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Introduction

Let $m$ be a fixed positive integer, and let $F_m$ denote the complete plane curve over the complex number field $\mathbb{C}$ with projective equation

$$X^m + Y^m + Z^m = 0.$$ 

This is called the Fermat curve of exponent $m$ over $\mathbb{C}$. Let $J_m$ denote the Jacobian of $F_m$.

The object of this paper is to give a characterization of the endomorphism ring $\text{End}(J_m)$ of $J_m$ when $m$ is relatively prime to 6. To do this, we first determine $\text{End}^0(J_m) = \text{End}(J_m) \otimes \mathbb{Q}$, and the action of $\text{Aut}(F_m)$ on $H_1(F_m) = H_1(F_m(\mathbb{C}), \mathbb{Z})$. Rohrlich has shown in the appendix of [9] that the latter homology group is a cyclic module over a suitable (commutative) integral group ring. $\text{End}^0(J_m)$ turns out to be a quotient ring of $\mathbb{Q}[\text{Aut}(F_m)]$. To prove this, we use the results of Koblitz–Rohrlich in [11]. We then use the fact that for a non-singular projective curve $X$ over $\mathbb{C}$ with Jacobian $J_X$,

$$\text{End}(J_X) = \{\alpha \in \text{End}^0(J_X) | \alpha(H_1(X(\mathbb{C}), \mathbb{Z})) \subseteq H_1(X(\mathbb{C}), \mathbb{Z})\},$$

to write down necessary and sufficient conditions for an element of $\text{End}^0(J_m)$ to be in $\text{End}(J_m)$. In particular, we find examples of endomorphisms of $J_m$ which are not induced from elements of the integral group ring $\mathbb{Z}[\text{Aut}(F_m)]$.

Fixing a primitive $m$-root $\zeta$ of unity in $\mathbb{Q}$, $G = \text{Aut}(F_m)$ is generated by:

$$\sigma: (X, Y, Z) \rightarrow (\zeta X, Y, Z), \quad \tau: (X, Y, Z) \rightarrow (X, \zeta Y, Z),$$

$$\iota: (X, Y, Z) \rightarrow (Y, X, Z), \quad \rho: (X, Y, Z) \rightarrow (Z, X, Y).$$

The natural homomorphism $G \rightarrow \text{Aut}(J_m)$ gives rise to

$$\Phi: \mathbb{Q}[G] \rightarrow \text{End}^0(J_m).$$

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For each integer \( k \geq 0 \), we let
\[
I_k(T) = \sum_{j=0}^{m-1} \binom{j}{k} T^j \in \mathbb{Z}[T].
\]

Let \( T \) be the left-sided ideal of the group ring \( \mathbb{Q}[G] \) generated by the following elements: \( I_0(\sigma), I_0(\tau), I_0(\sigma \tau), I_0(\sigma^{-1} \tau)(1 + i), I_0(\sigma^2 \tau)(1 + i \rho), I_0(\sigma^2 \tau)(1 + i \rho^{-1}) \).

We will prove, in Sections 1 and 2,

**THEOREM A.** The sequence
\[
0 \to T \to \mathbb{Q}[G] \to \text{End}^0(J_m) \to 0
\]
is exact. Moreover, \( \ker(\Phi) = T \) is the two-sided ideal of \( \mathbb{Q}[G] \) generated by \( I_0(\sigma) \) and \( I_0(\sigma^{-1} \tau)(1 + i) \).

In Section 3, we study the singular homology group \( H_1(F_m) \) and the action of \( G \) on it. Let \( I: [0, 1] \to F_m(\mathbb{C}) \) denote the one-simplex
\[
I: t \to (t^{1/m}, (1 - t)^{1/m}, \alpha),
\]
where the \( m \)th root is the real \( m \)th root, and \( \alpha = -1 \) if \( m \) is odd but \( \alpha \) is a primitive \( 2m \)th root of unity if \( m \) is even. Let \( g \) denote the one-cycle
\[
g = (\sigma^{(m+1)/2} - \sigma^{(m-1)/2})(\tau^{(m+1)/2} - \tau^{(m-1)/2})I \quad \text{if } m \text{ is odd}
\]
and
\[
g = (1 - \sigma^{m-1})(1 - \tau^{m-1}) \quad \text{if } m \text{ is even}.
\]

Denoting the subgroup of \( G \) generated by \( \sigma \) and \( \tau \) by \( G_m \), we have

**PROPOSITION B.** \( H_1(F_m) \) is a cyclic \( \mathbb{Z}[G_m] \)-module with \( g \) as a generator. Furthermore, in homology, \( \iota(g) = -g \) and \( \rho(g) = g \).

Using Theorem A and Proposition B, we prove that:

**THEOREM C.** Let \( X, Y, Z, \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathbb{Q}[G_m] \). Denoting the ideal of \( \mathbb{Q}[G_m] \) generated by \( I_0(\sigma), I_0(\tau) \) and \( I_0(\sigma \tau) \) by \( J \), then
\[
\Phi(X + Y \rho + Z \rho^2 + \tilde{X} \rho + \tilde{Y} \rho^2 + \tilde{Z} \rho^2) \in \text{End}(J_m)
\]
if and only if, for all \( r \) and \( s \) in \( \mathbb{Z}/m\mathbb{Z} \),
\[
X \sigma^r \tau^s - \tilde{X} \sigma^r \tau^s + Y \sigma^{-s} \tau^{-r} - \tilde{Y} \sigma^{-s} \tau^{-r} + Z \sigma^r \tau^{-r} - \tilde{Z} \sigma^{-s} \tau^{-r} \in \mathbb{Z}[G_m] + J.
\]

The next theorem shows that there are endomorphisms of \( J_m \) which are not in \( \Phi(\mathbb{Z}[G]) \) when \( m \) is relatively prime to 6. Let
\[
W = m^{-1} \{- I_1(\sigma)I_3(\tau) + [I_1(\sigma)I_3(\tau) - I_3(\sigma)I_1(\tau)] \rho + I_3(\sigma)I_1(\tau) \rho^2 \} \in \mathbb{Q}[G_m, \rho].
\]
THEOREM D

\[ \text{End}(J_m) \cap \Phi(Q[G_m, \rho]) = \Phi(Z[G_m, \rho, W]) \quad \text{and} \quad \Phi(W) \]
is not in \( \Phi(Z[G]) \). However,

\[ \text{End}(J_m) \cap \Phi(Q[G_m, i]) = \Phi(Z[G_m, i]). \]

In particular, since the restriction of \( \Phi \) to \( Q[G_m, \rho] \) is surjective when \( m = 5 \), we have the following theorem.

THEOREM E. When \( m = 5 \), we have

\[ \text{End}(J_5) = \Phi(Z[G_m, \rho, W]). \]

1. The kernel of \( \Phi \)

With the exception of Lemma 1.1, let \( m \) be relatively prime to 6. We also assume \( m > 3 \). In this section, we prove that the kernel of \( \Phi \) is the left-sided ideal \( T \) of \( Q[G] \) defined in the Introduction. Let \( A = I_0(\sigma) \), \( B = I_0(\tau) \), \( C = I_0(\sigma \tau) \), \( D = I_0(\sigma^{-1}\tau) \), \( E = I_0(\sigma^2(1 + i\rho)) \) and \( F = I_0(\sigma^2(1 + i\rho^2)) \) be in \( Q[G] \).

LEMMA 1.1. \( T \subseteq \text{Ker}(\Phi) \).

Proof. Since the following relations hold in \( Q[G] \): \( \rho A \rho^{-1} = B \), \( \rho B \rho^{-1} = C \), \( \rho D \rho^{-1} = E \), \( \rho E \rho^{-1} = F \), and \( \text{Ker}(\Phi) \) is a two-sided ideal in \( Q[G] \), it suffices to show that \( A \) and \( B \) are in \( \text{Ker}(\Phi) \).

Let \( X \) be the plane curve \( u + v^m + 1 = 0 \) and \( h: F_m \to X \) be the morphism \( h(x, y) = (-x^m, y) \). The induced homomorphism \( h^*: J_X \to J_{F_m} \) on Jacobians is the zero map since \( X \) has genus zero. Since \( h \) is a cyclic covering with \( \langle \sigma \rangle \) as Galois group, we have

\[ \Phi(A)(P) - (Q)) = h^*((h(P)) - (h(Q))) = 0 \quad \text{for points } P, Q \in F_m. \]

Hence \( \Phi(A) = 0 \).

Next, we consider the curve \( Y = F_{1,1,-2}^m \), with singular equation \( y^m = x(1-x) \). It is hyperelliptic with \( \iota: (x, y) \to (1-x, y) \) as its hyperelliptic involution. Let \( \phi: F_m \to Y \) be the canonical projection \( \varphi_{1,1,-2}^m \). Composing the homomorphisms

\[ J_m \xrightarrow{\phi} J_Y \xrightarrow{(1 + i)_*} J_Y \xrightarrow{\phi^*} J_m, \]

we obtain the endomorphism \( \Phi(D) \) of \( J_m \). Since \( (1 + i)_* = -1 \) in \( \text{End}(J_Y) \), we have that \( \Phi(D) = 0 \). □
Fₘ is the Fermat curve \( X^m + Y^m + Z^m = 0 \) defined over \( \mathbb{Q} \). Let \( x = X/Z \) and \( y = Y/Z \). A basis for the complex vector space \( H^0(F_m, \Omega^1) \) is the set
\[
\left\{ w_{r,s} = x^{r-1} y^{s-1} \frac{dx}{y^{m-1}} \middle| 0 < r, s, r + s < m \right\}.
\]

**LEMMA 1.2.** Let \( \alpha \in \mathbb{Z}[G_m] \) be such that \( \Phi(\alpha)^* w_{r,s} = 0 \) for all \( w_{r,s} \in H^0(F_m, \Omega^1) \). Then \( \alpha \in J \), where \( J \) is the ideal of the group ring \( \mathbb{Q}[G_m] \) generated by \( A, B \) and \( C \).

**Proof.** Let \( \alpha = f(\sigma^a, \tau^b) \), where \( f(x, y) \in \mathbb{Z}[x, y] \). Since \( (\sigma^k \tau^l)^* w_{r,s} = \zeta^{r + sl} w_{r,s} \), \( \Phi(\alpha)^* w_{r,s} = 0 \) for all \( w_{r,s} \) implies that for \( 0 < r, s, r + s < m \),
\[
f(\zeta^r, \zeta^s) = 0.
\]

Let \( (a, b) \) be a pair of positive integers with \( a, b < m \) and \( a + b \neq m \). Let \( c \in \mathbb{Z} \) be such that \( 0 < c < m \) and \( a + b + c = km \), where \( k = 1 \) or \( k = 2 \). If \( k = 1 \), (1.1) holds for \( (r, s) = (a, b) \). Suppose \( k = 2 \). Then \((m-a) + (m-b) + (m-c) = m \), whence \((m-a) + (m-b) < m \). Therefore \( f(\zeta^a, \zeta^b) = 0 \). Applying the automorphism in \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \) which sends \( \zeta \) to \( \zeta^{-1} \) to the latter equation, we obtain \( f(\zeta^a, \zeta^b) = 0 \).

Let \( I \) be the ideal of \( \mathbb{Q}[x, y] \) generated by \( I_0(x) \), \( I_0(y) \) and \( I_0(xy) \). The ring \( R = \mathbb{Q}[x, y]/I \) is a product of fields (hence reduced), since it is a quotient of \( \mathbb{Q}[x, y]/(x^m - 1, y^m - 1) \). Let
\[
Z(I) = \{ (u, v) \in \mathbb{Q}^2 \mid I_0(u) = I_0(v) = I_0(uv) = 0 \}.
\]

Then
\[
Z(I) = \{ (\zeta^a, \zeta^b) \mid 0 < a, b < m, a + b \neq m \}.
\]

By Hilbert's Nullstellensatz, \( f \in \sqrt{I}. \mathbb{Q}[x, y] \cap \mathbb{Q}[x, y] = I \). It follows that \( \alpha \in J \). \( \Box \)

Proceeding in the same way as we did in proving Lemma 1.2, we can prove the following lemma.

**LEMMA 1.3.** Let \( \alpha \in \mathbb{Z}[G_m] \) be such that \( \Phi(\alpha)^* w_{r,s} = 0 \) for all \( w_{r,s} \in H^0(F_m, \Omega^1) \) with \( r \neq s \), \( 2r + s \neq m \) and \( r + 2s \neq m \). Then
\[
\alpha \in J + (I_0(\sigma^{-1} \tau), I_0(\sigma \tau^2), I_0(\sigma^2 \tau)).
\]

We devote the remaining space in this section to determine \( \text{Ker}(\Phi) \).

Let \( U, V, W, X, Y, Z \in \mathbb{Z}[G_m] \) and \( \varphi = U + V \rho + W \rho^2 + X \iota + Y \iota \rho + Z \iota \rho^2 \in \mathbb{Z}[G] \)
be such that for all \( w_{r,s} \in H^0(F_m, \Omega^1) \),
\[
\Phi(\varphi)^* w_{r,s} = 0.
\]

(1.2)
We choose polynomials $U, V, W, X, Y, Z \in \mathbb{Z}[x, y]$ such that
\[
U = \tilde{U}(\sigma, \tau), \quad V = \tilde{V}(\sigma, \tau), \quad W = \tilde{W}(\sigma, \tau),
\]
\[
X = \tilde{X}(\sigma, \tau), \quad Y = \tilde{Y}(\sigma, \tau), \quad Z = \tilde{Z}(\sigma, \tau).
\]
From (1.2), it follows that $w_{r,s}$ is annihilated by
\[
\Phi(U)^* + \rho^*\Phi(V)^* + (\rho^2)*\Phi(W)^* + i^*\phi(X)^* +
\]
\[
+ \rho^i^*\Phi(Y)^* + (\rho^2)^i\Phi(Z)^*,
\]
or equivalently, for all $(r, s) \in \mathbb{Z}^2$ with $0 < r, s, r + s < m$,
\[
\tilde{U}(\zeta', \zeta^s)w_{r,s} + \tilde{V}(\zeta', \zeta^s)w_{s,m-r-s} + \tilde{W}(\zeta', \zeta^s)w_{m-r-s,r} -
\]
\[
- \tilde{X}(\zeta', \zeta^s)w_{s,r} - \tilde{Y}(\zeta', \zeta^s)w_{r,m-r-s} - \tilde{Z}(\zeta', \zeta^s)w_{m-r-s,s} = 0. \tag{1.3}
\]
When $r \neq s, r + 2s \neq m$ and $2r + s \neq m$, the set
\[
\{w_{r,s}, w_{s,m-r-s}, w_{m-r-s,s}, w_{s,r}, w_{r,m-r-s}, w_{m-r-s,s}\}
\]
is a linearly independent subset of $H^0(F_m, \Omega^1)$. Hence, from (1.3), $\tilde{U}, \tilde{V}, \tilde{W}, \tilde{X}, \tilde{Y}$ and $\tilde{Z}$ vanish at $(\zeta', \zeta^s)$ whenever $0 < r, s, r + s < m, r \neq s, r + 2s \neq m$ and $2r + s \neq m$. In other words, for these pairs $(r, s)$,
\[
\Phi(U)^*w_{r,s} = \Phi(V)^*w_{r,s} = \Phi(W)^*w_{r,s} = \Phi(X)^*w_{r,s}
\]
\[
= \Phi(Y)^*w_{r,s} = \Phi(Z)^*w_{r,s} = 0. \tag{1.4}
\]
When $r = s, (1.3)$ implies that
\[
(\tilde{U} - \tilde{X})(\zeta', \zeta^s)w_{r,r} + (\tilde{V} - \tilde{Y})(\zeta', \zeta^s)w_{r,m-2r} + (\tilde{W} - \tilde{Z})(\zeta', \zeta^s)w_{m-2r,r} = 0.
\]
Since $\{w_{r,r}, w_{r,m-2r}, w_{m-2r,r}\}$ is a linearly independent subset of $H^0(F_m, \Omega^1)$ (by virtue of the fact that $m$ is coprime to 3), we have for $0 < r \leq (m-1)/2$,
\[
\Phi(U - X)^*w_{r,r} = \Phi(V - Y)^*w_{r,r} = \Phi(W - Z)^*w_{r,r} = 0. \tag{1.5}
\]
By considering (1.3) in the cases when $r + 2s = m$ and $2r + s = m$, we obtain
\[
\Phi(U - Y)^*w_{m-2r,r} = \Phi(V - Z)^*w_{m-2r,r} = \Phi(W - X)^*w_{m-2r,r} = 0 \tag{1.6}
\]
for $0 < r < m$, and
\[
\Phi(U - Z)^*w_{m-2r} = \Phi(V - X)^*w_{m-2r} = \Phi(W - Y)^*w_{m-2r} = 0 \tag{1.7}
\]
for $0 < r < m$, respectively.

Let $\mathfrak{J}$ be the ideal of $\mathbb{Q}[G_m]$ generated by $I_0(\sigma), I_0(\tau), I_0(\sigma \tau), I_0(\sigma^{-1} \tau), I_0(\sigma^2 \tau)$ and $I_0(\sigma^2 \tau)$. We fix a basis $\{x_1, \ldots, x_{\ell_0}\}$ over $\mathbb{Q}$ for the ideal $\mathfrak{J}$ generated by $I_0(\sigma), I_0(\tau)$ and $I_0(\sigma \tau)$. Then we choose a basis
\[
\{\beta_1 I_0(\sigma^{-1} \tau), \ldots, \beta_1 I_0(\sigma^{-1} \tau)\} \cup \{\gamma_1 I_0(\sigma^2 \tau), \ldots, \gamma_2 I_0(\sigma^2 \tau)\} \cup
\]
\[
\cup \{\delta_1 I_0(\sigma^2 \tau), \ldots, \delta_3 I_0(\sigma^2 \tau)\}
\]
for \( \mathfrak{J}/\mathfrak{J} \), where each \( \beta_i, \gamma_j, \delta_k \in G_m \). We note that
\[
\{x_1, \ldots, x_{i_0}\} \cup \{\beta_1 I_0(\sigma^{-1} \tau), \ldots, \beta_i I_0(\sigma^{-1} \tau)\} \cup \{\gamma_1 I_0(\sigma \tau^2), \ldots, \gamma_{i_1} I_0(\sigma \tau^2)\} \\
\cup \{\delta_1 I_0(\sigma^2 \tau), \ldots, \delta_{i_1} I_0(\sigma^2 \tau)\}
\] (1.8)
is a \( \mathbb{Q} \)-basis for \( \mathfrak{J} \).

Lemma 1.3 applied to (1.4) gives \( U, V, W, X, Y, Z \in \mathfrak{J} \). Using the basis in (1.8), we can write in a unique way:
\[
U = \sum_{j=1}^{l_0} \lambda_{j, \alpha} x_j + \sum_{j+1}^{l_1} \lambda_{j, \beta} I_0(\sigma^{-1} \tau) + \sum_{j=1}^{l_3} \lambda_{j, \gamma} I_0(\sigma \tau^2) + \sum_{j=1}^{l_3} \lambda_{j, \delta} I_0(\sigma^2 \tau),
\]
where the \( \lambda_{j, \alpha} \)'s, \( \lambda_{j, \beta} \)'s, \( \lambda_{j, \gamma} \)'s and \( \lambda_{j, \delta} \)'s are in \( \mathbb{Q} \). We will write
\[
U_0 = \sum_{j=1}^{l_0} \lambda_{j, \alpha} x_j, \quad U_1 = \sum_{j=1}^{l_1} \lambda_{j, \beta} I_0(\sigma^{-1} \tau), \quad U_2 = \sum_{j=1}^{l_3} \lambda_{j, \gamma} I_0(\sigma \tau^2), \quad U_3 = \sum_{j=1}^{l_3} \lambda_{j, \delta} I_0(\sigma^2 \tau).
\]
Thus
\[
U = U_0 + U_1 I_0(\sigma^{-1} \tau) + U_2 I_0(\sigma \tau^2) + U_3 I_0(\sigma^2 \tau).
\] (1.9)
We write similar expressions for \( V, W, X, Y \) and \( Z \) as we did for \( U \) in (1.9).

Consider
\[
U - X = (U_0 - X_0) + (U_1 - X_1) I_0(\sigma^{-1} \tau) + (U_2 - X_2) I_0(\sigma \tau^2) + (U_3 - X_3) I_0(\sigma^2 \tau).
\]
By (1.5), \( U - X \) annihilates \( w_{r,r} \). Since each of \( I_0(\sigma \tau^2) \) and \( I_0(\sigma^2 \tau) \) annihilates \( w_{r,r} \), so does \( (U_1 - X_1) I_0(\sigma^{-1} \tau) \). In addition, \( I_0(\sigma^{-1} \tau) \) annihilates all \( w_{r,r} \) with \( r \neq s \). Thus \( (U_1 - X_1) I_0(\sigma^{-1} \tau) \) annihilates all \( w_{r,s} \in H^0(F_m, \Omega^1) \). By Lemma 1.2, \( U_1 - X_1 \in \mathfrak{J} \). By definition of \( U_1 \) and \( X_1 \), we have \( U_1 = X_1 \).

We can similarly prove the following equalities: \( U_2 = Y_2, U_3 = Z_3, V_1 = Y_1, V_2 = Z_2, V_3 = Z_3, W_1 = Z_1, W_2 = X_2, W_3 = Y_3 \). Therefore, \( \varphi \) is equal to
\[
U_0 + V_0 \rho + W_0 \rho^2 + X_0 \iota + Y_0 \iota \rho + Z_0 \iota \rho^2 + \\
+ U_1 I_0(\sigma^{-1} \tau)(1 + i) + U_2 I_0(\sigma \tau^2)(1 + i \rho) + U_3 I_0(\sigma^2 \tau)(1 + i \rho^{-1}) + \\
+ V_1 I_0(\sigma^{-1} \tau)(1 + i \rho) + V_2 I_0(\sigma \tau^2)(1 + i \rho) + V_3 I_0(\sigma^2 \tau)(1 + i \rho^{-1}) + \\
+ W_1 I_0(\sigma^{-1} \tau)(1 + i) \rho^2 + W_2 I_0(\sigma \tau^2)(1 + i \rho) \rho^2 + W_3 I_0(\sigma^2 \tau)(1 + i \rho^{-1}) \rho^2.
\]
Together with Lemma 1.1 and the following relations in the group \( G \): \( \rho \sigma \rho^{-1} = \tau, \rho \tau \rho^{-1} = (\sigma \tau)^{-1} = \rho^{-1} \sigma \rho, i \rho^{-1} = \rho^{-1} \), we have proved that \( T = \text{Ker}(\Phi) \).
2. Isogeny classes

As before, $F_m$ is the Fermat curve $X^m + Y^m + Z^m = 0$ defined over $\mathbb{Q}$, and $x = X/Z$ and $y = Y/Z$.

Let $r, s, t \in \mathbb{Z}$ with $0 < r, s, t < m$ and $r + s + t \equiv 0 \pmod{m}$. Then

$$w_{r,s,t} = x^{r-1}y^{s-1} \frac{dx}{y^{m-1}}$$

is a differential form of the second kind on $F_m$. The forms $w_{r,s,t}$ are eigenforms for the action of $G_m: (\sigma_j^t) w_{r,s,t} = \xi^{rj + sk} w_{r,s,t}$. Since the characters on $(\mathbb{Z}/m\mathbb{Z})^2$ are mutually distinct,

$$\Omega = \{w_{r,s,t} | 0 < r, s, t < m, r + s + t \equiv 0 \pmod{m}\}$$

is a basis of the deRham cohomology $H^1_{\text{DR}}(F_m)$. In the Hodge splitting

$$H^1_{\text{DR}}(F_m) \xrightarrow{\cong} H^0(F_m, \Omega^1) \oplus H^1(F_m, \mathcal{O}),$$

$H^0(F_m, \Omega^1)$ has $\Omega_1 = \{w_{r,s,t} \in \Omega | r + s + t = m\}$ as a basis.

We say that an abelian variety $A/K$ has CM by a commutative ring $R$ if there is given a homomorphism $R \to \text{End}_K(A)$ such that $H^1_{\text{DR}}(A)$ becomes a cyclic $R \otimes K$-module. Let $K = \mathbb{Q}(\zeta)$. Then $J_m/K$ has CM by $\mathbb{Z}[G_m]$, with the map

$$\mathbb{Z}[G_m] \to \text{End}_K(J_m)$$

induced by the inclusion $G_m \to \text{Aut}_K(F_m)$.

Let $S \in S_m$ be the class of $(a, b, c)$, where $a, b, c \in \mathbb{Z}$, $0 < a, b, c < m$ and $a + b + c = m$. We first consider the case when $(m, a, b, c) = 1$. Then

$$F_{a,b,c} = F_m/\langle \sigma^b \tau^{-a} \rangle$$

has irreducible equation

$$y^m = x^a(1 - x)^b,$$

and

$$\Omega_S = \Omega^{\langle \sigma^b \tau^{-a} \rangle}$$

descends to a basis of eigenforms for $H^1_{\text{DR}}(J^m_S)$ under the action of $\mathbb{Z}[G_m/\langle \sigma^b \tau^{-a} \rangle]$. Hence the Jacobian $J^m_S = J^m_{a,b,c}$ of $F^m_{a,b,c}$ has CM by $\mathbb{Z}[G_m/\langle \sigma^b \tau^{-a} \rangle]$.

Let $f_m(x)$ denote the $m$th cyclotomic polynomial over $\mathbb{Q}$, and let $\alpha$ be any generator of the cyclic group $G_m/\langle \sigma^b \tau^{-a} \rangle$. We define $A^m_S = (J^m_{a,b,c})^{\text{new}}$ to be the abelian variety obtained as a quotient of $J^m_S$ by the abelian subvariety $f_m(\alpha)J^m_S$.

In general, if $d = (m, a, b, c) = m/m'$, we let $a' = a/d$, $b' = b/d$, $c' = c/d$, and define

$$A^m_S = (J^m_{a',b',c'})^{\text{new}}.$$
Then it is well-known that the composition
\[ J_m \rightarrow \prod_{S \in \mathcal{S}_m} J^m_S \rightarrow \prod_{S \in \mathcal{S}_m} A^m_S \]
is an isogeny over \( \mathbb{Q} \): \( J_m \rightarrow \prod_{S \in \mathcal{S}_m} A^m_S \).

For \( S_1, S_2 \in \mathcal{S}_m \), we say that \( S_1 \) and \( S_2 \) are equivalent (written \( S_1 \sim S_2 \)) if \( A^m_{S_1} \) and \( A^m_{S_2} \) are isogeneous. If \( [S] \) denotes the equivalence class of \( S \in \mathcal{S}_m \), we set
\[ A^m_{[S]} = \prod_{S \in [S]} A^m_S. \]

\( A^m_{[S]} \) is well-defined up to the order of the factors. Let \( \lambda^m_{[S]} \) be the homomorphism
\[ \mathbb{Q}[G] \rightarrow \text{End}^0(A^m_{[S]}). \]

Then \( \lambda^m_{[S]} \) factors through the image of
\[ \mathbb{Q}[G] \rightarrow \text{End}^0(J^m_{[S]}), \text{ where } J^m_{[S]} = \prod_{S' \in [S]} J^m_{S'}. \]

Let us fix some terminology. (1) If \( R \) is a ring, then \( \Delta_n(R) \) is the subspace of the ring of \( (n \times n) \)-matrices \( M_n(R) \) with entries in \( R \) consisting of the diagonal matrices. (2) If \( r_1, \ldots, r_n \in R \), let \( \Delta(r_1, \ldots, r_n) \) be the diagonal matrix \( (r_{i,i}) \in \Delta_n(R) \) for which \( r_{i,i} = r_i \) for all \( i \). (3) Let \( I_n \) be the multiplicative unit of \( M_n(R) \). (4) If \( A \) is a simple abelian variety, then we associate to an endomorphism \( \phi \) of \( A^n \) the matrix \( U_\phi \in M_n(\text{End}(A)) \) if on closed points,
\[ \phi: \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} \rightarrow U_\phi \cdot \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}. \]

(5) Let \( A \) and \( B \) be abelian varieties over a field \( F \), and let \( \varphi: A \rightarrow B \) be an isogeny of degree \( n \). Then there is a unique isogeny \( \tilde{\varphi}: B \rightarrow A \) such that \( \tilde{\varphi} \varphi = n_A \) is multiplication by \( n \) on \( A \). \( \varphi \) induces the canonical isomorphism \( F_{\varphi}: \text{End}^0(A) \rightarrow \text{End}^0(B) \), which sends \( \alpha \in \text{End}(A) \) to \( n^{-1}(\alpha \varphi \tilde{\varphi}) \).

**Case 1.** \( A^m_{[S]} \) is non-simple.

In this case \([11]\), \( S \) is the class of a permutation of \( (1, w, -(1+w)) \), where \( w \in \mathbb{Z}/m\mathbb{Z} \) satisfies (a) \( w^2 + w + 1 = 0 \), or (b) \( w^2 = 1 \) and \( w \neq \pm 1 \).

In subcase (a), \( A^m_{[S]} = A^m_{1, w, w^2} \times A^m_{1, w^2, w} \). Let \( L = K^{(w)} \). Then \( A^m_{1, w, w^2} \) is isogeneous to a cube of a simple abelian variety \( B \) with CM by the ring of integers \( \mathcal{O}_L \), and the homomorphism
\[ \Phi_1: \mathbb{Q}[\sigma, \rho] \rightarrow \text{End}^0(A^m_{1, w, w^2}) \]
is surjective \([13]\). Since \( \iota(\sigma^w \tau^{-1}) \iota^{-1} = (\sigma^w \tau^{-1})^w \) in \( \text{Aut}(F_m) \), \( \iota \) induces an isomorphism \( F^m_{1, w^2, w} \rightarrow F^m_{1, w, w^2} \). Consider the isogeny \( f \), which is the composition
\[ A^m_{1, w, w^2} \times A^m_{1, w^2, w} \xrightarrow{1 \times 1} (A^m_{1, w, w^2})^2 \rightarrow B^6. \]
We claim that $F_{f_{\mathbb{S}}} : \mathbb{Q}[G] \to M_6(L)$ is surjective. This is the case because $F_{f_{\mathbb{S}}}$ sends $i, I_0(\sigma \tau^{-w^2}), I_0(\sigma \tau^{-w})$ to
\[
\begin{pmatrix}
0 & U_1 \\
U_2 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & mI_3
\end{pmatrix}
\begin{pmatrix}
mI_3 & 0 \\
0 & 0
\end{pmatrix}
\]
respectively (where $U_1$ and $U_2$ are units in $M_3(\mathcal{O}_L)$), and $\Phi_1$ is surjective.

In subcase (b), $A_{m}^{m_{\mathbb{S}}} = A_{1,w,-(1+w)}^m \times A_{1,-(1+w),w}^m \times A_{m,-(1+w),1,w}^m$. Since $\Phi_2 : \mathbb{Q}[\sigma, 1] \to \text{End}^0(A_{1,w,-(1+w)}^m)$ is surjective \cite{[13]}, and $\rho \in \text{Aut}(F_m)$ induces the isomorphisms $F_{1,w,-(1+w)}^m \xrightarrow{\rho} F_{m,-(1+w),1,w}^m \xrightarrow{\rho} F_{1,-(1+w),w}^m \xrightarrow{\rho} F_{1,w,-(1+w)}^m$, a proof similar to the one given above for subcase (a) shows that $\lambda_{\mathbb{S}}^m$ is surjective.

We have shown that

**LEMMA 2.1.** If $(m, S) = 1$ and $A_{S}^m$ is non-simple, then $\lambda_{\mathbb{S}}^m$ is surjective.

**Case 2.** $A_{S}^m$ is simple and $F_{S}^m$ is hyperelliptic.

Here, we use the results of Coleman \cite{[2]}: $S$ is the class of a permutation of $(1, 1, -2)$. Since the 3 distinct permutations of $(1, 1, -2)$ give rise to 3 distinct classes in $S_m$, we have $A_{m_{\mathbb{S}}}^m = A_{1,1,-2}^m \times A_{1,-2,1}^m \times A_{-2,1,1}^m$.

**LEMMA 2.2.** If $(m, S) = 1$ and $A_{S}^m$ is simple and $F_{S}^m$ hyperelliptic, $\lambda_{\mathbb{S}}^m$ is surjective.

**Proof.** $\sigma \in \text{Aut}(F_m)$ induce isomorphisms $F_{1,1,-2}^m \xrightarrow{\rho} F_{m,-2,1,1}^m \xrightarrow{\rho} F_{1,-2,1}^m$. Thus we identify $A_{m_{\mathbb{S}}}^m = A_{1,-2,1}^m \times A_{-2,1,1}^m \times A_{1,1,-2}^m$ with $(A_{1,-2,1}^m)^3$ via the isomorphism $(1 \times \rho \times \rho^2)$. Consider the composition

$\lambda = F_{1,1,-2}^m \rho \lambda_{\mathbb{S}}^m : \mathbb{Q}[G_{m}, \rho] \to M_3(K)$,

where we identify $\text{End}(A_{1,-2,1}^m)$ with $\mathbb{Z}[[\xi]]$ by mapping $\sigma$ to $\xi$. That $\lambda$ is surjective follows from the following:

$\lambda(\rho) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}$,

$\lambda(\rho^2) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}$,

$\lambda(\tau) = \Delta(\zeta, \xi, \xi^{-2})$,

$\lambda(I_0(\sigma^2 \tau)) = \Delta(0, m, 0)$,

$\lambda(I_0(\sigma \tau^{-1})) = \Delta(0, 0, m)$.

**Case 3.** $A_{S}^m$ is simple and $F_{S}^m$ is non-hyperelliptic.

Then $S$ is the class of $(a, b, c)$, where $a$, $b$ and $c$ are distinct elements in $\mathbb{Z}/m\mathbb{Z} - \{0\}$ with $a + b + c = 0$, and

$A_{m_{\mathbb{S}}}^m = A_{a,b,c}^m \times A_{a,c,b}^m \times A_{b,a,c}^m \times A_{b,c,a}^m \times A_{c,a,b}^m \times A_{c,b,a}^m$. 


We identify $A_{m}^{n}$ with $(A_{a,b,c})^{b}$ via the isomorphism
\[ g = 1 \times (p^2) \times 1 \times \rho \times \rho \times (p^2), \]
and fix an isomorphism $\text{End}(A_{a,b,c}) \to \mathbb{Z}[\zeta]$. Consider the composition
\[ \lambda = F_{\phi^{m}_{[S]} : \mathbb{Q}[G] \to M_{6}(K)}. \]
We have
\[ \lambda(I_{o}(\sigma^{b\tau} - \delta)) = \Delta(0, 0, 0, 0, 0, 0), \quad \lambda(I_{o}(\sigma^{\tau} - \delta)) = \Delta(0, m, 0, 0, 0, 0), \]
\[ \lambda(I_{o}(\sigma^{\tau}) - \delta) = \Delta(0, 0, 0, 0, 0, 0), \quad \lambda(I_{o}(\sigma^{b\tau}) - \delta) = \Delta(0, 0, 0, 0, 0, 0), \]
\[ \lambda(I_{o}(\sigma^{a\tau} - \delta)) = \Delta(0, 0, 0, 0, 0, 0), \quad \lambda(I_{o}(\sigma^{a\tau}) - \delta) = \Delta(0, 0, 0, 0, 0, 0). \]
Also, there exists an $\alpha \in G_{m}$ such that $\alpha$ has exact order $m$ in $\text{Aut}(F_{m}) \subseteq \text{Aut}(J_{m})$ since $(m, S) = 1$. Hence, $\Delta_{6}(K) \subseteq \text{Im}(\lambda) \subseteq M_{6}(K)$.
Furthermore, there are units $a_{j}$ and $b_{j}$ in $\mathbb{Z}[\zeta]$ such that
\[ \lambda(\rho) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{1} \\ 0 & a_{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{4} \\ a_{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{6} & 0 & 0 & 0 \end{array}\right), \quad \lambda(\iota) = \left(\begin{array}{cccccc} 0 & 0 & b_{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{2} & 0 \\ b_{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{4} \\ 0 & b_{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{6} & 0 & 0 \end{array}\right). \]
Finally, we note that $M_{6}(K)$ is the direct sum of the subspaces
\[ \Delta(K), \Delta(K)\lambda(\rho), \Delta(K)\lambda(\rho^{2}), \Delta(K)\lambda(\iota), \Delta(K)\lambda(\iota^{2}). \]
Hence, $\lambda$ is surjective.

**Lemma 2.3.** If $(m, S) = 1$, $A_{S}^{m}$ is simple and $F_{S}^{m}$ is non-hyperelliptic, then $\lambda_{[S]}^{m}$ is surjective.

We note that for any positive divisor $M$ of $m$, the morphism
\[ F_{m} \to F_{M}, \quad (X, Y, Z) \to (X^{m/M}, Y^{m/M}, Z^{m/M}) \]
induces an isomorphism
\[ F_{a,b,c}^{m} \approx F_{a+b}^{m} \quad \text{(where } a, b, a+b \in \mathbb{Z}/M\mathbb{Z} - \{0\}). \]
Together with this observation, Lemmas 2.1, 2.2 and 2.3 imply $\lambda_{[S]}^{m}$ is surjective for any $S \in S_{m}$. In what follows, we will prove that $\Phi: \mathbb{Q}[G] \to \text{End}^{0}(J_{m})$ is surjective. The isogeny $\varphi: J_{m} \to \Pi_{[S]} J_{S_{m}/m} \sim J_{m}^{[S]}$ induces an isomorphism $F_{\varphi}$ between $\text{End}^{0}(J_{m})$ and $\Pi_{[S]} J_{S_{m}/m} \sim \text{End}^{0}(A_{S}^{m})$. Consider $F = F_{\varphi} \Phi = (\lambda_{[S]}^{m})_{[S]} J_{S_{m}/m}$. For each $S' \in [S]$, let $g(S') \in G_{m}$ be such that $F_{S'} = F_{m}/\langle g(S') \rangle$. Then
\[ \lambda_{[S]}^{m} \left( \sum_{S' \in [S]} I_{o}(g(S')) \right) = m \quad \text{but} \quad \lambda_{[S]}^{m} \left( \sum_{S' \in [S]} I_{o}(g(S')) \right) = 0 \quad \text{for } [S] \neq [S]. \]
Since each $\lambda_{[S]}^{m}$ is surjective, $F$ is surjective.
3. The kernel of $\varphi$

Throughout this section, let $m = p$ be a prime. By Pic functoriality, we have from the canonical projection $F_p \to F_p^p$, the homomorphism $(\varphi_p^p)^*: J_p^p \to J_p$. Then $(\varphi_p^p)^*$ is the dual homomorphism to $(\varphi_p^p)_*$, and

$$\varphi = \prod_{S \in \mathcal{S}_p} (\varphi_p^p)^*: J_p \to \prod_{S \in \mathcal{S}_p} J_p^p$$

and

$$\hat{\varphi} = \sum_{S \in \mathcal{S}_p} (\varphi_p^p)^*: \prod_{S \in \mathcal{S}_p} J_p^m \to J_m$$

are dual homomorphisms by the next lemma.

**LEMMA 3.1.** Let $f: A \to B$ and $g: A \to C$ be homomorphisms of abelian varieties. Then, identifying $(B \times C)^\wedge$ with $\hat{B} \times \hat{C}$, the dual of $(f, g): A \to B \times C$ is $\hat{f} + \hat{g}: \hat{B} \times \hat{C} \to \hat{A}$.

**LEMMA 3.2.** Denoting the genus of $F_p$ by $g$, $\hat{\varphi} = p$ and $\deg(p) = p^g$.

**Proof.** The proof of the lemma can be found in Corollary 3.8 of [12].

Let $\mathcal{L}$ be a line bundle on an abelian variety $A$ over $\mathbb{C}$. For a point $x$ on $A$, let $T_x$ be the translation by $x$ map, and let

$$\phi_\mathcal{L}: A \to \hat{A}, x \mapsto \text{isomorphism class of } T_x^\wedge \mathcal{L} \otimes \mathcal{L}^{-1} \text{ in Pic}(A).$$

Then $\phi_\mathcal{L}$ is a homomorphism between $A$ and its dual $\hat{A}$ ([14], Section 8). Mumford ([14], Section 23) defined a skew-symmetric bihomomorphism

$$e_\mathcal{L}: K(\mathcal{L}) \times K(\mathcal{L}) \to \mathbb{G}_m,$$

where $K(\mathcal{L}) = \text{Ker}(\phi_\mathcal{L})$, with the property that if

$$e_n: A[n] \times \hat{A}[n] \to \mu_n$$

is the Weil $e_n$-pairing on $A$, then

$$x \in A[n], \quad y \in n^{-1}_A(K(\mathcal{L})) = \phi_\mathcal{L}(A[n])$$

imply

$$e_n(x, \phi_\mathcal{L}(y)) = e_\mathcal{L}(x, y). \quad (3.1)$$

**PROPOSITION 3.3.** Let $A$ and $B$ be principally polarized abelian varieties, and let $\varphi: A \to B$ be an isogeny which respects the principal polarizations of $A$ and $B$. If $\text{Ker}(\varphi) \subseteq A[n]$ and the order of $A[n]$ is the square of the order of $\text{Ker}(\varphi)$, then $\text{Ker}(\varphi)$ and $\text{Ker}(\hat{\varphi})$ are maximal isotropic subgroups in regard to the respective Weil $e_n$-pairings.

**Proof.** Let $\mathcal{M}$ be a line bundle on $B$ associated to a theta divisor $\Theta_B$ of $B$, and let $\mathcal{L} = \varphi^* \mathcal{M}$. Then $\mathcal{L}$ is a line bundle on $A$ associated to a theta divisor on $A$, and $\mathcal{L}^m \approx \varphi^*(\mathcal{M}^m)$. Applying the corollary to Theorem 2 in Section 23 of [14],

$$e_\mathcal{L}|_{\text{Ker}(\varphi) \times \text{Ker}(\varphi)} \equiv 1. \quad (3.2)$$
Since the order of $A[n]$ is the square of $\text{Ker}(\phi)$, from (3.1), (3.2) and Theorem 4 in Section 23 of [14], we conclude that $\text{Ker}(\phi)$ is a maximal isotropic subgroup of $A[n]$ with respect to the Weil $e_n$-pairing.

The dual $\hat{\phi}$ of $\phi$ respects the principal polarizations of $B$ and $A$, and $\text{Ker}(\hat{\phi})$ has the same order as $\text{Ker}(\phi)$. Therefore the same argument as above shows that $\text{Ker}(\hat{\phi})$ is maximal isotropic with respect to the Weil $e_n$-pairing on $B$. □

The following corollary answers a question of Rohrlich.

**COROLLARY 3.4.** The kernel of $\phi: J_p \rightarrow \prod_{S \in S_p} J_p^S$ is a maximal isotropic subgroup of $J_p[p]$ with respect to the Weil $e_p$-pairing on $J_p$. The same result holds for the kernel of $\phi$.

**Proof.** The homomorphism $J_p \rightarrow J^p_{r, -(1+r)}$ respects the principal polarizations of $J_p$ and $J^p_{r, -(1+r)}$ since it is induced from the covering $F_p \rightarrow F^p_{r, -(1+r)}$ by Albanese functoriality. Therefore $\phi$ respects the principal polarizations of $A = J_p$ and $B = \prod_{r=2}^{\infty} J^p_{r, -(1+r)}$. The corollary is then a direct application of Proposition 3.3. □

**LEMMA 3.5.** Consider the homomorphism

$$\lambda = (\lambda_{S})_{S \in S_p}: \mathbb{Q}[G] \rightarrow \text{End}^0\left(\prod_{\{S\} \in S_p/\sim} J^S_p\right) = \prod_{\{S\} \in S_p/\sim} \text{End}^0(J^S_p).$$

Then $p^2 \text{End}(J^S_p) \subseteq \lambda(\mathbb{Z}[G])$.

**Proof.** For each $[S] \in S_p$, as noted in Section 2, there is an element $\alpha_S \in \mathbb{Z}[G_p]$ for which $\lambda^S_{\{S\}}(\alpha_S) = p$ and $\lambda^S_{\{S\}}(\alpha_S) = 0$ for $[S] \neq [S']$. If we re-examine the proof to establish the surjectivity of $\lambda^S_{\{S\}}$, we see that $p \text{End}(J^S_p) \subseteq \lambda^S_{\{S\}}(\mathbb{Z}[G])$. Hence it follows that

$$\{0\} \times \cdots \times \{0\} \times p^2 \text{End}(J^S_p) \times \{0\} \times \cdots \times \{0\} \subseteq \text{Im}(\mathbb{Z}[G]).$$

This completes the proof of the lemma. □

Let $\phi: A \rightarrow B$ be an isogeny with kernel $K$ of exponent $m$. Given $\alpha \in \text{End}(A)$, there is a unique $\beta \in \text{End}(B)$ such that $\phi \circ \alpha = n\beta \circ \phi \iff \alpha(\text{Aut}^{-1}(K)) \subseteq K$. Thus given $\alpha \in \text{End}(A)$, there is a unique $\beta \in \text{End}(B)$ for which $\phi m\beta \alpha = m\beta \phi$. Thus implies that $F_{\phi\beta} : \text{End}^0(A) \rightarrow \text{End}^0(B)$ maps $m \text{End}(A)$ into $\text{End}(B)$.

**PROPOSITION 3.6.** $p^2 \text{End}(J_p) \subseteq \Phi(\mathbb{Z}[G])$.

**Proof.** Applying Maschke’s theorem ([5], Theorem 3.14) to the exact sequence in Theorem A, there is an idempotent $e \in \mathbb{Q}[G]$ such that (1) $T = \mathbb{Q}[G]e$, (2) the map $f: \mathbb{Q}[G] \rightarrow T \times \text{End}^0(J_p), X \rightarrow (Xe, \Phi(X))$ is an isomorphism. Clearly $\Sigma = \mathbb{Z}[G]e \times \text{End}(J_p)$ is a $\mathbb{Z}$-order in $\Sigma \otimes \mathbb{Q}$, $f(\mathbb{Z}[G]) \subseteq \Sigma$, and with the identification $f$, $\Phi$ becomes the projection map $T \times \text{End}^0(J_p) \rightarrow \text{End}^0(J_p)$,
(\(X, Y\)) \(\to Y\). Since \(G\) has order \(6p^2\), \(\Sigma\) is contained in \((6p^2)^{-1}Z[G]\). Applying \(\Phi\), we obtain \(\text{End}(J_{\rho}) \subseteq (6p^2)^{-1}\Phi(Z[G])\). Maintaining the notation of Lemma 3.5, we have \(\lambda = F_{\varphi}\Phi\). The remarks preceding the lemma together with Lemmas 3.2 and 3.8 imply
\[
F_{\varphi}(p^3 \text{End}(J_{\rho})) \subseteq p^2 \text{End} \left( \prod_{\sigma \in \mathbb{S}_p} J_{\sigma}^3 \right) \subseteq \lambda(Z[G]).
\]
Hence, \(p^3 \text{End}(J_{\rho})\) is contained in \(\Phi(Z[G])\). The g.c.d. of \(6p^2\) and \(p^3\) is \(p^2\), and the proposition follows.

4. Singular homology of Fermat curves

It is known (see the appendix in [9]) that \(H_1(F_m(C), \mathbb{Z})\) is a cyclic module over \(Z[G_m]\) with
\[
(1 - \sigma)(1 - t)I
\]
as a generator. Hence \(g\) as defined in Proposition B is also a generator.

By Lemma 1.1,
\[
A, B, C \in \text{Ann}_{Z[G_m]}(H_1(F_m(C), \mathbb{Z})),
\]
where \(A, B, C\) are as defined in Section 1.

We will determine, in what follows, generators for this ideal of \(Z[G_m]\).

A special case of Lemmas 5.2 and 5.3 is that the ideal \(J\) of \(Q[G_m]\) generated by \(A, B, C\) has dimension \((3m - 2)\) as a vector space over \(Q\). Fix a basis \(\{A_1, \ldots, A_{3m-2}\}\) for \(J\) and extend it to a basis \(\{A_1, \ldots, A_{3m-2}, B_1, \ldots, B_l\}\) of \(Q[G_m]\), where \(l + 3m - 2 = m^2\). Then \(\{B_1g, \ldots, B_lg\}\) spans \(H_1(F_m(C), Q)\) over \(Q\), and is therefore a basis because the genus of \(F_m\) is \(l/2\). In particular, the annihilator of \(H_1(F_m(C), Q)\) over \(Q[G_m]\) is \(J\).

Let \(\Delta = \Sigma' \tau^r \sigma^{-s} \in Q[G_m]\), where the sum \(\Sigma'\) is taken over \((r, s)\) with \(0 \leq r, s, r + s \leq m - 2\). We note that \(1 - \sigma\) is a unit in the ring \(R = Q[\sigma]/(I_0(\sigma))\) and that \((1 - \sigma^{-1})\Delta = I_0(\tau) - \sigma I_0(\sigma\tau)\) in \(Z[G_m]\). Thus, in \(R[\tau], \Delta R[\tau] \subseteq (I_0(\tau), I_0(\sigma\tau))R[\tau] \) and \((I_0(\sigma), \Delta)Z[G_m] \subseteq J \cap Z[G_m]\).

The latter inclusion induces an epimorphism
\[
Z[G_m]/(I_0(\sigma), \Delta) \to Z[G_m]/(J \cap Z[G_m]).
\]
By definition of \(\Delta\), there is a surjective mapping
\[
\sum_{0 \leq r \leq m - 2, 0 \leq s \leq m - 3} Z\sigma^r \tau^s \to Z[G_m]/(J \cap Z[G_m])
\]
between free \(Z\)-modules of rank \(2l\). Therefore, the latter map is an isomorphism and we have
PROPOSITION 4.1. The annihilator of the $\mathbb{Z}[G_m]$-module $H_1(F_m(C), \mathbb{Z})$ is the ideal of $\mathbb{Z}[G_m]$ generated by $I_0(\sigma)$ and $\Delta$.

It follows that $\{\sigma^r\tau^s g | 0 \leq r \leq m - 2, 0 \leq s \leq m - 3\}$ is a $\mathbb{Z}$-basis of $H_1(F_m(C), \mathbb{Z})$.

We recall that $H^0(F_m, \Omega^1)$ is spanned by

$$w_{r,s} = x^{r-1}y^{s-1}\frac{dx}{y^{m-1}} (1 \leq r, s, r + s \leq m - 1).$$

To prove that $i(g) = g$ and $\rho(g) = g$

in homology is equivalent to showing that

$$\int_{i(g)+g} w_{r,s} = \int_g (i^*w_{r,s} + w_{r,s}) = 0$$

and

$$\int_{\rho(g)-g} w_{r,s} = \int_g (\rho^*w_{r,s} - w_{r,s}) = 0$$

for all $r, s \geq 1$ and $r + s \leq m - 1$, i.e. that

$$\int_g w_{s,r} = \int_g w_{r,s}$$

(4.1)

and

$$\int_g w_{s,m-r-s} = \int_g w_{r,s}$$

(4.2)

for all $r, s$ as stated above.

If $B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt$ is the classical beta function, we have by Rohrlich's calculations in [9] that equations (4.1) and (4.2) are equivalent to

$$\frac{B(s/m, r/m)}{m} (1 - \zeta^s)(1 - \zeta^r) = \frac{B(r/m, s/m)}{m} (1 - \zeta^r)(1 - \zeta^s)$$

(4.3)

and

$$\alpha^{2r+s+m} \frac{B(s/m, 1-r+s/m)}{m} (1 - \zeta^s)(1 - \zeta^{-r-s}) = \frac{B(r/m, s/m)}{m} (1 - \zeta^r)(1 - \zeta^s)$$

(4.4)

respectively. (4.3) is trivially true. (4.4) follows from the identity

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.$$
5. Endomorphisms, 1

Let $X$, $Y$, $Z$, $\tilde{X}$, $\tilde{Y}$, $\tilde{Z} \in \mathbb{Q}[G_m]$, and

$$\alpha = X + Y\rho + Z\rho^2 + \tilde{X}\iota + \tilde{Y}\iota + \tilde{Z}\rho^2 \in \mathbb{Q}[G].$$

Then $\Phi(\alpha) \in \text{End}(J_m)$ if and only if, for all $r, s \in \mathbb{Z}/m\mathbb{Z}$, $\alpha(\sigma^r g) \in H_1(F_m(C), \mathbb{Z})$, where $g$ is as defined in Proposition B. Since $\rho(g) = g$ and $\iota(g) = -g$, Theorem C follows.

Let $I_k(T) \in \mathbb{Z}[T]$ and $W \in \mathbb{Q}[G_m, \rho]$ be as defined in the Introduction. Let $w = \Phi(W) \in \text{End}^0(J_m)$. The rest of this section is devoted to showing that $w$ is in $\text{End}(J_m)$ but not in $\Phi(\mathbb{Z}[G])$.

Since

$$I_k(T) = \frac{T^k}{k!} \frac{d^k}{dT^k} \left( \frac{1 - T^m}{1 - T} \right),$$

it follows, using Leibnitz's rule for derivatives and induction, that

**Lemma 5.1.** When $0 \leq k < m$, $(1 - T)I_k(T) = -\binom{m}{k} T^m + T I_{k-1}(T)$.

**Lemma 5.2.** Let $F$ be an arbitrary field, and let $\theta$ be the element $(1 - \sigma)(1 - \tau)(1 - \sigma \tau)$ of the group ring $F[G_m]$. Then $\dim_F \ker_F(\theta) = 3m - 2$, where $\ker_F(\theta)$ is the annihilator of $\theta$ in $F[G_m]$.

**Proof.** Let $X = \sum a_{r,s} \sigma^r \tau^s \in F[G_m]$. $X$ is in $\ker_F(\sigma - 1)$ if and only if $a_{r,s} = a_{r+1,s}$ for all $(r, s)$. Thus $\ker_F(\sigma - 1) = I_0(\sigma) F[G_m]$ has dimension $m$ over $F$. The same is true if $\sigma$ is replaced by $\tau$ or $\sigma \tau$. $X$ is in $\ker_F(\sigma - 1)$ and $(\tau - 1)F[G_m]$ if and only if $a_{r,s} = a_{r+1,s}$ for all $(r, s)$, and $\sum_s a_{r,s} = 0$ for all $r$. For such an $X$, all the $a_{r,s}$'s are uniquely determined once the $a_{0,s}$'s are known for $0 < s < m - 1$. So $\ker(\sigma - 1) \cap (\tau - 1)F[G_m]$ and $(\sigma - 1)(\tau - 1)F[G_m]$ have dimensions $m - 1$ and $(m^2 - m) - (m - 1)$ over $F$. Furthermore,

$$(\sigma - 1)F[G_m] \cap (\tau - 1)F[G_m]$$

has dimension

$$\dim(\sigma - 1)F[G_m] + \dim(\tau - 1)F[G_m] - \dim(\sigma - 1, \tau - 1)F[G_m] = (m - 1)^2.$$

Therefore,

$$(\sigma - 1)F[G_m] \cap (\tau - 1)F[G_m]$$

is equal to

$$(\sigma - 1)(\tau - 1)F[G_m].$$

Finally, we note that $X$ is in $\ker(\sigma \tau - 1)$ and $(\sigma - 1)(\tau - 1)F[G_m]$ if and only if
\(a_{r,s} = a_{r+1,s+1}\) for all \((r, s)\), and \(\Sigma_r a_{r,s} = 0\) for all \(s\). For such an \(X\), all the \(a_{r,s}\)’s are uniquely determined if \(a_{0,s} (0 \leq s \leq m-2)\) are known. Therefore

\[\text{Ker}_F(\sigma - 1) \cap (\sigma - 1)(\tau - 1)F[G_m] \quad \text{and} \quad \text{Ker}_F(\theta)\]

have dimensions \(m-1\) and \(m^2-(m-1)^2-(m-1))=3m-2\) respectively. \(\square\)

**Lemma 5.3.** Maintaining the notation of Lemma 6.2, \(\text{Ker}_F(\theta)\) is the ideal of \(F[G_m]\) generated by \(I_0(\sigma), I_0(\tau), I_0(\sigma \tau)\) and \(I_1(\sigma)I_1(\tau)\).

**Proof.** Let \(J_1, J_2\) and \(J_3\) be the principal ideals of \(F[G_m]\) generated by \(I_0(\sigma), I_0(\tau)\) and \(I_0(\sigma \tau)\) respectively, and let \(J_F = \sum_{i=1}^{k} J_i\). We claim that \(J_F\) has dimension \(3m-2\) and \(3m-3\) depending on whether \(m\) is relatively prime to the characteristic of \(F\) or not. We fix the bases \(\{\sigma' I_0(\sigma)\}, \{\sigma' I_0(\tau)\}, \{\sigma' I_0(\sigma \tau)\}\), where \(r\) ranges between 0 and \(m-1\) inclusive in each case, for \(J_1, J_2\) and \(J_3\) respectively.

Let

\[X = \sum_{0 \leq r < m-1} a_r \sigma^r I_0(\sigma) = \sum_{0 \leq r < m-1} b_r \sigma^r I_0(\sigma)\]

be in \(J_1 \cap J_2\), where each \(a_r, b_r \in F\). By comparing the coefficients of \(\tau', a_r = a_0\) for all \(r\). Hence, \(J_1 \cap J_2\) is \(F. I_0(\sigma)I_0(\tau)\), and

\[\{\sigma' I_0(\sigma), \tau' I_0(\sigma) | 0 \leq r \leq m - 1, 0 \leq s \leq m - 2\}\]

is an \(F\)-basis for \((J_1 + J_2)\).

Let \(a_r, b_r, c_r \in F\) be such that

\[Y = \sum_{0 \leq r \leq m - 1} \sigma' a_r I_0(\tau) + \sum_{0 \leq s \leq m - 2} b_s \tau^s I_0(\sigma) = \sum_{0 \leq r \leq m - 1} c_r \sigma^r I_0(\sigma \tau)\]

Comparing the coefficients of \(\sigma' \tau^k\) and \(\sigma' \tau^{m-1}\), where \(0 \leq k \leq m - 2\) and \(0 \leq r \leq m - 1\), we obtain \(a_r + b_k = c_{r-k}\) and \(a_r = c_{r+1}\). In particular, \(c_{m-1} = a_{m-1} + b_0 = c_0 + b_0\). By induction, \(c_{m-k} = c_0 + kb_0\) for \(1 \leq k \leq m\). If \(m\) is prime to the characteristic of \(F\), we conclude that \((J_1 + J_2) \cap J_3 = F. I_0(\sigma)I_0(\tau)\) and \(J_F\) has dimension \(3m - 2\).

Let \(m\) be a multiple of the characteristic of \(F\). Then, maintaining the notation of the previous paragraph, \(Y = (c_0 I_0(\sigma) - b_0 I_1(\sigma))I_0(\sigma \tau)\), since \(c_r = c_{m-(m-r)} = c_0 + (m-r)b_0\). Thus,

\[(J_1 + J_2) + J_3 \leq F. I_0(\sigma)I_0(\tau) \oplus F. I_1(\sigma)I_0(\sigma \tau)\]

Since \(\text{Ker}_F((\sigma - 1)(\tau - 1))\) and \((J_1 + J_2)\) have the same dimension (see the proof of Lemma 5.2), and the latter is contained in the former, they are equal. By Lemma 5.1, \(I_1(\sigma)I_0(\sigma \tau)\) is annihilated by \((\sigma - 1)(\tau - 1)\). Hence, \((J_1 + J_2) \cap J_3\) equals \(F. I_0(\sigma)I_0(\tau) \oplus F. I_1(\sigma)I_0(\sigma \tau)\), \(J_F\) has dimension \(3m - 3\) and a basis

\[\{\tau' I_0(\sigma), \sigma' I_0(\tau), \sigma' I_0(\sigma \tau) | 0 \leq r \leq m - 1, 0 \leq s \leq m - 2, 0 \leq t \leq m - 3\}\]

By Lemma 5.2, \(\theta\) annihilates \(Z = I_1(\sigma)I_1(\tau)\). We claim that \(Z\) is not in \(J_F\).
Suppose, on the contrary, that

\[ Z = \sum_{0 \leq r \leq m-1} a_r \sigma^r I_0(\sigma) + \sum_{0 \leq s \leq m-2} b_s \tau^s I_0(\tau) + \sum_{0 \leq t \leq m-3} c_t \sigma^t I_0(\sigma \tau). \]

Then a contradiction follows by comparing the coefficients of \( \sigma^r, \tau^s, \sigma^{m-2} \tau^{m-1} \) for \( 0 \leq r \leq m-1 \) and \( 1 \leq s \leq m-1 \). We omit the details of this routine calculation.

Let \( A \) be the ring \( \mathbb{Z}[G_m]/(J \cap \mathbb{Z}[G_m]) \). We recall (Proposition 4.1) that the ideal \( J \cap \mathbb{Z}[G_m] \) is generated by \( I_0(\sigma) \) and \( \Delta = \Sigma' \sigma^{-s} \), where the \( \Sigma' \) is taken over \( (r, s) \) with \( 0 \leq r, s, r + s \leq m - 2 \). Under the homomorphism \( \mathbb{Z}[G_m] \to \mathbb{Z}[\tau] \), in which \( \sigma \to 1 \) and \( \tau \to \tau \), the elements \( I_0(\sigma) \) and \( \Delta \) are mapped onto \( m \) and \( f(\tau) \) respectively, where

\[ f(\tau) = \sum_{l=0}^{m-2} (m - l - 1) \tau^l. \]

Then

\[ I_1(\tau) = -\tau f(\tau) = 0 \quad \text{in} \quad A/(1 - \sigma)A = (\mathbb{Z}/m \mathbb{Z})[\tau]/(f(\tau)) \]

and

\[ I_1(\tau) \in (1 - \sigma)A. \]

By symmetry,

\[ I_1(\sigma) \in (1 - \tau)A. \]

By Lemma 5.1,

\[ m = (\sigma - 1) I_1(\sigma) = (1 - \tau) I_1(\tau) \]

in \( A \). We conclude that

\[ I_1(\sigma) I_1(\tau) \in m \mathbb{Z}[G_m] + J. \]

**Lemma 5.4.** Let \( X \in \mathbb{Q}[G_m] \). Then \( X \in \mathbb{Z}[G_m] + J \) if and only if \( \theta X \in \mathbb{Z}[G_m] \).

**Proof.** Let \( l \) be a prime. Suppose that \( Y = lX \in \mathbb{Z}[G_m] \), and \( \theta X \in \mathbb{Z}[G_m] \). Then \( \theta Y = l(\theta X) \equiv 0 \pmod{l} \). By Lemma 5.3 and the remark before Lemma 5.4,

\[ Y \in l \mathbb{Z}[G_m] + J \quad \text{and} \quad X \in \mathbb{Z}[G_m] + J. \]

Assume now that

\[ l^n X \in \mathbb{Z}[G_m] \quad \text{and} \quad \theta X \in \mathbb{Z}[G_m]. \]

Then \( l^n - 1 X \in \mathbb{Z}[G_m] + J \). Choose \( Z \in J \) such that \( l^n - 1 (X - Z) \in \mathbb{Z}[G_m] \). Also \( \theta(X - Z) = \theta X \in \mathbb{Z}[G_m] \). By induction hypothesis, \( X - Z \in \mathbb{Z}[G_m] + J \). Hence we have proved that if \( l^n X \in \mathbb{Z}[G_m] \) and \( \theta X \in \mathbb{Z}[G_m] \), then \( X \in \mathbb{Z}[G_m] + J \).
We can now prove the following statement by induction on $k$ ($k \in \mathbb{Z}_{>0}$): if $kX \in \mathbb{Z}[G_m]$ and $\theta X \in \mathbb{Z}[G_m]$, then $X \in \mathbb{Z}[G_m] + J$, since we know it to be true for any prime power $k=l^n$.

Applying Corollary 5.4, we obtain

**COROLLARY 5.5.** Let $X \in \mathbb{Z}[G_m]$. Then $X \in m\mathbb{Z}[G_m] + J$ if and only if $\theta X \equiv 0 \pmod{m}$.

**LEMMA 5.6.** Let

$$X, Y, Z \in \mathbb{Q}[G_m].$$

Then

$$\Phi(X + Yp + Zp^2) \in \text{End}(J_m)$$

if and only if

$$X \sigma^r\tau^s + Y \sigma^{-s}\tau^r - Z \sigma^r\tau^s \in \mathbb{Z}[G_m] + J \quad \forall (r, s) \in \mathbb{Z}^2.$$

**Proof.** This follows directly from Theorem C. \qed

**PROPOSITION 5.7.** $w \in \text{End}(J_m)$.

**Proof.** Let

$$\eta_{r,s} = I_1(\sigma)I_3(\tau)(\sigma^{-s}\tau^{-s} - \sigma^r\tau^s) + I_3(\sigma)I_1(\tau)(\sigma^r\tau^r - \sigma^{-s}\tau^{-s}).$$

In view of Corollary 5.5 and Lemma 5.6, to prove the proposition, it suffices to verify that $\theta \eta_{r,s} \in m\mathbb{Z}[G_m]$ for all $(r, s) \in \mathbb{Z}^2$.

By Lemma 5.1, and using the fact that $T^a \equiv 1 + a(T - 1) \pmod{(T - 1)^2}$, we have that:

$$\theta I_1(\sigma)I_3(\tau)(\sigma^{-s}\tau^{-s} - \sigma^r\tau^s) \equiv (2s - r)I_0(\sigma)I_0(\tau) \pmod{m},$$

$$\theta I_3(\sigma)(\sigma^r\tau^r - \sigma^{-s}\tau^{-s}) \equiv (r - 2s)I_0(\sigma)I_0(\tau) \pmod{m}.$$

Therefore, $\theta \eta_{r,s} \equiv 0 \pmod{m}$, as required. \qed

**LEMMA 5.8.** Let $X, Y, Z \in \mathbb{Q}[G_m]$, and let $I$ be either an ideal of $\mathbb{Q}[G_m]$ or the subring $\mathbb{Z}[G_m] + J$. Suppose that

$$X \sigma^r\tau^s + Y \sigma^{-s}\tau^r - Z \sigma^r\tau^s \in I,$$  \hspace{1cm} (5.1)

for all $(r, s) \in \mathbb{Z}^2$. Then $(\sigma - 1)^2X$, $(\sigma - 1)(\tau - 1)X$, $(\tau - 1)^2X \in I$, with similar statements for $Y$ and $Z$.

**Proof.** From $X + Y + Z \in I$ and (5.1), we obtain

$$Y(\sigma^r + s\tau^s - 1) + Z(\sigma^s - r\tau^r + s - 1) \in I,$$  \hspace{1cm} (5.2)

for all $(r, s) \in \mathbb{Z}^2$. By setting

$$r \equiv -s \pmod{m}, \quad r \equiv 2s \pmod{m} \quad \text{and} \quad 2r \equiv s \pmod{m},$$

we have that
and using the hypothesis that \( m \) is coprime to 3, we obtain:

\[
Y(\sigma \tau - \sigma) + Z(1 - \sigma) \in I, \tag{5.3}
\]
\[
Y(\sigma - 1) + Z(\sigma \tau - 1) \in I, \tag{5.4}
\]
\[
Y(\sigma \tau - 1) + Z(\tau - 1) \in I. \tag{5.5}
\]

Setting \( r = s = 1 \), we get

\[
Y(\sigma^2 \tau - 1) + Z(\sigma^2 - 1) \in I. \tag{5.6}
\]

From (5.4) and (5.6), it follows that \( Y(\sigma \tau - 1) + Z(\tau - \tau) \in I \). Together with (5.5), the latter gives \((\tau - 1)^2 Z \in I\). By symmetry, \((\sigma - 1)^2 Z \in I\). Adding (5.3) and (5.4), we obtain \( Y(\sigma \tau - 1) + Z(\sigma \tau - \sigma) \in I \). Together with (5.5), \((\sigma - 1)(\tau - 1)Z \in I\) follows.

**COROLLARY 5.9.** Let

\[
A = \Phi(Z[G_m]) \quad \text{and} \quad B = \Phi(Z[G_m, \rho]).
\]

Then

\[
A^3 \to B, (X, Y, Z) \to X + Y \rho + Z \rho^2
\]

is a left \( A \)-module isomorphism. In particular, \( \ker(\Phi|_{Q[G_m, \rho]}) = JQ[G_m, \rho] \).

**Proof.** Let \( \alpha = X + Y \rho + Z \rho^2 \in \ker(\Phi) \), with \( X, Y, Z \in Q[G_m] \). Since \( \alpha \) acts as the zero endomorphism, we have

\[
X \sigma \tau^s + Y \sigma^{-1} \tau^{-s} + Z \sigma^s \tau^{-s} \in J \quad \forall (r, s) \in \mathbb{Z}^2.
\]

By Lemma 5.8, \((\sigma - 1)(\tau - 1)X \in J\). Since \((\sigma - 1)(\tau - 1)\) is a unit in \( Q[G_m]/J \), we have that \( X \in J \). Likewise, \( Y \) and \( Z \) are in \( J \). This proves the lemma.

**COROLLARY 5.10.** \( m(\text{End}(J_m) \cap \Phi(Q[G_m, \rho])) \subseteq \Phi(Z[G_m, \rho]). \)

**Proof.** This follows directly from Lemmas 5.7 and 5.8 (taking \( l = Z[G_m] + J \)), and the fact that there is a \( y \in Z[G_m] \) such that \( m \equiv (\sigma - 1)(\tau - 1)y \pmod{J} \).

**COROLLARY 5.11.** The element \( W \in Q[G_m, \rho] \) is not in \( Z[G] + \ker(\Phi) \).

**Proof.** Let \( m \geq 7 \), and let \( \tilde{\theta} \) be \( \theta(\sigma^{-1} - 1)(\sigma \tau - 1)(\sigma^2 \tau - 1) \) in \( Q[G_m] \). By Theorem A, \( \tilde{\theta} \) annihilates \( \ker(\Phi) \). Suppose that \( W = X + Y \), where \( X \in Z[G_m] \) and \( Y \in \ker(\Phi) \). Then

\[
\tilde{\theta}W = \tilde{\theta}X \in Z[G_m] \quad \text{and} \quad \tilde{\theta}I_1(\sigma)I_3(\tau) \in mZ[G_m].
\]

The coefficient of \( \sigma \) in \( \sigma^{-1} \tau^{-3} \tilde{\theta}I_1(\sigma)I_3(\tau) \) is

\[
c \equiv 1 - 5 \binom{m-1}{3} + 6 \binom{m-2}{3} - \binom{m-3}{3} \pmod{m},
\]

whence \( 6c \) is congruent to \( 12(m-2) \) or \(-48 \pmod{m} \). In particular, \( c \) is not
divisible by \( m \), a contradiction. This proves that \( W \) is not in \( Z[G] + \text{Ker}(\Phi) \) for \( m \geq 7 \).

Now let \( m = 5 \). Suppose again that \( W \in Z[G] + \text{Ker}(\Phi) \). Then

\[
\omega \in \Phi(Z[G]) \subseteq \text{End}(J_5) \quad \text{and} \quad \omega = x + iy
\]

for some \( x, y \in \Phi(Z[G_5, \rho]) \). From \( I_0(\sigma^{-1}\tau)(1 + i) = 0 \) in \( \text{End}(J_5) \), we have

\[
I_0(\sigma^{-1}\tau)\omega = I_0(\sigma^{-1}\tau)(x - y) \in \Phi(Z[G_5, \rho]).
\]

By Corollary 5.9,

\[
\sigma^{-1}\tau^{-1}I_0(\sigma^{-1}\tau)I_1(\sigma)I_3(\tau) \in 5\Phi(Z[G_5]).
\]

This is not the case by an explicit computation using the following facts

1. \( I_0(\sigma) = \Delta = 0 \) in \( \text{End}(J_5) \),
2. \( \{\sigma^r \tau^s | 0 \leq r \leq 3, 0 \leq s \leq 2\} \) is a free \( Z \)-basis of \( \Phi(Z[G_5]) \).

This contradiction shows, as before, that \( W \) is not in \( Z[G] + \text{Ker}(\Phi) \). \qed

6. Endomorphisms of \( J_m, \text{II} \)

Proceeding as in Proposition 5.6, we can also show that the image \( v \) of

\[
V = m^{-1}I_1(\sigma)I_2(\tau)(\rho - 1) \in \mathbb{Q}[G_m, \rho]
\]

under \( \Phi \) is in \( \text{End}(J_m) \). Alternatively, we can deduce this fact as follows. Let \( \bar{W} = mW \) and \( \bar{V} = mV \). Then

\[
(1 - \tau)^2\bar{W} \equiv \tau^2I_1(\sigma)I_1(\tau)(\rho - 1) \pmod{m\mathbb{Z}[G_m, \rho]},
\]

\[
(1 - \tau)\bar{V} \equiv \tau I_1(\sigma)I_1(\tau)(\rho - 1) \pmod{m\mathbb{Z}[G_m, \rho]}.
\]

Therefore

\[
(1 - \tau)\{\tau\bar{V} - (1 - \tau)\bar{W}\} \equiv 0 \pmod{m\mathbb{Z}[G_m, \rho]}.
\]

Let

\[
\tau\bar{V} - (1 - \tau)\bar{W} = X + Y\rho + Z\rho^2,
\]

with \( X, Y, Z \in \mathbb{Z}[G_m] \). Then

\[
(1 - \tau)X \equiv (1 - \tau)Y \equiv (1 - \tau)Z \equiv 0 \pmod{m\mathbb{Z}[G_m]}.
\]

A direct calculation shows that the annihilator of \( (1 - \tau) \) in \( (\mathbb{Z}/m\mathbb{Z})[G_m] \) is the ideal generated by \( \sum_{j=0}^{m-1} \tau^j \). Therefore,

\[
\tau\bar{V} - (1 - \tau)\bar{W} \in m\mathbb{Z}[G_m, \rho] + \mathbb{JQ}[G_m, \rho]
\]

and

\[
V - \tau^{-1}(1 - \tau)W \in \mathbb{Z}[G_m, \rho] + \mathbb{JQ}[G_m, \rho].
\]
We will now show that
\[ \text{End}(J_m) \cap \Phi(Z[G_m, \rho]) = \Phi(Z[G_m, \rho, W]). \]

Let
\[ \alpha = X + Y \rho + Z \rho^2 \in \mathbb{Q}[G_m, \rho] \] with \( X, Y, Z \in \mathbb{Q}[G_m] \).

be such that
\[ \Phi(\alpha) \in \text{End}(J_m). \]

By Lemma 5.6, we may assume that \( X + Y + Z = 0 \). By Lemma 5.8, we have that
\[ (\sigma - 1)^2 X, \quad (\sigma - 1)(\tau - 1)X \quad \text{and} \quad (\tau - 1)^2 X \]
are in \( Z[G_m] \) + \( J \), with similar statements for \( Y \) and \( Z \).

We choose \( \tilde{X}, \tilde{Y} \) and \( \tilde{Z} \) in \( Z[G_m] \) such that
\[ \tilde{X} \equiv mX \pmod{J}, \quad \tilde{Y} \equiv mY \pmod{J}, \quad \text{and} \quad \tilde{Z} \equiv mZ \pmod{J}. \]

Then
\[ (\sigma - 1)^2 \theta \tilde{X} \equiv (\sigma - 1)(\tau - 1)\theta \tilde{X} \equiv (\tau - 1)^2 \theta \tilde{X} \equiv 0 \pmod{mZ[G_m]} \].

We wish to show that there are integers \( a_X, b_X \) and \( c_X \) such that
\[ \theta \tilde{X} \equiv a_X I_0(\sigma)I_1(\tau) + b_X I_1(\sigma)I_0(\tau) + c_X I_0(\sigma)I_0(\tau) \pmod{mZ[G_m]} \].

Let
\[ \theta \tilde{X} = \sum_{0 \leq r, s \leq m - 1} a_{r,s} \sigma^r \tau^s \in \mathbb{Z}[G_m], \]
and define \( a_X = a_{0,1} - a_{0,0}, b_X = a_{1,0} - a_{0,0} \) and \( c_X = a_{0,0} \).

From
\[ (\sigma - 1)^2 \theta \tilde{X} \equiv 0 \pmod{mZ[G_m]}, \quad (\tau - 1)^2 \theta \tilde{X} \equiv 0 \pmod{mZ[G_m]} \]
and
\[ (\sigma - 1)(\tau - 1)\theta \tilde{X} \equiv 0 \pmod{mZ[G_m]}, \]

we obtain the following congruences respectively
\begin{align*}
    a_{r+2,s} - 2a_{r+1,s} + a_{r,s} &\equiv 0 \pmod{m}, \\
    a_{r,s+2} - 2a_{r,s+1} + a_{r,s} &\equiv 0 \pmod{m}, \\
    a_{r+1,s+1} + a_{r,s} &\equiv a_{r,s+1} + a_{r+1,s} \pmod{m}.
\end{align*}

By double induction on \((r, s)\), we can prove that the above congruences imply that
\[ a_{r,s} \equiv a_X \cdot s + b_X \cdot r + c_X \pmod{m} \quad \forall (r, s) \quad \text{with} \quad 0 \leq r, s \leq m - 1. \]
We omit the details here. We conclude that
\[ \theta \bar{X} \equiv a_{x}I_{0}(\sigma)I_{1}(\tau) + b_{x}I_{1}(\sigma)I_{0}(\tau) + c_{x}I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}]). \]

Similarly, there are integers \( a_{y}, a_{z}, b_{y}, b_{z}, c_{y}, c_{z} \) such that
\[ \theta \bar{Y} \equiv a_{y}I_{0}(\sigma)I_{1}(\tau) + b_{y}I_{1}(\sigma)I_{0}(\tau) + c_{y}I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}]), \]
\[ \theta \bar{Z} \equiv a_{z}I_{0}(\sigma)I_{1}(\tau) + b_{z}I_{1}(\sigma)I_{0}(\tau) + c_{z}I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}]). \]

Using Lemma 5.1,
\[ a_{y}I_{0}(\sigma)I_{1}(\tau)(\sigma^{r+s}\tau^{2s-r} - 1) = a_{y}I_{0}(\sigma)I_{1}(\tau)(\tau^{2s-r} - 1) \]
is congruent modulo \( m \mathbb{Z}[G_{m}] \) to
\[ a_{y}I_{0}(\sigma)I_{1}(\tau)((2s - r)(r - 1)) \equiv a_{y}(r - 2s)I_{0}(\sigma)I_{0}(\tau). \]

Similarly,
\[ b_{y}I_{1}(\sigma)I_{0}(\tau)(\sigma^{r+s}\tau^{2s-r} - 1) \equiv -b_{y}(r + s)I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}])}. \]

Therefore,
\[ \theta \bar{Y}(\sigma^{r+s}\tau^{2s-r} - 1) \equiv -\{a_{y}(2s-r) + b_{y}(r+s)\}I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}]}, \]
and
\[ \theta \bar{Z}(\sigma^{2r-s}\tau^{r+s} - 1) \equiv -\{a_{z}(r+s) + b_{z}(2r-s)\}I_{0}(\sigma)I_{0}(\tau) \pmod{\mathbb{Z}[G_{m}]). \]

From
\[ \theta \bar{Y}(\sigma^{r+s}\tau^{2s-r} - 1) + \theta \bar{Z}(\sigma^{2r-s}\tau^{r+s} - 1) \equiv 0 \pmod{\mathbb{Z}[G_{m}]}, \]

it follows that
\[ a_{y}(2s-r) + b_{y}(r+s) + a_{z}(r+s) + b_{z}(2r-s) \equiv 0 \pmod{m}. \]  
(6.4)

Setting \((r,s)=(-1,1)\) and \((r,s)=(2\lambda, \lambda)\), where \( \lambda \in \mathbb{Z} \) is a solution of \( 3\lambda \equiv 1 \pmod{m} \), in (6.4), we obtain that
\[ a_{y} - b_{z} \equiv 0 \pmod{m}, \quad b_{y} + a_{z} + b_{z} \equiv 0 \pmod{m}. \]  
(6.5)

It is clear that (6.4) and (6.5) are equivalent.

By Lemma 5.1 again, we note that \( \theta I_{1}(\sigma)I_{3}(\tau) \) and \( \theta I_{1}(\tau)I_{3}(\sigma) \) are congruent to
\[ I_{0}(\sigma)I_{1}(\tau) - 2I_{0}(\sigma)I_{0}(\tau) \quad \text{and} \quad I_{0}(\tau)I_{1}(\sigma) - 2I_{0}(\sigma)I_{0}(\tau) \pmod{m \mathbb{Z}[G_{m}].} \]
respectively. Let \( \gamma \in \mathbb{Z} \) be such that \( \gamma \equiv 2a_{z} + 2b_{z} + c_{z} \pmod{m} \). Then
\[ \theta(\bar{Z} - a_{z}I_{1}(\sigma)I_{3}(\tau) - b_{z}I_{1}(\tau)I_{3}(\sigma) - \gamma_{z}I_{1}(\sigma)I_{2}(\tau)) \equiv 0 \pmod{m \mathbb{Z}[G_{m}]). \]

By Corollary 5.5,
\[ \bar{Z} \equiv a_{z}I_{1}(\sigma)I_{3}(\tau) + b_{z}I_{1}(\tau)I_{3}(\sigma) + \gamma_{z}I_{1}(\sigma)I_{2}(\tau) \pmod{m \mathbb{Z}[G_{m}] + J}). \]
Similarly, there is $\gamma \in \mathbb{Z}$ such that 
\[ \tilde{Y} \equiv b_2 I_1(\sigma)I_3(\tau) - (a_2 + b)I_1(\tau)I_3(\sigma) + \gamma I_1(\sigma)I_2(\tau)(\mod(m\mathbb{Z}[G_m] + J)). \]
Since $\tilde{X} + \tilde{Y} + \tilde{Z} = 0$ (by assumption), $\tilde{X}$ is congruent modulo $m\mathbb{Z}[G_m]$ to 
\[ -(a_2 + b)I_1(\sigma)I_3(\tau) + a_2 I_1(\tau)I_3(\sigma) - (\gamma + \gamma_Z)I_1(\sigma)I_2(\tau). \]
Hence,  
\[ \alpha \equiv b_2 W - a_2 W\rho^2 + \gamma Y + \gamma Z V(\rho + 1)(\mod(Z[G_m, \rho] + JQ[G_m, \rho])). \]
By the remarks at the beginning of this section,
\[ \text{End}(J_m) \cap \Phi(\mathbb{Z}[G_m, \rho]) = \Phi(\mathbb{Z}[G_m, \rho, W]). \]
This proves the first statement of Theorem D.

**COROLLARY 6.1.** Let 
\[ \Sigma = \Phi(\mathbb{Z}[G_m, \rho, W]) \quad \text{and} \quad B = \Phi(\mathbb{Z}[G_m, \rho]). \]
Then the quotient group $Q = \Sigma/B$ is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank 4.

**Proof.** We have shown that the following map is surjective
\[ f: (\mathbb{Z}/m\mathbb{Z})^4 \to Q, (a, b, c, d) \to aw + bw\rho^2 + cv + dv\rho. \]
Let $a, b, c, d \in \mathbb{Z}$ be such that
\[ aw + bw\rho^2 + cv + dv\rho \in Z[G_m, \rho] + \mathbb{JQ}[G_m, \rho]. \tag{6.6} \]
By Corollary 5.9, we can collect terms in $Q[G_m]$ 
\[ -a I_1(\sigma)I_3(\tau) + b I_1(\sigma)I_3(\tau) - I_1(\tau)I_3(\sigma)) - c I_1(\sigma)I_2(\tau) \in m\mathbb{Z}[G_m] + J. \]
Multiplying throughout by $m\theta$, we get 
\[ (2a - 2b) I_0(\sigma)I_0(\tau) + (b - a) I_0(\sigma)I_1(\tau) - b I_0(\sigma)I_1(\tau) \in m\mathbb{Z}[G_m]. \]
Comparing coefficients of $\tau$ and $\tau^2$, we obtain $2a \equiv 2b \equiv c (\mod m)$. Looking at coefficients of $\sigma$ and $\sigma^2$, $a \equiv 0 (\mod m)$.

Next we collect terms in $Q[G_m, \rho]$ in (6.6), and we use 
\[ a \equiv b \equiv c (\mod m), \]
to get 
\[ d I_1(\sigma)I_2(\tau) \in m\mathbb{Z}[G_m] + J. \]
Multiplying by $\theta$, we conclude that $d \equiv 0 (\mod m)$.

We end this section by showing that, when $m$ is odd, 
\[ \text{End}(J_m) \cap \Phi(\mathbb{Z}[G_m, l]) = \Phi(\mathbb{Z}[G_m, l]). \]
Let \( X, Y \in \mathbb{Q}[G_m] \) be such that \( \Phi(X + Y) \in \text{End}(J_m) \). Then, for all \( r \in \mathbb{Z} \),
\[
X \sigma^r - Y \tau^r \in \mathbb{Z}[G_m] + J.
\]
This is equivalent to
\[
X - Y, (\sigma - \tau)X \in \mathbb{Z}[G_m] + J.
\]  \hspace{1cm} (6.7)

Let \( M \) denote
\[
\{Z \in \mathbb{Q}[G_m] | (\sigma - \tau)Z \in \mathbb{Z}[G_m] + J\}.
\]
We claim that
\[
M = \mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau).
\]

Recall that
\[
\Delta = \sum_{0 \leq r, s, p, q \leq s - 2} \tau^r \sigma^s \tau^p \sigma^q \in \mathbb{Z}[G_m]
\]
and \( I_0(\sigma) \) generates the ideal \( J \cap \mathbb{Z}[G_m] \). Since \( m \) is odd by hypothesis, in the ring
\[
\mathbb{Q}[G_m]/(I_0(\sigma), \sigma - \tau), I_0(\sigma \tau) = 0
\]
and the equality
\[
(1 - \sigma^{-1})\Delta = I_0(\tau) - \sigma I_0(\sigma \tau)
\]
in \( \mathbb{Z}[G_m] \) implies that \( (1 - \sigma^{-1})\Delta = 0 \). Furthermore, \( 1 - \sigma \) is a unit in \( \mathbb{Q}[\sigma]/(I_0(\sigma)) \) and so we have \( \Delta \in (I_0(\sigma), \sigma - \tau)\mathbb{Q}[G_m] \). It then follows from
\[
(I_0(\sigma))\mathbb{Q}[\sigma] \cap \mathbb{Z}[\sigma] = \mathbb{Z}. I_0(\sigma)
\]
that
\[
\Delta \in (I_0(\sigma), \sigma - \tau)\mathbb{Q}[G_m] \cap \mathbb{Z}[G_m] = (I_0(\sigma), \sigma - \tau)\mathbb{Z}[G_m].
\]

In particular, the ring
\[
R = \mathbb{Z}[G_m]/(J \cap \mathbb{Z}[G_m], \sigma - \tau) = \mathbb{Z}[G_m]/(I_0(\sigma), \sigma - \tau) = \mathbb{Z}[\sigma]/(I_0(\sigma))
\]
is a free \( \mathbb{Z} \)-module.

We define a homomorphism \( \phi : M \to R \) as follows. Let \( Z \in M \) be such that \( (\sigma - \tau)Z = a + k \), where \( a \in \mathbb{Z}[G_m] \) and \( k \in J \). We then define \( \phi(Z) = a \). Clearly \( \phi \) is well-defined and a homomorphism, and \( \text{Ker}(\phi) \) contains \( \mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau) \).

We wish to show that they are equal.

Let \( Z \in \text{Ker}(\phi) \). Write \( (\sigma - \tau)Z = (\sigma - \tau)a + k \), for some \( a \in \mathbb{Z}[G_m] \) and some \( k \in J \). Then \( a = \phi(Z) = 0 \) in \( R \) implies that \( a = a_1 I_0(\sigma) + a_2 (\sigma - \tau) \) for some \( a_1, a_2 \in \mathbb{Z}[G_m] \). Then
\[
(\sigma - \tau)(Z - a_2) = a_1 I_0(\sigma) + k.
\]
To show that 

$$Z \in \mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau)$$

is equivalent to showing that

$$Z - a_2 \in \mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau).$$

Hence, we can replace $Z$ by $Z - a_2$, and assume that $(\sigma - \tau)Z \in J$.

For $X \in \mathbb{Q}[G_m]$, let $\tilde{X}$ be its image in $\mathbb{Q}[G_m]/J$. Since $\mathbb{Q}[G_m]$ is a product of fields, it follows that

$$\text{Ker}(\sigma - \tau) = (\text{Ker}(\sigma - \tau) + J)/J.$$ 

Therefore, $Z \in \text{Ker}(\sigma - \tau) + J$. Thus we have shown that the kernel of $\phi$ is $\mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau)$. So $\phi$ induces a monomorphism

$$M/(\mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau)) \rightarrow R$$

from a torsion $\mathbb{Z}$-module into a torsion-free $\mathbb{Z}$-module. This implies that $M = \mathbb{Z}[G_m] + J + \text{Ker}(\sigma - \tau)$, and our claim is established.

An easy calculation shows that

$$\text{Ker}(\sigma - \tau) = (I_0(\sigma^{-1}\tau))\mathbb{Q}[G_m].$$

Thus

$$X + Y(1 + i) = (X - Y) + Y(1 + i) \quad \text{with} \quad Y \in \mathbb{Z}[G_m] + J + (I_0(\sigma^{-1}\tau))\mathbb{Q}[G_m]$$

and

$$X - Y \in \mathbb{Z}[G_m] + J.$$ 

By Lemma 1.6, $I_0(\sigma^{-1}\tau)(1 + i)$ is in $\text{Ker}(\Phi)$. We conclude that

$$\Phi(X + Y(1 + i)) \in \Phi(\mathbb{Z}[G_m, i]).$$

This completes the proof of Theorem D.

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References


