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Equisingular deformations below the Newton boundary

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1. Introduction

Let \((V, 0) \subseteq (\mathbb{A}_k^3, 0)\) be a normal 2-dimensional singularity \((k = \text{fixed algebraically closed field with char } k = 0)\). Suppose an embedded good resolution \((X, Y, D_v, E_v)\) to be given, then the following functors on \(\mathcal{C}\) (category of local Artinian algebras of finite type over \(k\)) are defined:

\[
E_{EX}(A \in \mathcal{C}) := \{\text{isomorphism classes of } A\text{-deformations } \hat{Y}, \hat{D}_v \subset \hat{X} \text{ of } Y, D_v \subset X \text{ such that } \hat{X} \text{ blows down to } (\mathbb{A}_k^3, 0)\}
\]

\[
E_{SY}(A \in \mathcal{C}) := \{\text{isomorphism classes of } A\text{-deformations } \hat{E}_v \subset \hat{Y} \text{ of } E_v \subset Y \text{ such that } \hat{Y} \text{ blows down to } (V, 0)\}.
\]

In an obvious way we get natural transformations \(ESE_X \xrightarrow{\cong} ESE_Y \xrightarrow{\beta} \text{Def}_v\), and following Wahl [Wa2] the deformations in the image of \(\beta\) (denoted by \(ES\)) will be called “equisingular” (this notion does not depend on the choice of the resolution \(\pi\)).

(1.1) Are there suitable embedded resolutions \(X\) such that all equisingular deformations are induced by the corresponding functor \(ESE\)?

In case of plane curve singularities Wahl used the embedded resolution obtained by successive blowing ups of \((A^2, 0)\). In his paper [Wa1] he showed

\[
ES(A) = \frac{ESE(A)}{A\text{-automorphisms of } (\mathbb{A}_k^2, 0) \times_k A}.
\]

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in particular the natural transformation \( \alpha : ESE \to ES \) is surjective for all \( A \in \mathcal{C} \).
(Note the different notations: In [Wa1] the functors \( ESE \) and \( ES \) are denoted by \( ES \) and \( ES \), respectively.)

If, in dimension 2, the singularity \( (V, 0) \subseteq (A^3_k, 0) \) is given by a polynomial \( f \) that is non-degenerate on the Newton boundary \( \Gamma(f) \), we can use embedded resolutions given by subdividing the dual Newton polyhedron \( \Sigma_0 \) (cf. [Va]).

Under the additional assumption that each of the coordinate axes of \( \mathbb{R}^3 \) meets exactly one top dimensional face of \( \Gamma(f) \), the existence of a "good" subdivision \( \Sigma < \Sigma_0 \) making \( \alpha : ESE_{X_1} \to ES_{Y_1} \) surjective on \( \mathcal{C} \) is shown in [Al], (5.4).

In sections 2 and 3 of this paper we will show for arbitrary non-degenerate polynomials \( f \), that any equisingular first order deformation \( \xi \) of \( (V, 0) \) can be split into equisingular parts \( \xi = \xi_1 + \xi_2 + \xi_3 \) such that each of the \( \xi_i \) is induced by an equisingular deformation of an embedded resolution \( \pi_i : X_i \to (A^3_k, 0) \) \( (i = 1, 2, 3) \) (cf. Theorem (3.4)).

This means we can give a support to the above question on the infinitesimal level in the following sense:

There are three embedded resolutions, one for each coordinate axis, such that all equisingular deformations are induced by the corresponding functors \( ESE_i \) \( (i = 1, 2, 3) \) together.

(1.2) In section 4 we fix an arbitrary smooth subdivision \( \Sigma < \Sigma_0 \) and compute the image \( \text{Im}(ESE_{X_1}(k[\varepsilon]) \to \text{Def}_R(k[\varepsilon])) \) (cf. Proposition (4.6)). This together with Theorem (3.4) imply our main result – an algorithm for computing all equisingular first-order deformations in \( \text{Def}_R(k[\varepsilon]) \) (cf. Theorem (5.1)). None of the smooth subdivisions \( \Sigma < \Sigma_0 \), but only the starting f.r.p.p. decomposition \( \Sigma_0 \) itself is used there, hence, this algorithm seems to be an easy method to determine \( ES(k[\varepsilon]) \) by computers. In particular, for each equation \( f \) we can decide if there are equisingular deformations below \( \Gamma(f) \) or not.

Finally, an example is given in (5.3).

2. Resolutions related to the coordinate axes of \( \mathbb{R}^3 \)

(2.1) We will use the notations of [Al]:

Let \( M := \mathbb{Z}^3 \), \( N := \mathbb{Z}^3 \) which are regarded as being dual to each other by the canonical pairing \( \langle \cdot, \cdot \rangle \). Let \( f \in (x)^2k[x] \) \( (x = (x_1, x_2, x_3)) \) be an irreducible, complete polynomial (i.e. \( f(0, \ldots, x_i, \ldots, 0) \neq 0 \) for \( i = 1, 2, 3 \)); suppose moreover, that \( f \) is non-degenerate on its Newton boundary \( \Gamma(f) \subseteq M_R \).

The dual of \( \Gamma(f) \) gives a finite rational partial polyhedral (f.r.p.p.) decomposition \( \Sigma_0 < R^3_{\geq 0} \subseteq N_R \). We consider this only by regarding the intersection

\[
\Delta := R^3_{> 0} \cap \left\{ (a_1, a_2, a_3) \in R^3_{\geq 0} \middle| \sum_{i=1}^{3} a_i = 1 \right\} \subseteq N_R;
\]
However, rational elements of $\Delta$ will always be given with integer coordinates that are relatively prime. The vertices of $\Delta$ are in a 1 to 1-correspondence to the coordinate functions $x_1, x_2, x_3$ and will be denoted by $e^1, e^2$ and $e^3$, respectively.

Subdivisions of $\Sigma_0$ will always be considered not to change the boundary $\partial \Delta$. Finally, we denote by $\Omega\langle D \rangle, \Omega\langle D + Y \rangle, \Theta\langle -D \rangle, \Theta\langle -D - Y \rangle$ the corresponding sheaves with logarithmic poles or their duals. (In [Al] the latter two were called $S'$ and $S$, respectively.)

(2.2) DEFINITION. Let $a, b \in \Delta \cap \mathbb{Q}^3$. We denote by $P_b(a)$ the point of $\Delta \cap \mathbb{Q}^3$ characterized by the following conditions:

1. $P_b(a) \in \overline{ab}$.
2. If $P_b(a)$ is given by integer coordinates that are relatively prime, then \{P_b(a), b\} will be a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^3 \subseteq N_\mathbb{R}$.
3. The distance between $a$ and $P_b(a)$ is minimal.

REMARK. (i) Denote by
$$d := \det(a, b) := \gcd\left(\begin{array}{cc} a_i & a_j \\ b_i & b_j \end{array}\right), \quad 1 \leq i < j \leq 3$$
the "volume" of the corresponding cone $\langle a, b \rangle \subseteq \mathbb{R}^3$. Then, $P_b(a)$ is given by
$$P_b(a) := \frac{a + k \cdot b}{d} \in \Delta \cap \mathbb{Q}^3 \quad \text{with} \quad 0 \leq k < d$$
$$d|a_i + k \cdot b_i \quad \text{for all} \quad i.$$ (ii) As in [Al] (5.2) we abbreviate $P_{e^i}(a)$ by $P_i(a)$.

(2.3) DEFINITION. A smooth f.r.p.p. subdivision $\Sigma \subset \Sigma_0$ is called "good for $e^i$" if the following condition is satisfied: Whenever an element $a \in \Sigma_0^{(1)}$ (i.e. a vertex of the $\Sigma_0$-partition of $\Delta$) is contained, with $e^i$, in a common cone $a \in \Sigma_0$, then there will be a cone $\tilde{a} \in \Sigma$ containing $P_i(a)$ and $e^i$. 

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**Diagram:**

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**Diagram:**

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REMARK. In section 5 of [Al] the property of $\Sigma$ to be good for all $e^i$ ($i = 1, 2, 3$) is called "good" and is shown to be sufficient for making the natural transformation $ESE_{X_1} \to ESE_{X_2}$ smooth. Nevertheless, in most of the cases drawn above (i.e., with non-simplicial "boundary-cones" of $\Sigma_0$) such subdivisions $\Sigma$ do not exist. (There are exceptions: In Example (5.3) the fact $P_2(a) \in \overline{b^3}$ yields good subdivisions of $\Sigma_0$.)

(2.4) Let $i \in \{1, 2, 3\}$ be fixed. Now, we give a construction of a $\Sigma_i < \Sigma_0$ which will be good for $e^i$:

**STEP 1.** Connecting $e^i$ with all $a \in \Sigma_0^{(1)}$ contained, with $e^i$, in a common $\Sigma_0$-cone. We get a new $\Sigma_0$ with the 2-skeleton $\Sigma_0^{(2)}$.

**STEP 2.** Dividing all line segments $\overline{ab} \in \Sigma_0^{(2)}$ into the "canonical primitive sequence" (cf. [O], (3.5)(i)): $b^0 := b$,

$$b^{v+1} := P_{b^v}(a) \quad (v \geq 0)$$

stops with $b^n = a$. (If $\overline{ab}$ was already smooth, then $b^1 = P_a(a) = a$. Hence, in difference to [O], nothing is to do with that line segment.) We obtain a new $\Sigma_0^{(2)}$.

**STEP 3.** By a non-canonical division of the 3-dimensional cones of $\Sigma_0$ we obtain a smooth subdivision $\Sigma_i < \Sigma_0$ with $\Sigma_i^{(2)} \cap |\Sigma_0^{(2)}| = \Sigma_0^{(2)} \cap |\Sigma_0^{(2)}|$. Let $\Sigma' := \Sigma_0^{(2)} \cap |\Sigma_0^{(2)}|$ (that means the actual subdivision of the 2-skeleton of $\Sigma_0$); this f.r.p.p.-decomposition does not depend on $i \in \{1, 2, 3\}$!

Denoting by $X_i := X_{\Sigma_i}$ and $W := X_{\Sigma'}$ the corresponding torus embeddings after a flat base change $\times A^3_k(A^3_k, 0)$, the following geometric situation results:

$$\begin{array}{ccc}
W \xrightarrow{\text{open}} X_i & \xrightarrow{\pi_i} & (A^3_k, 0) \\
\downarrow & & \downarrow \\
Y_i & \xrightarrow{(\text{strict transform})} & (V, 0)
\end{array}$$

are embedded resolutions of $(V, 0) \subseteq (A^3_k, 0)$ for $i = 1, 2, 3$.

(2.5) **PROPOSITION.** Whenever $\Sigma < \Sigma_0$ is a smooth subdivision with $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$, the strict transform $Y_{\Sigma} \subseteq X_{\Sigma}$ of $(V, 0) \subseteq (A^3_k, 0)$ is contained in $W = X_{\Sigma'}$; therefore, all such resolutions $\pi: Y_{\Sigma} \to (V, 0)$ coincide and will be denoted by $\pi: Y^\# \to (V, 0)$.

In particular, the above three resolutions $\pi_i: Y_i \to (V, 0)$ are equal to $Y^\#$.

**Proof.** Step 1. There is a 1 to 1-correspondence between the top dimensional cones $\beta \in \Sigma_0$ and the vertices $r^\theta \in \Gamma(f)$:

$$\langle a, r^\theta \rangle = m(a) := \min_{r \in \Gamma_+(f)} \langle a, r \rangle$$
Now, let $\Sigma < \Sigma_0$ be an arbitrary smooth subdivision, and let $\alpha \in \Sigma$ be an arbitrary cone ($\alpha \not= \partial \Delta$); we fix the following notations:

$$
\alpha < \alpha'(\in \Sigma) \subseteq \beta(\in \Sigma_0)
$$

$$
\langle a^1, \ldots, a^k \rangle < \langle a^1, a^2, a^3 \rangle \quad (1 \leq k \leq 3).
$$

Then, $X_{a'} \subseteq X_\Sigma$ is an open subset with the coordinates $y_1, y_2, y_3$, and we will look for the condition for $Y_\Sigma$ to meet the closed orbit $\text{orb}(\alpha) \subseteq X_\Sigma$:

$$
(\text{orb}(a) \cap X_{a'}) \cap Y_\Sigma \neq \emptyset
$$

iff there is a triple $(y_1, y_2, y_3) \in Y_\Sigma$ with $y_1 = \cdots = y_k = 0$

iff there is a non-trivial monomial $y_1^{a_1+1} \cdots y_3^{a_3}$ in the equation $f_\beta(y) := \prod_{i=1}^{3} y_i^{-m(a)}f(x) = x^{-r_\beta}f(x)$ of $Y_\Sigma$ in $X_{a'}$.

iff there exists an $r \in \text{supp} f$ (different from $r_\beta$) with $\langle a, r \rangle = \langle a, r_\beta \rangle = m(a)$ for all $a \in \bar{\Sigma}$

iff there exists a vertex $r' \in \Gamma(f)$ ($\beta \neq \gamma$) with $\langle a, r' \rangle = \langle a, r_\gamma \rangle = m(a)$ for all $a \in \bar{\Sigma}$.

The latter condition does not depend on the special choice of $a'$.

Finally, we obtain:

$$
\overline{\text{orb}(\alpha)} \cap Y_\Sigma \neq \emptyset \quad \text{iff } \alpha \text{ is contained in two different top dimensional } \Sigma_0\text{-cones.}
$$

**STEP 2.** If $\Sigma \cap |\Sigma(2)| = \Sigma'$, we obtain

$$
\overline{\text{orb}(\alpha)} \cap Y_\Sigma = \emptyset \quad \text{for all } \alpha \not= |\Sigma(2)|, \text{ i.e. } \alpha \not\in \Sigma'.
$$

In particular, $W = X_\Sigma \setminus \bigcup_{\alpha \in \Sigma} \overline{\text{orb}(\alpha)}$ contains $Y_\Sigma$. \qed

**REMARK.** As we have seen in (2.4), $Y^*$ arises after a canonical subdivision of $\Sigma(2)$. This resolution is the minimal one in the category of resolutions $Y_\Sigma$ obtained by smooth subdivisions $\Sigma < \Sigma_0$.

**Proof.** By induction only the following has to be checked:

Is $a = a^0, \ldots, a^N = b$ any smooth subdivision (i.e. $\det(a^v, a^{v+1}) = 1$) of a line segment $\overline{ab}$, then $P_\delta(a)$ is among the elements $a^v$ ($v = 0, \ldots, N$).

We can assume $b = e^1$; let $d := \gcd(a_2, a_3)$.
Now, any point \( c \) of the line \( ae^1 \) (in \( \Delta \cap Q^3 \)) can be written as

\[
c\left(\frac{p}{q}\right) = c(p, q) = \left( p, q \cdot \frac{a_2}{d}, q \cdot \frac{a_3}{d} \right) \quad \text{with} \quad p, q \in \mathbb{N}; \quad (p, q) = 1,
\]

and this notation admits the following properties:

1. \( c: \mathbb{Q}_{\geq 0} \cup \{\infty\} \to ae^1 \) is order preserving
2. \( c(\infty) = c(1, 0) = e^1; \ c(0) = c(0, 1) \) is the intersection of \( ae^1 \) with \( \langle e^2, e^3 \rangle \)
3. \( a = c(a_1/d) \); and \( c(r) \) belongs to the line segment \( \overline{ae^1} \) if and only if \( r \geq a_1/d \)
4. \( c(p_1/q_1), c(p_2/q_2) \) yield a smooth segment if and only if
   \[
   \begin{vmatrix}
   p_1 & p_2 \\
   q_1 & q_2
   \end{vmatrix} = \pm 1 \quad \text{(i.e.} \quad \frac{p_1}{q_1} - \frac{p_2}{q_2} = \pm \frac{1}{q_1 q_2} \text{).}
   \]
5. \( c(r), e^1 \) yield a smooth segment if and only if \( r \in \mathbb{Z}; \ P_1(a) = c([a_1/d] + 1) \).

Let \( a' = c(p_v/q_v), a' + 1 = c(p_{v+1}/q_{v+1}) \) be adjacent elements in the above smooth subdivision of \( \overline{ae^1} \). Then, by (4) the open interval \( (p_v/q_v, p_{v+1}/q_{v+1}) \) can not contain any integer \( g \)

\[
\left( \frac{p_{v+1}}{q_{v+1}} - \frac{p_v}{q_v} = \frac{p_{v+1}}{q_{v+1}} - g \right) + \left( g - \frac{p_v}{q_v} \right) \geq \frac{1}{q_{v+1}} + \frac{1}{q_v} > \frac{1}{q_v q_{v+1}} \quad \text{(would give a contradiction)}
\]

in particular, we obtain \([a_1/d] + 1 \notin (p_v/q_v, p_{v+1}/q_{v+1})\).

(2.6) Finally, we want to state here the most important property of resolutions that are good for \( e^1 \).

Let \( \Sigma < \Sigma_0 \) be a smooth subdivision, then by section 5 of [AI] we get the following diagram (with exact rows and columns)

\[
\begin{array}{ccccccc}
0 & \rightarrow & ESE_x(k[\varepsilon]) & \xrightarrow{\chi} & ESY(k[\varepsilon]) \\
& & \downarrow & & \downarrow \\
H^1(X, \Theta_X(-D - Y)) & \longrightarrow & H^1(Y, \Theta_Y(-E)) & \longrightarrow & H^2(X, \Theta_X(-D)(-Y)) & \longrightarrow & 0 \\
\phi \downarrow \otimes_x \chi_j & & \psi \downarrow \otimes_y \chi_j & & \phi \downarrow \otimes_x \chi_j \\
H^1(Y, \mathcal{O}_X^3) & \longrightarrow & H^2(X, \mathcal{O}_X(-Y)^3)
\end{array}
\]

\((k[\varepsilon]) \) denotes the object \( k[\varepsilon] := k[\varepsilon]/\varepsilon^2 \) of \( \mathcal{O} \).
The sheaf $\Theta_X\langle -D \rangle$ as well as the map $\Phi$ split into

$$\Theta_X\langle -D \rangle = \sum_{j=1}^{3} S_j$$

with $S_j := \Theta_X\langle -D \rangle \cap \mathcal{O}_{\text{torus}} \frac{\partial}{\partial x_j}$

(denoted by $S'_j$ in [Al])

and

$$\Phi = \sum_{j=1}^{3} \Phi_j$$

with $\Phi_j; H^2(X, S_j(-Y)) \xrightarrow{x_j} H^2(X, \mathcal{O}_X(-Y))$.

Now, the proof of Proposition (5.4) of [Al] yields:

*If $\Sigma < \Sigma_0$ is good for $e^t$, then $\Phi_i$ will be injective.*

3. Dividing $E_{S,Y}(k[\epsilon])$ into a sum

(3.1) Let $\tilde{\Sigma} < \Sigma < \Sigma_0$ be two smooth subdivisions with a canonical $[\Sigma_0^2]$-part $\Sigma'$; let $\sigma: \tilde{X} \to X$ be the corresponding map of the torus embeddings. We get an injection $\Theta_{\tilde{X}} \hookrightarrow \sigma^*\Theta_X$, and the canonical map

$$\bigoplus_{\mu} \mathcal{N}_{\tilde{D}_\mu\tilde{X}} \to \bigoplus_{\nu} \mathcal{N}_{\sigma^*\tilde{D}_\nu\tilde{X}} = \bigoplus_{\nu} \sigma^*\mathcal{N}_{D_\nu X}$$

induces the injection

$$i: \Theta_{\tilde{X}}\langle -\tilde{D} \rangle \hookrightarrow \sigma^*\Theta_X\langle -D \rangle.$$ 

(Locally we can illustrate the situation as follows: Let $\sigma^*: B \to A$ be the ring homomorphism corresponding to $\sigma$; $D_\nu \subseteq X$ correspond to equations $g_\nu \in B$, $\tilde{D}_\mu \subseteq \tilde{X}$ correspond to equations $f_\mu \in A$, and the pull backs $\sigma^*D_\nu$ are given by

$$\sigma^*(g_\nu) = \prod_{\mu} f_{v,\mu}^{c_{\mu}}.$$ 

Then, the map $\bigoplus_{\mu} \mathcal{N}_{\tilde{D}_\mu\tilde{X}} \to \bigoplus_{\nu} \mathcal{N}_{\sigma^*\tilde{D}_\nu\tilde{X}}$ is given by

$$\bigoplus_{\mu} \text{Hom} \left( \frac{f_\mu}{f_\mu^2}, \frac{A}{f_\mu^2} \right) \to \bigoplus_{\nu} \text{Hom} \left( \frac{\sigma^*(g_\nu)}{\sigma^*(g_\nu)^2}, \frac{A}{\sigma^*(g_\nu)^2} \right)$$

$$\left[ f_\mu \mapsto a_\mu \in A \right] \mapsto \left[ \sigma^*(g_\nu) \mapsto \sum_{\mu} c_{v,\mu} \frac{\sigma^*(g_\nu)}{f_{v,\mu}} a_{v,\mu} \right].$$
Is $D: A \to A$ a derivation (i.e. $D \in \Theta_{\tilde{X}}$), we can compute the image of $D$ in the above sheaves as

$$[f_{\mu} \mapsto D(f_{\mu})]_{\mu} \mapsto \left[ \frac{\sigma^*(g_{\nu})}{f_{\nu,\mu}} D(f_{\nu,\mu}) \right]_v$$

$$D \left( \prod_{\mu} f^c_{\nu,\mu} \right) = D(\sigma^*g_{\nu}).$$

Finally, $D(\sigma^*g_{\nu}) = D(g_{\nu})$ if $D$ is considered as an element of $\sigma^*\Theta_X$ (i.e. as a derivation $B \to A$).

The two sheaves are both contained in $\bigoplus_{j=1}^3 \mathcal{O}_{\text{torus}} \frac{\partial}{\partial x_j}$;

therefore,

$$S^j(\tilde{X}) = \sigma^*S^j(X) \cap \Theta_{\tilde{X}} \langle -\tilde{D} \rangle,$$

and the injection $i$ will split into $i_j: S^j(\tilde{X}) \subset \sigma^*S^j(X)$ ($j = 1, 2, 3$). Now, the exact sequence

$$0 \to \Theta_X \langle -D \rangle \langle -Y^\# \rangle \to \Theta_Y \langle -D - Y^\# \rangle \to \Theta_Y \langle -E \rangle \to 0$$

together with the analogous one for $\tilde{X}$ yields the surjections

$$H^2(\tilde{X}, \bigoplus_{j=1}^3 S^j(\tilde{X})(-Y^\#)) \to H^2(\tilde{X}, \Theta_{\tilde{X}} \langle -E \rangle) \to H^2(\tilde{X}, \bigoplus_{j=1}^3 S^j(X)(-Y^\#))$$

(a) Existence of $H^2(i)$:

$$H^2(\tilde{X}, (\sigma^*S^j(X)(-Y^\#)) = H^2(\tilde{X}, \sigma^*[S^j(X)(-Y^\#)]) \quad (\sigma \text{ is an isomorphism on } W, \text{ hence, } \sigma^*(Y^\#) = Y^\#)$$

$$= H^2(X, (R^+\sigma_*)\sigma^*[S^j(X)(-Y^\#)])$$

$$= H^2(X, S^j(X)(-Y^\#))$$
(S(X)(- Y#) is invertible, and $\mathcal{O}_X \to (\mathbb{R}^+ \cdot \sigma^*) \mathcal{O}_X$ is a quasiisomorphism).

(b) The horizontal maps are surjective by

$$H^2(\tilde{X}, \Theta_{\tilde{X}} \langle -D - Y^# \rangle) = H^2(X, \Theta_X \langle -D - Y^# \rangle) = 0 \quad (\text{cf. [Al], (4.2)).}$$

(3.2) LEMMA. There exists a universal surjection of $k$-vectorspaces

$$\psi: H^1(Y^#, \Theta_{Y^#} \langle -E \rangle) \to \bigoplus_{j=1}^{3} P_j$$

such that for arbitrary smooth subdivisions $\Sigma < \Sigma_0$ (with $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$) the surjection $H^1(Y^#, \Theta_{Y^#} \langle -E \rangle) \to H^2(X, \bigoplus_{j=1}^{3} S_j^{(X)}(- Y^#))$ will factorize (uniquely) through $P_j \to H^2(X, S_j^{(X)}(- Y^#))$ ($j = 1, 2, 3$).

Proof. Two smooth subdivisions of $\Sigma_0$ coinciding in $|\Sigma_0^{(2)}|$ admit a common finer one without changing the part in $|\Sigma_0^{(2)}|$.

Therefore, we can define $P_j$ as the limit of the successive surjections

$$H^2(\tilde{X}, S_j^{(X)}(- Y^#)) \to H^2(X, S_j^{(X)}(- Y^#))$$

described in (3.1); by $\dim_k H^1(Y^#, \Theta_{Y^#} \langle -E \rangle) < \infty$, this process stops after finitely many steps. □

(3.3) Now, we can define some subspaces of $ES_{Y^#}(k[\varepsilon])$ and $H^1(Y^#, \Theta_{Y^#} \langle -E \rangle):$ For $j = 1, 2, 3$ let

$$F_j := \psi^{-1}(P_j) \subseteq H^1(Y^#, \Theta_{Y^#} \langle -E \rangle) \quad \text{and} \quad F_j := F_j \cap ES_{Y^#}(k[\varepsilon]).$$

PROPOSITION.

(1) $\sum_{j=1}^{3} F_j = H^1(Y^#, \Theta_{Y^#} \langle -E \rangle)$.

(2) For arbitrary smooth subdivisions $\Sigma < \Sigma_0$ with $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$ the surjection

$$H^1(Y^#, \Theta_{Y^#} \langle -E \rangle) \to H^2(X, \bigoplus_{j=1}^{3} S_j^{(X)}(- Y^#))$$

maps $F_j$ onto $H^2(X, S_j^{(X)}(- Y^#))$.

(3) $\sum_{j=1}^{3} F_j = ES_{Y^#}(k[\varepsilon])$.

(4) If the image of $F_j$ under the map $ES_{Y^#}(k[\varepsilon]) \to \text{Def}_R(k[\varepsilon])$ is denoted by $\tilde{F}_j \subseteq ES(k[\varepsilon])$, then $\bigcap_{j=1}^{3} \tilde{F}_j$ contains all "above $-\Gamma(f)$ - deformations" of $f$ (i.e. deformations $\tilde{f}$ with $\Gamma(\tilde{f}) = \Gamma(f)$).

Proof. The parts (1) and (2) are clear by definition or by the previous Lemma. For (3) and (4) take a smooth subdivision $\Sigma < \Sigma_0$ ($\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$) that realizes
\( \bigoplus_{j=1}^{3} P_j, \text{i.e.} \ H^2(X, S_j(-Y^*)) = P_j \ (j = 1, 2, 3). \) Now, we regard the diagram of (2.6):

**Part (3).** Let \( \xi \in \text{ES}_Y(k[\varepsilon]) \) with \( \xi = \xi_1 + \xi_2 + \xi_3 \ (\xi_j \in F_j). \) Then, \( \varphi = \Phi \circ \psi \) maps each of the \( \xi_j \) into the corresponding factor of \( H^1(Y^*, \mathcal{O}_{Y^*})^3, \) and we get

\[
0 = \varphi(\xi) = \varphi(\xi_1) + \varphi(\xi_2) + \varphi(\xi_3) \in H^1(Y^*, \mathcal{O}_{Y^*})^3,
\]

hence \( \varphi(\xi_1) = \varphi(\xi_2) = \varphi(\xi_3) = 0. \)

But this means \( \xi_j \in \text{ES}_Y(k[\varepsilon]), \) i.e. \( \xi_j \in F_j \ (j = 1, 2, 3). \)

**Part 4.** By construction of the subspaces \( F_j \subseteq \text{ES}_Y(k[\varepsilon]) \) we have

\[
\text{Im} (\text{ES}_X(k[\varepsilon]) \to \text{ES}_Y(k[\varepsilon])) = \text{Im} (H^1(X, \Theta_X \langle -D - Y^* \rangle) \to H^1(Y^*, \Theta_Y \langle -E \rangle)) = \text{Ker} \psi = \bigcap_{j=1}^{3} F_j, \text{ hence }
\]

\[
\text{Im} (\text{ES}_X(k[\varepsilon]) \to \text{ES}_Y(k[\varepsilon])) \subseteq \bigcap_{j=1}^{3} \bar{F}_j.
\]

On the other hand, all "above-\( \Gamma(f) \)-deformations" can be lifted to elements of \( \text{ES}_X(k[\varepsilon]) \) (cf. [Al], (2.1)).

(3.4) Finally, let us return to the situation of section 2: We had three embedded resolutions

\[
\pi_i: X_i \longrightarrow (\mathbb{A}^3_k, 0)
\]

\[
\pi: Y^* \longrightarrow (V, 0),
\]

and each \( X_i \) was good for \( e^i \) (\( i = 1, 2, 3 \)). Regarding the corresponding diagram

\[
\begin{array}{cccc}
0 & \to & \text{ES}_X(k[\varepsilon]) & \to & \text{ES}_Y(k[\varepsilon]) \\
\to & \to & \downarrow & \to & \downarrow \\
H^1(X_i, \Theta_X \langle -D^{(i)} - Y^* \rangle) & \to & H^1(Y^*, \Theta_Y \langle -E \rangle) & \to & H^2(X_i, \bigoplus_{j=1}^{3} S_j^{(X_{ij})}(-Y^*)) \\
\phi & \circ & \Psi & \circ & \Phi \\
H^1(Y^*, \mathcal{O}_{Y^*}^{3\#}) & \to & H^2(X_i, \mathcal{O}_{X_i}(-Y^*)^3)
\end{array}
\]

of (2.6), we obtain
THEOREM. For \( i = 1, 2, 3 \), the subspace \( F_i \subseteq \text{ES}_X(\kappa[\varepsilon]) \) is contained in \( \text{Im}(\varepsilon_i) \). In particular, the map between the tangent spaces

\[
\bigoplus_{i=1}^3 \text{ESE}_X(\kappa[\varepsilon]) \xrightarrow{\oplus \varepsilon_i} \text{ES}_X(\kappa[\varepsilon])
\]

is surjective.

Proof. Let \( \varepsilon \in F_i \subseteq \text{ES}_X(\kappa[\varepsilon]) \), then, by Proposition (3.3), the image \( \psi_i(\varepsilon) \) is contained in the factor \( H^2(X, S^1_i(\mathbb{Y}^\#)) \).

Now, \( \Phi_i^{(0)} \) is injective (cf. (2.6)), and from \( \phi(\zeta) = 0 \) we obtain the vanishing of \( \psi_i(\zeta) \), i.e. \( \zeta \) is contained in the image of \( \text{ESE}_X(\kappa[\varepsilon]) \).

(3.5) REMARK. (1) Equisingular deformations coming from some ESE are equimultiple (cf. [Ka], (4.6)). So the above theorem illustrates the fact that over \( \kappa[\varepsilon] \) equisingularity alone is sufficient for equimultiplicity (cf. [Ka], (2.8)).

(2) In [Al] the smoothness of ESE and the surjectivity of some \( \text{ESE}(\kappa[\varepsilon]) \to \text{ES}(\kappa[\varepsilon]) \) are used to show the smoothness of ES. It is not possible to apply this schema of proof to our situation – the surjectivity of \( \bigoplus_{i=1}^3 \text{ESE}_X(\kappa[\varepsilon]) \to \text{ES}_X(\kappa[\varepsilon]) \) cannot be lifted to any Artinian rings \( A \) of higher order because there is no possibility of defining the sum of two \( A \)-deformations!

4. Computation of \( \text{Im}(\text{ESE}_X(\kappa[\varepsilon]) \xrightarrow{\phi} \text{Def}_R(\kappa[\varepsilon])) \) (for a fixed embedded resolution \( X \))

For this section we fix an arbitrary smooth f.r.p.p. subdivision \( \Sigma < \Sigma_0 \) with the corresponding good resolution \( \pi: X \to \mathbb{A}^3_k \).

(4.1) The connecting morphism of the cohomology sequences of

\[
0 \to \Theta_X \langle -D - Y \rangle \to \Theta_X \langle -D \rangle \to \mathcal{N}_{Y|X} \to 0
\]

yields the following diagram:
This diagram can be identified with

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\langle \text{monomials} \geq \Gamma(f) \rangle & \rightarrow & \langle \text{monomials} \geq \Gamma(f)/(f) \rangle \\
\downarrow & & \downarrow \\
k[x] & \rightarrow & R \\
\downarrow & & \downarrow \\
k[x]/\langle \text{monomials} \geq \Gamma(f) \rangle & \rightarrow & R/\langle \text{monomials} \geq \Gamma(f) \rangle \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

(The first columns are identified according to [Al] (2.2) – we take

\[
k[x] = H^0 \left( X \setminus D, \mathcal{O}_X \left( - \sum_{a \in \Sigma(i)} m(a)D_a \right) \right) \xrightarrow{\sim/_{f}} H^0(X \setminus D, \mathcal{O}_X(Y))
\]

and

\[
H^1(X, \mathcal{O}_X(Y)) = H^1(X, \mathcal{N}_{Y/X}) = 0;
\]

for the right hand side we use (2.5)(γ) and (4.2) of [Al] – in the latter one the vanishing of \( H^2(X, \Theta_X \langle -D - Y \rangle) \) has been proved.)

**DEFINITION.** For \( k[x] = H^0(X \setminus D, \mathcal{O}_X(Y)) \) we denote by \( k[f](\xi) \) the image of \( \xi \) in

\[
\frac{k[x]}{\langle \text{monomials} \geq \Gamma(f) \rangle} = H^0_b(X, \mathcal{O}_X(-\Sigma m(a)D_a)) \cong H^0_b(X, \mathcal{O}_X(Y)).
\]

Taking the canonical section of \( k[x] \rightarrow k[x]/\langle \text{monomials} \geq \Gamma(f) \rangle \), we get

\[
\xi_{<\Gamma(f)} = \sum_{r \geq 0} \xi \cdot x^r.
\]

**Proposition.** (1) For \( i = 1, 2, 3 \) the vertices \( e^i \in \Delta \) correspond to the non exceptional divisors \( orb(e^i) \subseteq X \). Denote these divisors also by \( e^i \), and let

\[
\varphi_i : H^0_b(X, \mathcal{O}_X(e^i)) \rightarrow H^0_b(X, \mathcal{O}_X(-\Sigma m(a)D_a))
\]
be the multiplication by \( x_i(\partial f/\partial x_i) \). Under the isomorphism

\[
H^1_D(X, \mathcal{O}_X(\Sigma m(\alpha)D_\alpha)) \xrightarrow{\sim} H^1_D(X, \mathcal{O}_X(Y)) \xrightarrow{\sim} H^1_D(X, \mathcal{N}_{Y/X}),
\]

then

\[
\text{Coker} \gamma = \text{Coker} \left( \bigoplus_{i=1}^3 \varphi_i \right).
\]

(2) Let \( \xi = \Sigma_{r \geq 0} \xi_r \cdot \mathbf{x}^r \in k[\mathbf{x}] \) define an element of \( \text{Def}_k(k[\mathbf{e}]) \) (the infinitesimal deformation \( f(x, \mathbf{e}) = f(x) - \varepsilon \xi(x) \)). Then, this deformation is induced by \( E_\mathbf{x}X(k[\mathbf{e}]) \) if and only if

\[
\xi \in \Gamma(f) \in \text{Im} \left( \bigoplus_{i=1}^3 \varphi_i \right).
\]

Proof. (1) By the second diagram of (4.1) it holds

\[
\text{Coker} \gamma = \frac{H^1_D(X, \mathcal{N}_{Y/X})}{\text{Ker} \psi} = \text{Coker}(H^1_D(X, \Theta_X(-D)) \to H^1_D(X, \mathcal{N}_{Y/X})).
\]

On the other hand, we can lift the surjection \( \Theta_X(-D) \to \mathcal{N}_{Y/X} \) to the homomorphism \( \Theta_X(-D) \to \mathcal{O}_X(Y) \) given by \( \eta \mapsto [\eta(f)/f] \):

(i) In local coordinates (take the same notations as in the proof of (2.5): \( f = \mathbf{x}^\alpha \cdot F_\alpha \)) we obtain

\[
\frac{\eta(f)}{f} = \frac{\eta(F_\alpha)}{f} + \frac{\eta(x^\alpha)}{x^\alpha}.
\]

Since \( \eta \in \Theta_X(-D) \), the section \([\eta(x^\alpha)/x^\alpha]\) is regular on \( X \), and \( \eta(f)/f \) is indeed an element of the sheaf \( \mathcal{O}_X(Y) \).

(ii) The projections \( \Theta_X(-D) \to \mathcal{N}_{Y/X} \) and \( \mathcal{O}_X(Y) \to \mathcal{N}_{Y/X} \) are locally given by

\[
\eta \mapsto \left[ F_\alpha \in \frac{f_\alpha}{f_\alpha^2} \mapsto \eta(F_\alpha) \in \frac{\mathcal{O}_X}{f_\alpha} \right]
\]

and

\[
a \mapsto \left[ F_\alpha \in \frac{f_\alpha}{f_\alpha^2} \mapsto af_\alpha \in \frac{\mathcal{O}_X}{f_\alpha} \right].
\]
respectively. Then, the congruence
\[ \eta(f) \cdot f_x = \eta(f_x) + \frac{\eta(x^t)}{x^t} \cdot f_x \equiv \eta(f) \pmod{f_x} \]
shows that the diagram
\[ \begin{array}{ccc}
\mathcal{O}_X(Y) \\
\downarrow \\
\Theta_X(-D) \rightarrow \mathcal{N}_{Y|X} \end{array} \]
commutes.

Since \( H^1_b(X, \mathcal{O}_X(Y)) \rightarrow H^1_b(X, \mathcal{N}_{Y|X}) \) is an isomorphism, we obtain
\[ \text{Coker } \gamma = \text{Coker}(H^1_b(X, \Theta_X(-D)) \rightarrow H^1_b(X, \mathcal{O}_X(Y))). \]

Finally, the first claim follows by the equation
\[ \eta(f) = \sum_{i=1}^3 \left( x_i \frac{\partial f}{\partial x_i} \right) \frac{\eta(x_i)}{x_i} \]
and taking the isomorphism
\[ \Theta_X(-D) \tilde{\rightarrow} \bigoplus_{i=1}^3 \mathcal{O}_X(e^i) \]
\[ \eta \mapsto \left( \frac{\eta(x_1)}{x_1}, \frac{\eta(x_2)}{x_2}, \frac{\eta(x_3)}{x_3} \right). \]

(2) \( \xi \in k[x] = H^0(X \setminus D, \mathcal{O}_X(-\Sigma m(a)D_a)) \cong H^0(X \setminus D, \mathcal{O}_X(Y)) \) maps onto \( 0 \in \text{Coker } \gamma \) if and only if
\[ \xi |_{\Gamma(f)} \in H^1_b(X, \mathcal{O}_X(-\Sigma m(a)D_a)) \cong H^1_b(X, \mathcal{O}_X(Y)) \]
vanishes in \( \text{Coker}(\bigoplus_{i=1}^3 \varphi_i). \)

(4.3) Our next task will be to describe the maps \( \varphi_i \) by the methods of torus embeddings. For this purpose it is useful to regard the dual version of these maps:
\[ \varphi_i^*: H^2(X, \omega_X(\Sigma m(a)D_a)) \rightarrow H^2(X, \omega_X(-e^i)), \]
and the homomorphisms are still given by multiplication by $x_i(\partial f / \partial x_i)$. Now, for $r, t \in M$ we define the following sets:

$$A_r := \{a \in \Delta / \langle a, -r \rangle \leq -m(a) \} = \{a \in \Delta / \langle a, r \rangle \geq m(a) \},$$

$$B^\infty_{i,t} := \{a \in \Delta / \langle a, -t \rangle \leq \psi_i(a) \} \quad \text{with} \quad \psi_i(a) := \begin{cases} 0 & \text{for } a \in \Sigma^{(i)}, a \neq e^i \\ 1 & \text{for } a = e^i \end{cases}$$

(linear on the $\Sigma$-cones),

$$H_t := \{a \in \Delta / \langle a, t \rangle < 0 \}.$$ 

Then, the convex sets $(\Delta \setminus H_t)$ are contained in $B^\infty_{i,t}$, and the maps $\varphi^*_i$ are equal to some homomorphisms

$$\varphi^*_i : \bigoplus_{r \in M} H^1(A_r, \kappa) \longrightarrow \bigoplus_{t \in M} H^1(B^\infty_{i,t}, \kappa) \quad (i = 1, 2, 3).$$

(As we are really interested in the dual of, for instance, $H^2(X, \omega_X(\Sigma m(a)D_a))$, the notations are chosen such that $A_r$ describes the cohomology of the $-r$(th) factor of this sheaf. The relations "$\leq"$ or "$\geq"$ — instead of the strict ones — in the definitions of $A_r$ and $B^\infty_{i,t}$ are induced by taking $\omega_X$(divisor) instead of $\mathcal{O}_X$(divisor).)

But, what does $\varphi^*_i$ look like? We have to make some general remarks concerning the computation of cohomology on torus embeddings:

(4.4) Denote by $j : T \hookrightarrow X$ a torus embedding in the sense of $[Ke]$.

1. Let $L \subseteq j^*\mathcal{O}_Y$ be an $M$-graded invertible sheaf with order function $\omega : \Sigma \to \mathbb{R}$; for $r \in M$ let

$$A_r := \{a \in \Delta / \langle a, r \rangle < \Phi(a) \}.$$ 

Then, if $\alpha \in \Sigma$ is an arbitrary cone, we obtain

$$L(r)|_{X_{\alpha}} = \begin{cases} \kappa & (\forall a \in \alpha : \langle a, r \rangle \geq \Phi(a)) \\
0 & (\exists a \in \alpha : \langle a, r \rangle < \Phi(a)) \end{cases},$$

hence $L(r)|_{X_{\alpha}} = H^0(\alpha, \alpha \cap A_r) \otimes \kappa$. In particular, the sheaf $L(r)$ and the pair $(\Delta, A_r)$ yield exactly the same Cech complexes.

2. Let $L^1, L^2 \subseteq j^*\mathcal{O}_Y$ be $M$-graded invertible sheaves with $\Phi^1, \Phi^2$ and $A^1_r, A^2_r$ as before. Assume that there is an $s \in M$ with $x^s \cdot L^1 \subseteq L^2$ (equivalent: $\Phi^1 + s \geq \Phi^2$ as functions on $\Delta$).
Then, for each \( r \in M \) there is an inclusion \( A_{r+s}^2 \subseteq A_r^1 \), which provides the commutative diagram

\[
\begin{array}{ccc}
\Gamma(X_\alpha, \mathbb{L}^1(r)) & \xrightarrow{x^s} & \Gamma(X_\alpha, \mathbb{L}^2(r + s)) \\
\downarrow & & \downarrow \\
H^0(\alpha, \alpha \cap A_r^1) & \subseteq & H^0(\alpha, \alpha \cap A_{r+s}^2).
\end{array}
\]

Again by taking Cech cohomology we obtain a description of the multiplication by \( x^s \) on the cohomological level:

\[
\begin{array}{ccc}
H^n(X, \mathbb{L}^1) & \xrightarrow{x^s} & H^n(X, \mathbb{L}^2) \\
\downarrow & & \downarrow \\
\bigoplus_{r \in M} H^n(\Delta, A_r^1) & \xrightarrow{\varphi} & \bigoplus_{r \in M} H^n(\Delta, A_r^2)
\end{array}
\]

(\( \varphi \) is induced by the inclusion \( A_{r+s}^2 \subseteq A_r^1 \); in particular, \( \varphi \) is homogeneous of degree \( s \)).

(3) Let \( \mathbb{L}^i, A_i^j (i = 1, 2) \) as before, assume that there is a Laurent polynomial \( g(x) \in k[M] \) with \( g(x) \cdot \mathbb{L}^1 \subseteq \mathbb{L}^2 \).

Then, by \( M \)-gradation of both sheaves \( \mathbb{L}^1 \) and \( \mathbb{L}^2 \), this fact is equivalent to

\[ x^s \cdot \mathbb{L}^1 \subseteq \mathbb{L}^2 \text{ for all } s \in \text{supp } g. \]

Hence, the method of (2) can be applied to describe the maps

\[
H^n(X, \mathbb{L}^1) \xrightarrow{g(x)} H^n(X, \mathbb{L}^2).
\]

(4.5) The third part of the previous general remark applies exactly to the special maps \( \phi^e \) regarded in (4.3). Denoting by \( \Delta^r \subseteq \Delta \) the union of all closed \( \Sigma \)-cones not containing \( e^r \), we obtain the following

**LEMMA.** (1) \( H^1(A_r, k) = \begin{cases} k \cdot x^{-r} & \text{ (for } r \geq 0 \text{ and } r < \Gamma(f) \text{)} \\ 0 & \text{ (otherwise) } \end{cases} \),

and the perfect pairing with

\[
H^1_p(X, \mathcal{O}_X(-\Sigma m(a)D_a)) = \bigoplus_{r \geq 0} k \cdot x^r \\
\bigoplus_{r < \Gamma(f)} k \cdot x^r
\]
is built in the obvious way.

(2) For \( i = 1, 2, 3 \) and \( t \in M \) the cohomology group \( H^1(B_{it}, k) \) is equal to

(i) \( H_0(\Delta_t \cap \mathbb{H}_i) \cdot x^{-t} \) (for \( t_i = -1 \) and \( t_j \geq 0 \) for all \( j \neq i \)),

(ii) \( \frac{H_0(\Delta_t \cap \mathbb{H}_i)}{H_0(\{e^j\})} \cdot x^{-t} \) (for \( t_i = -1, t_j \leq -1 \) (\( j \neq i \)) and the remaining component \( \geq 0 \)),

(iii) 0 (for \( t_i \neq -1 \) or \( t \neq -(1, 1, 1) \)).

(3) Let \( f(x) = \sum_{\text{supp } f} \lambda_x \cdot x^s \) be the explicit description of our starting equation. Let \( r, i \) and \( t \) be such that \( H^1(A_r, k) \), \( H^1(B_{it}, k) \neq 0 \) (i.e. \( r \geq 0 \), \( r < \Gamma(f) \) and \( t_i = -1, t \neq -(1, 1, 1) \), respectively).

Then, the \( x^{-t} \) part of \( \varphi_t^*(x^{-s}) \) is given by

\[
s_t \cdot \lambda_s \cdot [H_0(\{a^s\})] \in H_0(\Delta_t^\Sigma \cap \mathbb{H}_i)
\]

with \( s := t + r \) (because of \( (t) = s + (-r) \)) and \( a^s \in \Sigma_0^{(1)} \) such that \( \langle a^s, r \rangle < m(a^s) \).

In particular, this part of \( \varphi_t^*(x^{-s}) \) vanishes, unless \( s \geq \Gamma(f) \).

Proof. (1) \( A_r = \Delta \setminus \{a \in \Delta/\langle a, r \rangle < m(a)\} = \Delta \setminus (\text{convex set}) \), and the above conditions for \( r \) arise by \( r \geq 0 \) iff \( \partial \Delta \subseteq A_r \) and

\[ r < \Gamma(f) \quad \text{iff} \quad A_r \neq \Delta. \]

(2) \( \Delta \setminus \mathbb{H}_i \subseteq B_{it}^\Sigma \), and the only vertex of \( \Sigma^{(1)} \) in which both sets can differ is \( e^i \). Hence, the non-vanishing of \( H^1(B_{it}^\Sigma, k) \) implies \( e^i \notin \Delta \setminus \mathbb{H}_i \), \( e^i \in B_{it}^\Sigma \), and we obtain \( t_i = \langle e^i, t \rangle = -1 \).

Assuming this from now on, we see that \( B_{it}^\Sigma \) contains exactly the same elements of \( \Sigma^{(1)} \) as \( \Delta \setminus [\Delta_t^\Sigma \cap \mathbb{H}_i] \). In particular, both subsets of \( \Delta \) (consisting of open or closed halfspaces in every cone of \( \Sigma \)) are homotopy equivalent and yield the same cohomology. Without loss of generality we take \( i = 1 \) and consider the above three cases:

(i) \( t_2, t_3 \geq 0 \): Then, \( \partial \Delta \subseteq \Delta \setminus [\Delta_t^\Sigma \cap \mathbb{H}_i] \), and

\[
H^1(\Delta \setminus [\Delta_t^\Sigma \cap \mathbb{H}_i], k) = H_0(\Delta_t^\Sigma \cap \mathbb{H}_i)
\]

follows by the Alexander duality.

(ii) \( t_2 \leq -1, t_3 \geq 0 \): This means \( e^1, e^3 \in (\Delta \setminus [\Delta_t^\Sigma \cap \mathbb{H}_i]), e^2 \notin (\Delta \setminus [\Delta_t^\Sigma \cap \mathbb{H}_i]) \) and
therefore, the connected component $C$ of $e^2$ in $\Delta^2 \cap H_i$ has no influence on the cohomology:

$$H^1(\Delta \setminus [\Delta^2 \cap H_i], k) = H^1(\Delta \setminus ((\Delta^2 \cap H_i) \setminus C), k)$$

$$= H_0((\Delta^2 \cap H_i) \setminus C) = \frac{H_0(\Delta^2 \cap H_i)}{H_0(\{e^2\})}.$$  

(The middle equality again follows by the Alexander duality.)

(iii) $t_2, t_3 \leq -1$: By $H_i = \Delta$ we obtain

$$\Delta \setminus [\Delta^2 \cap H_i] = \Delta \setminus \Delta^2,$$

and this set can be contracted to the point $e^1$.

(3) The linear map $H^1(A_r, k) \to H^1(B_{i,t}^\infty, k)$ is constructed by the inclusion $B_{i,t}^\infty \subseteq A_r$ (cf. (4.4)); in dual terms this means that $H_0(\Delta \setminus A_r) \to H_0(\Delta^2 \cap H_i)/\cdots$ is induced by

$$(\Delta \setminus A_r) \subseteq (\Delta \setminus B_{i,t}^\infty) \sim (\Delta^2 \cap H_i):$$

Take an element $a^* \in \Sigma^{(1)}_0$ with $\langle a^*, r \rangle < m(a^*)$ (i.e. $a^* \in \Delta \setminus A_r$); assuming $s \geq \Gamma(f)$, we obtain

$$\langle a^*, t \rangle = \langle a^*, r \rangle - \langle a^*, s \rangle > 0 \quad \text{(i.e. } a^* \in H_i),$$

and $x^{-r}$ maps onto the corresponding connected component in $\Delta^2 \cap H_i$ (multiplied by the coefficient of $x^r$ in $x_i(\partial f/\partial x_i)$).

REMARK. As $\Delta \setminus A_r$ is convex, we obtain for (3):

$$H_0(\{a^*\}) \in H_0(\Delta \setminus B_{i,t}^\infty) = H_0(\Delta^2 \cap H_i)$$

does not depend on the choice of $a^* \in \Sigma^{(1)}_0$ with $\langle a^*, r \rangle < m(a^*)$.

(4.6) Now, we are in the position to determine the deformations of

$$\text{Im}(\text{ESE}_x(k[\varepsilon]) \to \text{Def}_R(k[\varepsilon]))$$

exactly:

DEFINITION. (1) We choose (and fix) a map

$$a: \{r \geq 0/r < \Gamma(f)\} \to \Sigma^{(1)}_0 \setminus \{e^1, e^2, e^3\}$$

$$r \mapsto a(r) \text{ with } \langle a(r), r \rangle < m(a(r)).$$
(r < \Gamma(f)) means that r sits below some faces of \Gamma(f). Now, by the map \alpha, one of these is selected. \alpha(r) plays exactly the role of \alpha^* in the proposition above, and we have seen that all constructions are independent of the special choice the map \alpha.

(2) For i = 1, 2, 3 let \mathcal{M}_i := \{r \in M / r \geq 0, \Gamma(f) - e_i \leq r < \Gamma(f)\} (\{e_1, e_2, e_3\} denotes the canonical \mathbb{Z}\text{-basis of } M).

Recall the definitions

\mathbb{H}_i := \{a \in \Delta / \langle a, t \rangle < 0\} (\text{for } t \in M)

and

\Delta_i^\mathcal{X} := \bigcup \{\bar{a}/ \bar{a} \in \Sigma, e^i \notin \bar{a}\} \subseteq \Delta.

**Proposition.** (I) Given the following data

(1) \(i \in \{1, 2, 3\}\),
(2) \(t \in M\) with:
   (a) \(t_i = -1\)
   (b) (i) \(t_v \geq 0\) (i.e. \(e^i \notin \mathbb{H}_i\)) for all \(v \neq i\), or
   (ii) \(t_j \leq -1 (i \neq j)\) and the remaining component is \(\geq 0\),
   (c) there exists an \(r \in \mathcal{M}_i\) with \(r - t \geq \Gamma(f)\) and \(\langle \alpha(r), t + e_i \rangle \geq 0\),
(3) a connected component \(C\) of \(\Delta_i^\mathcal{X} \cap \mathbb{H}_i\) not containing any of the vertices \(e^1, e^2, e^3\),

then, the deformation defined by

\[
\sum_{r \in \mathcal{M}_i, \alpha(r) \in \mathcal{C}} (r_i + 1)\lambda_{r-t} \cdot x^t = \left( x^{t+e_i} \frac{\partial f}{\partial x_i} \right)_{M, \alpha^{-1}(C)}
\]

comes from \(\text{ESE}_X(k[\varepsilon])\).

(II) \(\text{Im}(\gamma) \subseteq \text{Def}_k(k[\varepsilon])\) as a \(k\text{-vectorspace}\) is spanned by the above-\(\Gamma(f)\)-deformations and all deformations constructed in the above way.
Proof. By Proposition (4.2), $\text{Im}(\gamma)$ is spanned by the above-$\Gamma(f)$-deformations together with the images of the maps $\varphi_i$ ($i = 1, 2, 3$). However, in Lemma (4.5)(2) it is shown that the data $\{i, t, C\}$ meeting (1), (2a), (2b) and (3) of the claim form a $k$-basis of

$$\bigoplus_{i=1}^{3} H^2_\Delta(X, \mathcal{O}_X(e')) = \bigoplus_{i=1}^{3} \bigoplus_{r \in M} H^1(B^E_r, k) \quad \text{(or its $k$-dual)};$$

finally, part (3) of the same Lemma gives

$$\varphi_i(\{i, t, C\})_{k \cdot x'} = \begin{cases} (r_i + 1)\lambda_{r-i} & \text{(for } a(r) \in C) \\ 0 & \text{(otherwise)} \end{cases}.$$

It remains to prove that we are able to restrict ourselves to $r \in M_i$ (instead of $r \geq 0, r < \Gamma(f)$) and that the additional assumption (2c) for $t$ can be made:

Let $\{i, t, C\}$ be as before and take an $r \geq 0, r < \Gamma(f)$ such that $\varphi_i(\{i, t, C\})_{k \cdot x'} \neq 0$.

CLAIM. $\langle a(r), t \rangle \geq -a(r)_i$.

$\langle a(r), t \rangle < -a(r)_i$ would imply that there is an $j \neq i$ with $t_j \leq -1$ (cf. case (ii)), and we would obtain the following situation:

$$\mathbb{H}_i := \{a \in \Delta/\langle a, t \rangle < -a_i\} \subseteq \mathbb{H}_i$$

contains $a(r)$ and $e^j$, but not the vertex $e'$. Hence, there is no cone $be^j \in \Sigma$, $b \notin \mathbb{H}_i$, $(b \notin \mathbb{H}_i)$ meeting $\overline{a(r)e^j}$, and $a(r)$ and $e^j$ must be contained in the same connected component of $\Delta_i \cap \mathbb{H}_i$.

\[
\begin{array}{c}
\text{Therefore, the $x^{-t}$-part of } \varphi^*_x(x^{-t}) \text{ would be killed by dividing out } \text{H}_0(e^j) \subseteq \text{H}_0(\Delta^E_i \cap \mathbb{H}_i) \text{ to get } H^1(B^E_r, k).
\end{array}
\]
Now, \((r - t) \in \text{supp } f\) implies \(r - t \geq \Gamma(f)\); in particular, we obtain

\[\langle a(r), r - t \rangle \geq m(a(r))\]

and therefore

\[\langle a(r), r \rangle \geq m(a(r)) - a(r).\]

Finally, each face \(a\) of \(\Gamma(f)\) with \(\langle a, r \rangle < m(a)\) could be taken instead of \(a(r)\), and we obtain \(r \in M_i\).

REMARK. Condition (2c) guarantees that there are only a few (in particular a finite number of) \(r \in M\) fulfilling (2).

(4.7) In (2.6)-(2.8) of [Al] we already tried to describe the image of \(\gamma\).

For elements \(\xi \in H^1(X, \Theta_X \langle -D - Y \rangle)\) (given explicitly by a 1-cocycle \(\{\xi_{ab}\}\)) the induced deformation \(\gamma(\xi) \in \Gamma(f)\) was computed directly. Now, we want to give a short dictionary to understand this formulae in the cohomological language used here. This language is more suitable to see what really happens.

(i) For \(i = 1, 2, 3\) we obtain elements \(\xi(x_i) \in H^1(X, \mathcal{O}_X(-\Sigma_{a \geq 0} a_i D_a))\) (given by \(\xi_{ab}(x_i)\) in [Al]).

(ii) The exact sequence

\[0 \to \mathcal{O}_X \left( - \sum_{a > 0} a_i D_a \right) \to \mathcal{O}_X \to \mathcal{O}_{\Sigma a_i D_a} \to 0,\]

together with \(H^1(X, \mathcal{O}_X) = 0\), shows that \(\xi(x_i)\) can be lifted to an element \(b_i \in H^0(X, \mathcal{O}_{\Sigma a_i D_a})\).

(In [Al] these sections are given locally by \(b_i \in \mathcal{O}_X\):)

\(\xi_{ab}(x_i) = b_i^\alpha - b_i^\beta\) for every two cones \(\alpha, \beta \in \Sigma\).

(iii) Multiplying by \(\partial f / \partial x_i\) provides a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_X \left( - \sum_{a > 0} a_i D_a \right) & \to & \mathcal{O}_X & \to & \mathcal{O}_{\Sigma a_i D_a} & \to & 0 \\
\downarrow & & \downarrow \langle \partial f / \partial x_i \rangle & & \downarrow \ast & & \downarrow \langle \partial f / \partial x_i \rangle & & \downarrow \ast \\
0 & \to & \mathcal{O}_X \left( - \sum m(a) D_a \right) & \to & \mathcal{O}_X & \to & \mathcal{O}_{\Sigma m(a) D_a} & \to & 0.
\end{array}
\]

Therefore, we obtain \(\Sigma_{i=1}^3 (\partial f / \partial x_i) b_i \in H^0(X, \mathcal{O}_{\Sigma m(a) D_a})\) — still written as a local \(\mathcal{O}_X\)-section in [Al].
Finally, we recall the isomorphism

\[ H^0_{(D)}(X, \mathcal{O}_{\Sigma m(a)D_a}) \cong H^1_D(X, \mathcal{O}_X(-\Sigma m(a)D_a)) = \frac{k[x]}{\langle \text{monomials} \geq \Gamma(f) \rangle}. \]

5. An algorithm to determine the equisingular deformations below \( \Gamma(f) \)

(5.1) Analogously to Proposition (4.6) it is possible to compute all deformations of \( \text{ES}(k[\varepsilon]) \subseteq \text{Def}_R(k[\varepsilon]) \). The corresponding algorithm does not use any of the smooth subdivisions of \( \Sigma_0 \) regarded before, but only the starting f.r.p.p. decomposition \( \Sigma_0 \) itself.

Let \( \Delta_i := \cup \{ \tilde{x}/\tilde{x} \in \Sigma_0, \ e^{i} \tilde{x} \} \subseteq \Delta \ (i = 1, 2, 3) \) and take the definition of \( M_i \subseteq M, a: M_i \to \Sigma^{(1)}_0 \) and \( H_t \) of (4.6).

**THEOREM.** (I) Given the following data

1. \( i \in \{1, 2, 3\} \),
2. \( t \in M \) with:
   1. \( t_i \geq 0 \) (i.e. \( e^i \notin \mathbb{H}_i \)) for all \( v \neq i \), or
   2. \( t_j \leq -1 \) (i.e. \( e^j \notin \mathbb{H}_i \)) and the remaining component is \( \geq 0 \),
   3. there exists an \( r \in M_i \) with \( r - t \geq \Gamma(f) \) and \( \langle a(r), t + e_i \rangle \geq 0 \),
3. a connected component \( C \) of \( \Delta_i \cap \mathbb{H}_i \), not containing any of the vertices \( e^1, e^2, e^3 \),

then, the deformation defined by

\[ \sum_{r \in M_i} (r_i + 1) \lambda_{r-t} : x^r = \left( x^{r+e_i}, \frac{\partial F}{\partial x_i} \right)_{M_i \cap a^{-1}(C)} \]

is contained in \( \text{ES}(k[\varepsilon]) \).

(II) \( \text{ES}(k[\varepsilon]) \subseteq \text{Def}_R(k[\varepsilon]) \) as a \( k \)-vectorspace is spanned by the above-\( \Gamma(f) \)-deformations and all deformations constructed in the above way.

**Proof.** Take the three resolutions \( \Sigma_v \ (v = 1, 2, 3) \) of (2.4). Then, by Theorem (3.4) and Proposition (4.6) the above claim were valid if the \( \Delta_i \) would be replaced by \( \Delta_i^{(r)} \) and the resulting elements of \( \text{ES}(k[\varepsilon]) \) were put together for \( v = 1, 2, 3 \).

**STEP 1.** Each deformation that is induced by a \( \Delta_i^{(r)} \cap \mathbb{H}_i \) can also be obtained by using \( \Delta_i \cap \mathbb{H}_i \).

Let \( i, v \in \{1, 2, 3\}, t \in M \) be fixed. By construction it is clear that \( \Delta_i \subseteq \Delta_i^{(r)} \), hence \( \Delta_i \cap \mathbb{H}_i \subseteq \Delta_i^{(r)} \cap \mathbb{H}_i \).

Now, both sets contain the same elements of \( \Sigma_0^{(1)} \), and the connected components of \( \Delta_i^{(r)} \cap \mathbb{H}_i \) (restricted to \( \Delta_i \cap \mathbb{H}_i \)) are built by taking the union of several complete components of \( \Delta_i \cap \mathbb{H}_i \).
For the deformations induced by $\Delta_i \cap H_i$, this means that they split into sums of deformations induced by $\Delta_i \cap H_i$.

**STEP 2.** The connected components of $\Delta_i \cap H_i$ and $\Delta_i^2 \cap H_i$ correspond to each other and contain the same elements of $\Sigma_0^{(1)}$.

Let $a, b \in \Sigma_0^{(1)} \cap [\Delta_i \cap H_i]$ be contained in different components of $\Delta_i \cap H_i$, then they can be separated by a line segment $\overline{ce}$ (contained in a cone of $\Sigma_0$) with $c \notin H_i$.

By the construction of $\Sigma_1$ (cf. (2.4)), this f.r.p.p. decomposition contains $P_i(c)e^t$ as one cone of the canonical partition of $\overline{ce}$. Because of $t = -1$, $\langle c, t \rangle \geq 0$ implies $\langle P_i(c) t \rangle \geq 0$, and $P_i(c)e^t$ will separate $a$ and $b$ as elements of $\Delta_i^2 \cap H_i$.

(The opposite direction was already done in step 1.)

**REMARK.** (1) $\overline{ES(k[e])}/\langle\text{monomials } \geq \Gamma(f)\rangle$ is generated by the columns of the following matrix $A$:

The rows correspond to elements $r \in \bigcup_{i=1}^{3} M_i$,

the columns correspond to triples $(i, t, C)$ with (I), (1)–(3) of the above Theorem, and

$$a_{r,(i,t,C)} := \begin{cases} \langle r_i + 1 \rangle \cdot \lambda_{r-t} & \text{for } a(r) \in C \\ 0 & \text{otherwise} \end{cases}$$

(Of course, this matrix does not depend on the special choice of the function $a: M_i \to \Sigma_0^{(1)}$.)

To make the computation of all possible $t$ easier, it is useful to give coarser restrictions than those of (I)(2):

Let $i = 1$, hence $t_1 = -1$. 

\[ r - t \geq \Gamma(f) \text{ implies } t \leq r; \text{ together with } \langle a(r), t + e_1 \rangle \geq 0 \text{ this gives the following conditions for } t: \]

There exists an \( r \in M_1:\]

\[
- \frac{a_3(r)}{a_2(r)} \leq t_2 \leq r_2
\]

\[
- \frac{a_2(r)}{a_3(r)} \leq t_3 \leq r_3
\]

with \( t_2 \) and \( t_3 \) not both negative.

(2) All type-(i)-deformations in \( \overline{ES}(k[\varepsilon]) \) (cf. (I.2.b) of the Theorem) consist of pieces of trivial deformations (i.e. trivial deformations in which some terms are dropped). If, moreover, \( \Delta_i \cap H_i \) is connected, then the corresponding deformation will be really trivial.

(3) Compare with Theorem (5.8) of [Al]: If the sets \( \Delta_i \) are convex, there will be no type-(ii)-deformations, and all deformations of type (i) will be trivial. Hence, all equisingular deformations are above \( \Gamma \).

(5.2) COROLLARY. The \( k \)-vectorspace \( \overline{ES}(k[\varepsilon])/(\text{monomials } \geq \Gamma(f)) \) and, in particular, the fact whether \( \overline{ES} \) is exactly the functor of above-\( \Gamma(f) \)-deformations or not, are independent of the coefficients \( \lambda_s \) of \( f \) with

\[
\langle a, s \rangle \geq m(a) + \max\{a_1, a_2, a_3\} \quad \text{for all } a \in \Sigma_0^{(1)} \setminus \{e^1, e^2, e^3\}.
\]

Proof. Let \( \lambda_s \) be a coefficient of \( f \) that appears in the matrix \( A \) (defined in the previous remark). If

\[
a_{r, (s, t, c)} = s_1 \cdot \lambda_s \quad (s = r - t),
\]

then we take the element \( a := a(r) \in \Sigma_0^{(1)} \setminus \{e^1, e^2, e^3\} \), and now we obtain

\[
\langle a, r \rangle < m(a) \quad \text{(by definition of } a(r)),
\]

\[
\langle a, t \rangle \geq -a_i \quad \text{(by (I)(2c) of the Theorem)},
\]

hence,

\[
\langle a, s \rangle < m(a) + a_i. \]

(5.3) EXAMPLE. Let \( f(x, y, z) := x^5 + y^6 + z^5 + y^3z^2 \) (cf. [Al], §3); we get
(1) \( \Delta_1 \) is convex, hence, the case \( i = 1 \) yields only trivial deformations (cf. remark (5.1)(3)).

(2) Let \( i = 2 \).

Computation of \( M_2 \):

\[ r \in M_2 \text{ iff } r \geq 0, \]

\[
12r_1 + 10r_2 + 15r_3 \geq 60 - 10 = 50, \]

\[ r_1 + r_2 + r_3 \geq 5 - 1 = 4 \quad \text{and} \quad [12r_1 + 10r_2 + 15r_3 < 60 \quad \text{or} \quad r_1 + r_2 + r_3 < 5]. \]

We obtain

\[ M_2 = \{(0, 0, 4); (0, 1, 3); (0, 2, 2); (1, 0, 3); (1, 1, 2); (2, 0, 2); (3, 0, 1); (0, 4, 1); (0, 5, 0); (1, 3, 1); (1, 4, 0); (2, 2, 1); (2, 3, 0); (3, 2, 0); (4, 1, 0)\}. \]

Conditions for \( t \):

For \( a(r) (r \in M_2) \) there are only two candidates: \((12, 10, 15)\) and \((1, 1, 1)\). We obtain the conditions

\[ -\frac{15}{2} \leq t_1 \leq 4; \quad t_2 = -1; \]

\[ -1 \leq t_3 \leq 4 \quad (t_1, t_3 \text{ not both negative}). \]

Connected components of \( \Delta_2 \cap \mathbb{H}_i \):

The only possibility for \( \Delta_2 \cap \mathbb{H}_i \) to have at least two components is

\[ e^3, (1, 1, 1) \in \mathbb{H}_i, \]

\[ e^1, (12, 10, 15) \notin \mathbb{H}_i. \]
Hence,

\[0 \leq t_1 \leq 4, \quad t_2 = t_3 = -1,\]
\[t_1 + (-1) + (-1) < 0, \quad 12t_1 + (-10) + (-15) \geq 0,\]

and we get a contradiction.

(3) Let \(i = 3\).

By the same methods as in the previous case we obtain

\[M_3 = M_2 \cup \{(0, 3, 1); (1, 2, 1); (2, 1, 1); (3, 1, 0); (4, 0, 0)\}\]

and the conditions for \(t\):

\[-1 \leq t_1 \leq 4, \quad -\frac{12}{10} \leq t_2 \leq 5 \quad \text{and} \quad t_3 = -1 \quad (t_1, t_2 \text{ not both negative}).\]

**Connected components of** \(\Delta_3 \cap \mathbb{H}_i;\):

The only possibility for \(\Delta_3 \cap \mathbb{H}_i\) to have at least two components is

\[e^2, (12, 10, 15) \in \mathbb{H}_i,\]
\[e^1, (1, 1, 1) \notin \mathbb{H}_i.\]

Hence,

\[0 \leq t_1 \leq 4, \quad t_2 = t_3 = -1,\]
\[12t_1 + (-10) + (-15) < 0, \quad t_1 + (-1) + (-1) \geq 0.\]

We obtain the only solution \(t^0 = (2, -1, -1)\), and our matrix \(A\) consists of exactly one column \((i = 3, t^0, C)\) (with \(C \subseteq \Delta_3 \cap \mathbb{H}_{10}\) is the connected component containing the vertex \((12, 10, 15)\)).

**Computation of the entries of** \(A:\)

\[a_{r,(3,t^0,C)} := \begin{cases} (r_3 + 1)\lambda_r \cdot t^0 & \text{for } a(r) \in C \\ 0 & \text{otherwise} \end{cases}.\]

Which elements \(r \in \bigcup_{i=1}^{3} M_i \subseteq \{r \geq 0/r < \Gamma(f)\}\) have the property \(r - t^0 \geq \Gamma(f)\)?

\[\langle (12, 10, 15), t^0 \rangle = -1, \quad \langle (1, 1, 1), t^0 \rangle = 0;\]
therefore, such an $r$ has to meet the following conditions:

$$12r_1 + 10r_2 + 15r_3 = 60 - 1 = 59$$
$$r_1 + r_2 + r_3 \geq 5,$$

and the only solution is $r^0 = (2, 2, 1)$.

That means the only non-vanishing element in our matrix $A$ is

$$(r_3^0 + 1) \cdot \lambda_{r^0-r^0} = 2 \cdot \lambda_{(0,3,2)}$$

(in the row corresponding to $r^0 = (2, 2, 1)$).

Since $(0, 3, 2) \in M$ represents a vertex of $\Gamma(f)$, the coefficient $\lambda_{(0,3,2)}$ can never vanish ($\lambda_{(0,3,2)} = 1$ in our special example). Therefore, we have proved

$$\overline{\text{ES}}(k[\epsilon]) / \langle \text{monomials} \geq \Gamma(f) \rangle = k \cdot x^2 y^2 z,$$

not only for the special equation $f$, but for all equations having this special Newton boundary.

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References


