n(F_n(E)_{\mathbf{C}}) \geqslant -\log \gamma(E). \tag{3.3}$$

Inequalities (3.2) and (3.3) establish Theorem 1.2.

We now prove Theorem 1.6. With the notations of the theorem, let $M(n)$ be the set of polynomials f with integral coefficients in $z_1, \ldots, z_d$ which have degree $\leqslant [nr_k]$ in z_k for all k. For v finite, one checks that $F_n(E_v) = \mathbf{Z}_v \otimes_{\mathbf{Z}} M(n)$. Choose the Haar measure $\psi = \Pi_v \psi_v$ so $\psi_v(F_n(E_v)) = 1$ if v is finite. Let $\psi_{v(\infty)}$ (resp. μ) be induced by the usual Euclidean Haar measure on the coefficients of elements of $\mathbf{R} \otimes_{\mathbf{Z}} M(n)$ (*resp.* $\mathbf{C} \otimes_{\mathbf{Z}} M(n)$). Then $\lambda_n(\mathbf{E}, D) = \psi_{v(\infty)}(F_n(E_{v(\infty)}))$. Since each E_i in $E_{v(\infty)} = E_1 \times \cdots \times E_d$ is stable under complex conjugation, one finds as in the proof of Theorem 1.2 that

$$2 \log \psi_{v(\infty)}(F_n(E_{v(\infty)})) = \log \mu(F_n(E_{v(\infty)})_{\mathbf{C}}) + O(n^d)$$

as $n \to \infty$, where $F_n(E_{v(\infty)})_{\mathbf{C}}$ is the set of $f \in \mathbf{C} \otimes_{\mathbf{Z}} M(n)$ with sup norm less than one on $E_{v(\infty)}$. The rest of the proof follows the procedure used to show Theorem 1.2. One first bounds $\mu(F_n(E_{v(\infty)})_{\mathbf{C}})$ from above using the fact that $|f(Z)|_{v(\infty)} < 1$ if $f \in F_n(E_{v(\infty)})_{\mathbf{C}}$ and $Z = (z_1(j(0)), \ldots, z_d(j(d)))$, where $0 \leqslant j(k) \leqslant [r_k n]$ for all k and $z_k(0), \ldots, z_k([r_k n])$ are points of E_k chosen to maximize $\Pi_{i \neq j} |z_k(i) - z_k(j)|$. One then bounds $\mu(F_n(E_{v(\infty)})_{\mathbf{C}})$ from below by constructing a large convex symmetric subset of $F_n(E_{v(\infty)})_{\mathbf{C}}$. This convex set can be taken to be the set of linear combinations $\Sigma_\alpha b_\alpha g_\alpha$ in which the b_α are complex numbers such that $\Sigma_\alpha |b_\alpha| < 1$, g_α is the product $g_{1,\alpha(1)}(z_1) \cdots g_{d,\alpha(d)}(z_d)$ of single variable polynomials, and $g_{k,j}(z)$ is a polynomial of degree j with sup norm $\leqslant 1$ on E_k and large real leading coefficient. The details are very similar to those in the proof of Theorem 1.2, so we omit them.

To prove Theorem 1.7, let a and b be positive real numbers. Replace $M(n)$ in the proof of Theorem 1.6 by the set of Laurent polynomials $c_{-[an]} z^{-[an]} + \cdots + c_{[bn]} z^{[bn]}$ in one indeterminate z with real coefficients c_i. As in the proof of Theorem 1.6, one must find suitable upper and lower bounds for $\mu(F_n(E_{v(\infty)})_{\mathbf{C}})$, where now $E_{v(\infty)}$ is the circle $E(r)$ of radius $r > 0$. To bound $\mu(F_n(E_{v(\infty)})_{\mathbf{C}})$ from above, evaluate Laurent polynomials $f \in F_n(E_{v(\infty)})_{\mathbf{C}}$ at numbers of the form $r\zeta$ with ζ a root of unity and then use Vandermonde determinants. To bound $\mu(F_n(E_{v(\infty)})_{\mathbf{C}})$ from below use the convex symmetric subset of $F_n(E_{v(\infty)})_{\mathbf{C}}$ formed by the Laurent polynomials $e_{-[an]}(z/r)^{-[an]} + \cdots + e_{[bn]}(z/r)^{[bn]}$ for which the e_i are complex numbers such that $\Sigma_i |e_i| < 1$.

4. Necessary and sufficient conditions for E to have a capacity relative to D

We assume the notations of Definition 1.3. For each $n > 0$ let $W(n)$ be a basis for $H^0(L^{\otimes n}) = H^0(X, O_X([nD]))$ as a vector space over K. Let $\mathbf{E} = \Pi_v E_v$ be an adelic set. Recall that $\sup(f_v, E_v) = \sup(|f_v(z)|_v : z \in E_v\}$ for $f_v \in H^0(L^{\otimes n})_v = H^0(L^{\otimes n}) \otimes_K K_v$. Via the map $g \to g \otimes 1$ we will view $H^0(L^{\otimes n})$ as a subset of $H^0(L^{\otimes n})_v$ and of $H^0(L^{\otimes n})_{\mathbf{A}} = H^0(L^{\otimes n}) \otimes_K \mathbf{A}_K$. Define $c(W(n), v) = \max\{\sup(g, E_v): g \in W(n)\}$.

From Definition 1.3, one sees that $\mathbf{E}$ will have a capacity relative to D if and only if the following is true:

(4.1) *If* $n \gg 0$ *then* $c(W(n), v) < \infty$ *for all* v *and* $c(W(n), v) \leqslant 1$ *for almost all* v.

We now give some sufficient conditions for (4.1) to hold. Let S be a non-empty finite set of places of K containing the archimedean places of K, and let $O(S)$ be the ring of elements of K which are regular off of S. Suppose that $\mathbf{X}$ is an irreducible regular scheme which is flat and proper over $U = \operatorname{Spec}(O(S))$ and which has general fibre X. If v is a non-archimedean place of K let $\mathbf{O}_v$ be the ring of $z \in \mathbf{C}_v$ for which $|z|_v \leqslant 1$. Let D' be an effective Weil divisor on X. For v not in S define $E_{D',v}$ to be the set of $z \in X(\mathbf{C}_v)$ such that the Zariski closures of z and D' in $\mathbf{X} \times_U \operatorname{Spec}(\mathbf{O}_v)$ are disjoint.

THEOREM 4.1. *Suppose that* $\mathbf{E} = \Pi_v E_v$ *and* D' *satisfy the following conditions*:

(4.2) *If* v *is infinite then the closure of* E_v *in the complex topology is compact and disjoint from the points of* D' *over* $\mathbf{C}_v$.

(4.3) *If* v *is not in* S *then* $E_v \subseteq E_{D',v}$.

(4.4) *For each non-archimedean place* v *in* S *there is a very ample divisor* D''_v *on* X *whose support contains that of* D' *and for which* $\sup\{|g(z)|_v : z \in E_v\}$ *is finite for* $g \in H^0(X, O_X(D''_v))$.

Then $\mathbf{E}$ *has a capacity relative to each real linear combination* $D = \Sigma_i r_i D'_i$ *of the set* $\{D'_i\}_i$ *of irreducible components of* D'.

Proof. Since D' is effective, $H^0(X, L^{\otimes n}) = H^0(X, O_X([nD]))$ will be contained in $H^0(X, O_X(n'D'))$ if n' is sufficiently large relative to n. Hence the criterion in (4.1) shows it will be enough to consider the case in which $D = D'$. If v is archimedean then (4.2) implies that $c(W(n), v)$ in (4.1) is finite because each $g \in W(n)$ defines a continuous function on a compact set containing E_v. Suppose v is not in S and $z \in E_v \subseteq E_{D',v} = E_{D,v}$. Let $\mathbf{D}$ be the Zariski closure of D in $\mathbf{X}$. Because $H^0(X, L^{\otimes n}) = H^0(\mathbf{X}, O_{\mathbf{X}}(n\mathbf{D})) \otimes_{O(S)} K$, we can assume that the basis $W(n)$ of $H^0(X, L^{\otimes n})$ is contained in $H^0(\mathbf{X}, O_{\mathbf{X}}(n\mathbf{D}))$. Then (4.3) implies each $g \in W(n)$ is regular in a neighborhood of the section (z) of $\mathbf{X} \times_U \operatorname{Spec}(\mathbf{O}_v) \to \operatorname{Spec}(\mathbf{O}_v)$ which is the Zariski closure of z in $\mathbf{X} \times_U \operatorname{Spec}(\mathbf{O}_v)$. Hence $g(z)$ is the restriction of a global section of the structure sheaf of (z), so $g(z) \in \mathbf{O}_v$. Thus $|g(z)|_v \leqslant 1$ so $c(W(n), v) \leqslant 1$ in (4.1). Finally suppose v is a non-archimedean place in S and that (4.4) holds. To prove $c(W(n), v)$ is bounded for large n we are free to replace $D = D'$ by D''_v. Thus we may assume that $D = D' = D''_v$ is very ample. By assumption, $|f(z)|_v$ is bounded for $z \in E_v$ and f in a basis $W(1)$ for $H^0(X, O_X(D))$

over K. Suppose $n > 0$ and $g \in H^0(X, L^{\otimes n}) = H^0(X, O_X(nD))$. It follows from the argument showing [5, p. 126, Ex. 5.14(a)] that g is integral over the K-subalgebra of $K(X)$ which is generated by the elements of $W(1)$. (Hartshorne assumes that he is working over an algebraically closed ground field, but this assumption is not necessary.) Because the v-adic absolute values of the elements of $W(1)$ are bounded over E_v, it follows that $|g(z)|_v$ is bounded for $z \in E_v$. Hence $c(W(n), v)$ is finite, so Theorem 4.1 is proved.

REMARK 4.2. The case in which D' is very ample and $D' = D''_v$ for $v \in S$ is discussed just after Definition 1.3. Theorem 4.1 applies to more general situations, however, in which D' may not be ample.

Acknowledgements

I would like to thank R. Rumely for many important comments about this paper. Thanks also go to S. Bloch, E. Friedman, H. Gillet, E. Kani, W. McCallum, C. Soulé and D. Zagier for useful remarks.

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