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Wedge cancellation of certain mapping cones

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0. Introduction

In his thesis I. Bokor proved the following.

THEOREM. *Suppose given* α_i, $\beta_j \in \pi_{4n-1}(S^{2n})$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant H$, *and assume that they are elements of infinite order for* $i \leqslant m, j \leqslant m'$, *and elements of finite order for* $i > m, j > m'$. *Denote by* C_{α_i}, C_{β_j} *the corresponding mapping cones. Then*

$$\bigvee_{i=1}^{H} C_{\alpha_i} \simeq \bigvee_{j=1}^{H} C_{\beta_j}$$

if and only if

(i) $m = m'$.
(ii) $C_{\alpha_i} \simeq C_{\sigma(i)}$, $i = 1, \ldots, m$, *for some permutation* σ *of* $\{1, 2, \ldots, m\}$.
(iii) $\vee_{i=m+1}^{H} C_{\alpha_i} \simeq \vee_{j=m+1}^{H} C_{\beta_j}$.

In this paper we want to study the case of spaces with one cell in dimensions 0 and $4n$ and a finite number of cells in dimension $2n$. That is to say, we are going to consider mapping cones of maps $S^{4n-1} \to \vee^k S^{2n}$. It turns out that, for $k \geqslant 2$, the previous theorem fails; see section 4 for an example. Nevertheless, the result is true if we consider p-local mapping cones. Recall that the p-localization of a mapping cone C_α is homotopy equivalent to the mapping cone of the p-localization $\alpha_{(p)}$ of the map α: $C_{\alpha(p)}$. Moreover, each map $S^n_{(p)} \to \vee^k S^r_{(p)}$ is the p-localization of a map $S^m \to \vee^k S^n$, and

$$[S^m_{(p)}, \vee^k S^n_{(p)}] \cong [S^m, \vee^k S^n]_{(p)}$$

A basic reference for p-localization of groups and spaces is [4].

We shall prove the following.

THEOREM 1. *Let* p *be a fixed prime integer. Given* f_i, g_j: $S^{4n-1}_{(p)} \to \vee^k S^{2n}_{(p)}$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant M$, *assume that they represent elements in* $\pi_{4n-1}(\vee^k S^{2n})_{(p)}$ *of* *infinite order if* $i \leqslant m, j \leqslant m'$, *and elements of finite order if* $i > m, j > m'$. *Then*

$$\bigvee_{i=1}^{H} C_{f_i} \simeq \bigvee_{j=1}^{M} C_{g_j}$$

if and only if

(i) $H = M$, $m = m'$.
(ii) $\vee_{i=m+1}^{H} C_{f_i} \simeq \vee_{j=m+1}^{H} C_{g_j}$.
(iii) $C_{f_{\sigma(j)}} \simeq C_{g_j}$, $j = 1, \ldots, m$, *for some permutation* σ *of* $\{1, 2, \ldots, m\}$.

In some special cases, however, a wedge cancellation property similar to the one in Theorem 1 holds for non-local mapping cones. In Section 4 we study two such cases and we obtain the Bokor theorem as a Corollary.

1. The tools

We summarize in this section some facts that we shall use to prove our results.
Suppose

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \longrightarrow & C_f \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \delta} \\ B & \xrightarrow{g} & Y & \longrightarrow & C_g \end{array}$$

is a homotopy commutative diagram such that φ, δ are homotopy equivalences and A is a Moore space $K'(G, n)$. If $n \geqslant 2$ and Y is 2-connected, then there is a homotopy equivalence $\psi\colon A \to B$ completing the diagram; see [3]. In this paper f and g will always be elements in $[\vee S^{4n+1}, \vee S^{2n}]$ or in $[\vee S^{4n-1}_{(p)}, \vee S^{2n}_{(p)}]$. In the second case, for instance, each homotopy equivalence

$$C_f \simeq C_g$$

Arises from a homotopy commutative diagram

$$\begin{array}{ccc} \vee^H S^{4n-1}_{(p)} & \xrightarrow{f} & \vee^K S^{2n}_{(p)} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} \\ \vee^H S^{4n-1}_{(p)} & \xrightarrow{g} & \vee^K S^{2n}_{(p)} \end{array}$$

where φ and ψ are homotopy equivalences. Observe that $[\vee^k S^{2n}_{(p)}, \vee^k {}^{2n}_{(p)}]$ is isomorphic to the ring of $k \times k$ matrices over $\mathbb{Z}_{(p)}$, the p-localization of the ring $\mathbb{Z}$, and the homotopy class of φ is determined by a matrix (φ_{ij}) with entries

$$\varphi_{ij} = \text{degree}\left(S^{2n}_{(p)} \xrightarrow{in_j} \bigvee^k S^{2n}_{(p)} \xrightarrow{\varphi} \bigvee^k S^{2n}_{(p)} \xrightarrow{q_i} S^{2n}_{(p)} \right)$$

where in_j and q_i are the obvious inclusion and projection. We shall denote this matrix by φ. Actually, we systematically employ this abuse of terminology and denote by the same symbol a map and its matrix. It is clear that φ is a homotopy equivalence if and only if $\det \varphi$ is a unit in $\mathbb{Z}_{(p)}$. Analogously, ψ is determined up to homotopy by a $H \times H$ matrix with determinant a unit in $\mathbb{Z}_{(p)}$.

The non-local case can be treated in a similar way. In particular, the homotopy equivalences correspond to unimodular integer matrices.

The machinery that follows is due to I. Bokor. Its interest lies in the fact that it reduces the homotopy commutativity of the above diagram to some "matricial" formulas.

We know, by the Hilton–Milnor Theorem, that

$$\pi_{4n-1}\left(\bigvee^k S^{2n}\right) \cong \bigoplus^k \pi_{4n-1}(S^{2n}) \oplus \bigoplus_{i<j} \pi_{4n-1}(S^{4n-1})$$

the direct summands $\pi_{4n-1}(S^{4n-1})$ are embedded in $\pi_{4n-1}(\vee^k S^{2n})$ by composition of certain Whitehead products: $[\iota_i, \iota_j]$. The direct summands $\pi_{4n-1}(S^{2n})$ are embedded by the inclusions. Recall that

$$\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus T$$

where $\mathbb{Z}$ is generated by $[\iota_i, \iota_i]$, if $n \neq 1, 2, 4$, and by the Hopf map η_i, *if* $n = 1, 2, 4$. Hence, any $\alpha \in \pi_{4n-1}(\vee^k S^{2n})$, $n \neq 1, 2, 4$, can be written in the form

$$\alpha = \sum_{i=1}^k (a_{ii}[\iota_i, \iota_i] + \alpha^i) + \sum_{i<j}^k a_{ij}[\iota_i, \iota_j],$$

where $\alpha^i \in T$ and $a_{ii}, a_{ij} \in \mathbb{Z}$. For $n = 1, 2, 4$, we get a similar expression replacing $[\iota_i, \iota_i]$ by the Hopf map η_i. Observe that the Hopf invariant of the map $S^{4n-1} \xrightarrow{\alpha} \vee^k S^{2n} \xrightarrow{q_i} S^{2n}$ is

$$\begin{aligned} H(q_i \circ \alpha) &= 2a_{ii} \quad \text{if} \quad n \neq 1, 2, 4 \\ &= a_{ii} \quad \text{if} \quad n = 1, 2, 4. \end{aligned}$$

We define $H(\alpha)$ as the $k \times k$ symmetric matrix with entries a_{ij}, $i \neq j$, off the diagonal and the Hopf invariants $H(q_i \circ \alpha)$ on the diagonal.

Now, the suspension homomorphism

$$\Sigma: \pi_{4n-1}\left(\bigvee^{k} S^{2n}\right) \to \pi_{4n}\left(\bigvee^{k} S^{2n+1}\right) \cong \bigoplus^{k} \pi_{4n}(S^{2n+1})$$

induces a monomorphism on $\oplus^k T$. Thus $(\alpha^1, \ldots, \alpha^k) \in \oplus^k T$ is determined by its image $(\Sigma\alpha^1, \ldots, \Sigma\alpha^k)$. This image coincides with $\Sigma\alpha$ when $n \neq 1, 2, 4$. We shall again abuse the terminology and denote by $\Sigma\alpha$ the matrix

$$\Sigma\alpha = \begin{bmatrix} \Sigma\alpha^1 \\ \vdots \\ \Sigma\alpha^k \end{bmatrix}$$

for all n.

Clearly $H(\alpha)$ and $\Sigma\alpha$ characterise the homotopy class α.

Given maps $\psi: S^{4n-1} \to S^{4n-1}$ and $\varphi: \vee^k S^{2n} \to \vee^k S^{2n}$ I. Bokor proved that

$$\begin{aligned} H(\varphi \circ \alpha) &= \varphi H(\alpha)\varphi^t, & H(\alpha \circ \psi) &= \psi H(\alpha), \\ \Sigma(\varphi \circ \alpha) &= \varphi \Sigma\alpha, & \Sigma(\alpha \circ \psi) &= \psi \Sigma\alpha. \end{aligned} \tag{1.1}$$

Here φ^t denotes the transpose of the matrix φ, and ψ is the degree of the map $\psi: S^{4n-1} \to S^{4n-1}$.

We shall also deal with maps

$$\alpha = (\alpha_1, \ldots, \alpha_H) \in \left[\bigvee^{H} S^{4n-1}, \bigvee^{K} S^{2n}\right] \cong \bigoplus^{H} \pi_{4n-1}\left(\bigvee^{K} S^{2n}\right).$$

Then we define $H(\alpha)$ as the $K \times HK$ matrix obtained by juxtaposing the matrices $H(\alpha_i)$

$$H(\alpha) = (H(\alpha_1) \cdots H(\alpha_H)).$$

Similarly

$$\Sigma\alpha = (\Sigma\alpha_1 \cdots \Sigma\alpha_H).$$

Given maps $\psi: \vee^H S^{4n-1} \to \vee^H S^{4n-1}$ and $\varphi: \vee^K S^{2n} \to \vee^K S^{2n}$, we obtain

$$\begin{aligned} H(\varphi \circ \alpha) &= \varphi H(\alpha)(I_H \otimes \varphi^t), & \Sigma(\varphi \circ \alpha) &= \varphi \Sigma\alpha, \\ H(\alpha \circ \psi) &= H(\alpha)(\psi \otimes I_K), & \Sigma(\alpha \circ \psi) &= \psi \Sigma\alpha. \end{aligned}$$

Here $(\psi_{ij}) = \psi$, I_K is the $K \times K$ identity matrix and

$$(I_H \otimes \varphi^t) = \begin{bmatrix} \varphi^t & 0 & \cdots & 0 \\ 0 & \varphi^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi^t \end{bmatrix}$$

$$(\psi \otimes I_K) = \begin{bmatrix} \psi_{11} I_K & \cdots & \psi_{1H} I_K \\ \vdots & & \vdots \\ \psi_{H1} I_K & \cdots & \psi_{HH} I_K \end{bmatrix}$$

Assume finally that we have maps α_i, $\beta_j : S^{4n-1} \to \vee^k S^{2n}$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant H$. Take $\alpha = \alpha_1 \vee \cdots \vee \alpha_H$ and $\beta = \beta_1 \vee \cdots \vee \beta_H$, so that $C_\alpha \simeq \vee^H C_{\alpha_i}$ and $C_\beta \simeq \vee^H C_{\beta_j}$. It is easy to see that

$$H(\alpha) = \begin{bmatrix} H(\alpha_1) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & H(\alpha_2) & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & H(\alpha_H) \end{bmatrix}$$

$$\Sigma\alpha = \begin{bmatrix} \Sigma\alpha_1 & 0 & \cdots & 0 \\ 0 & \Sigma\alpha_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Sigma\alpha_H \end{bmatrix}$$

and similarly for $H(\beta)$ and $\Sigma\beta$. Consider now a diagram

$$\begin{array}{ccc} \vee^H S^{4n-1} & \xrightarrow{\alpha} & \vee^H(\vee^k S^{2n}) \\ \psi \big\downarrow & & \big\downarrow \varphi \\ \vee^H S^{4n-1} & \xrightarrow{\beta} & \vee^H(\vee^k S^{2n}) \end{array} \tag{1.2}$$

Suppose $\psi = (\psi_{ij})$ with $\psi_{ij} \in \mathbb{Z}$, and

$$\varphi = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1H} \\ \vdots & & \vdots \\ \phi_{H1} & \cdots & \phi_{HH} \end{bmatrix}$$

where ϕ_{ij} is the $k \times k$ matrix corresponding to the obvious map $\vee^k S^{2n} \xrightarrow{in_j} \vee^H(\vee^k S^{2n}) \xrightarrow{\phi} \vee^H(\vee^k S^{2n}) \xrightarrow{q_i} \vee^k S^{2n}$. From (1.1) we easily get that the diagram (1.2) is homotopy commutative if and only if the following conditions hold.

$$\varphi_{ji} H(\alpha_i)\phi^t_{ji} = \psi_{ji} H(\beta_j) \quad \text{for all } i, j.$$
$$\phi_{ji} H(\alpha_i)\phi^t_{li} = 0 \qquad \text{if } l \neq j. \tag{I}$$

$$\phi_{ji}\Sigma\alpha_i = \psi_{ji}\Sigma\beta_j \quad \text{for all } i, j. \tag{II}$$

Analogous conditions can be obtained for the p-localization at any prime p.

2. The proof of Theorem 1

$\Leftarrow$. It is obvious.

$\Rightarrow$. (i) $H = M$ is clear (by comparing the $4n$-dimensional homology).

Consider now $f = \vee^H f_i$, $g = \vee^H g_j$. We have $C_f \simeq C_g$. This homotopy equivalence arises from a homotopy commutative diagram

$$\begin{array}{ccc} \vee^H S^{4n-1}_{(p)} & \xrightarrow{f} & \vee^H(\vee^k S^{2n}_{(p)}) \\ \psi \downarrow & & \downarrow \varphi \\ \vee^H S^{4n-1}_{(p)} & \xrightarrow{g} & \vee^H(\vee^H S^{2n}_{(p)}) \end{array} \tag{2.1}$$

where ψ and φ are homotopy equivalences and the conditions (I) and (II) in Section 1 hold.

When f_i is of finite order $H(f_i) = 0$. So it follows from (I) in Section 1 that

$$\psi_{ji} = 0 \quad \text{if} \quad j \leqslant m' \qquad \text{and} \qquad i > m$$

But $\det\psi \neq 0$. Thus $m \geqslant m'$. By the symmetry we also have $m' \geqslant m$, so that $m = m'$.

(ii) Write the matrices ψ and φ in the form

$$\psi = \begin{pmatrix} \Psi_1 & 0 \\ \Psi_3 & \Psi_4 \end{pmatrix} \qquad \varphi = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}$$

where Ψ_1 is a $m \times m$ matrix and Φ_1 is a $mk \times mk$ matrix. Let

$$\varphi^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

In particular, we have

$$B_1\Phi_1 + B_2\Phi_3 = I \qquad \text{and} \qquad B_3\Phi_2 + B_4\Phi_4 = I.$$

Applying Lemma 6.4 in [1] it follows that there are matrices C and A such that

$$\bar{\Phi}_1 = \Phi_1 + CB_2\Phi_3 \qquad \text{and} \qquad \bar{\Phi}_4 = \Phi_4 + A\Phi_2$$

are units in the corresponding rings of matrices over $\mathbb{Z}_{(p)}$.

Consider the diagrams

$$\begin{array}{ccc} \vee^m S^{4n-1}_{(p)} & \xrightarrow{\vee_1^m f_i} & \vee^m(\vee^k S^{2n}_{(p)}) \\ \Psi_1 \downarrow & & \downarrow \bar{\Phi}_1 \\ \vee^m S^{4n-1}_{(p)} & \xrightarrow{\vee_1^m g_j} & \vee^m(\vee^k S^{2n}_{(p)}) \end{array}$$

$$\begin{array}{ccc} \vee^H_{m+1} S^{4n-1}_{(p)} & \xrightarrow{\vee^H_{m+1} f_i} & \vee^H_{m+1}(\vee^k_{m+1} S^{2n}_{(p)}) \\ \Psi_4 \downarrow & & \downarrow \bar{\Phi}_4 \\ \vee^H_{m+1} S^{4n-1}_{(p)} & \xrightarrow{\vee^H_{m+1} g_j} & \vee^H_{m+1}(\vee^k_{m+1} S^{2n}_{(p)}) \end{array} \tag{2.2}$$

We are going to prove that each is homotopy commutative, checking the conditions (I) and (II) in each case. The conditions (I) on the second diagram are obvious since $H(f_i) = 0 = H(g_j)$ if $i, j > m$. To check (II) let us write $\bar{\Phi}_4 = \Phi_4 + A\Phi_2$ in the form

$$\bar{\phi}_{ji} = \phi_{ji} + \sum_{l=1}^{m} A_{jl}\phi_{li} \quad \text{for all } i, j > m$$

Then

$$\bar{\phi}_{ji}\Sigma f_i = \phi_{ji}\Sigma f_i = \psi_{ji}\Sigma g_j$$

since $\phi_{li}\Sigma f_i = \psi_{li}\Sigma_{gl}$ and $\psi_{li} = 0$ for $l \leqslant m$, $i > m$. So the second diagram is homotopy commutative and, since Ψ_4 and $\bar{\Phi}_4$ are homotopy equivalences, it induces

$$\bigvee_{m+1}^{H} C_{f_i} \simeq \bigvee_{m+1}^{H} C_{g_j}.$$

In order to prove the homotopy commutativity of the first diagram, let us write $\bar{\Phi}_1 = \Phi_1 + CB_2\Phi_3$ in the form

$$\bar{\phi}_{ji} = \phi_{ji} + \sum_{\substack{l \leqslant m \\ r > m}} C_{jl} B_{lr} \phi_{ri} \qquad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m.$$

Now

$$\bar{\phi}_{ji} H(f_i) \bar{\phi}^t_{ji} = \phi_{ji} H(f_i) \phi_{ji} + \sum_{\substack{l \leqslant m \\ r > m}} C_{jl} B_{lr} \phi_{ri} H(f_i) \phi^t_{ri} B^t_{lr} C^t_{jl}$$

$$= \psi_{ji} H(g_j) + \sum_{\substack{l \leqslant m \\ r > m}} C_{jl} B_{lr} (\psi_{ri} H(g_r)) B^t_{lr} C^t_{jl} \tag{2.3}$$

$$\bar{\phi}_{ji} \Sigma f_i = \phi_{ji} \Sigma f_i + \sum_{\substack{l \leqslant m \\ r > m}} C_{jl} B_{lr} \phi_{ri} \Sigma f_i$$

$$= \psi_{ji} \Sigma g_j + \sum_{\substack{l \leqslant m \\ r > m}} \psi_{ri} C_{jl} B_{lr} \Sigma g_r \tag{2.4}$$

Observe that $B_1\Phi_2 + B_2\Phi_4 = 0$. Hence, if $l \leqslant m$, $i > m$,

$$\sum_{r=1}^{H} B_{lr} \phi_{ri} = 0$$

Thus

$$0 = \sum_{r=1}^{H} B_{lr} \phi_{ri} H(f_i) \phi^t_{ri} B^t_{lr} = \sum_{r=1}^{H} B_{lr} (\psi_{ri} H(g_r)) B^t_{lr}$$

$$= \sum_{r > m} \psi_{ri} B_{lr} H(g_r) B^t_{lr}$$

$$0 = \sum_{r=1}^{H} B_{lr} \phi_{ri} \Sigma f_i = \sum_{r=1} \psi_{ri} B_{lr} \Sigma f_i = \sum_{r > m} \psi_{ri} B_{lr} \Sigma g_r.$$

Since $\psi_{ri} = 0$ if $i \leqslant m$, $r > m$. But $\{\psi_{ri};\, r, i > m\}$ are the entries of the unit matrix Ψ_4. So, for $l \leqslant m$,

$$\sum_{r > m} \psi_{ri} B_{lr} H(g_r) B^t_{lr} = 0, \quad \forall i > m \Rightarrow B_{lr} H(g_r) B^t_{lr} = 0, \quad \forall r > m$$

$$\sum_{r > m} \psi_{ri} B_{lr} \Sigma g_r = 0, \quad \forall i > m \Rightarrow B_{lr} \Sigma g_r = 0, \quad \forall r > m.$$

Hence, from (2.3) and (2.4) we obtain

$$\bar{\phi}_{ji}H(f_i)\bar{\phi}_{ji}^t = \psi_{ji}H(g_i) \quad \text{and} \quad \bar{\phi}_{ji}\Sigma f_i = \psi_{ji}\Sigma g_j$$

and the first diagram in (2.2) is homotopy commutative. Since Ψ_1 and Φ_1 are homotopy equivalences, we get

$$\bigvee_1^m C_{f_i} \simeq \bigvee_1^m C_{g_j}.$$

(iii) Without loss of generality we may now assume that all the elements f_i and g_j are of infinite order. We argue by induction on the highest rank, say r, of the matrices $H(f_i)$ and $H(g_j)$.

Assume that rank $H(g_j) = r$ if and only if $1 \leqslant j \leqslant t$. Since $\det \psi \neq 0$; it follows that, for each $j \leqslant t$, there is an integer $\sigma(j)$ such that $\psi_{j\sigma(j)} \neq 0$. Thus, by (I),

$$\phi_{j\sigma(j)}H(f_{\sigma(j)})\phi_{j\sigma(j)}^t = \psi_{j\sigma(j)}H(g_j) \Rightarrow \text{rank } H(f_{\sigma(j)}) = r$$
$$\text{rank } \phi_{j\sigma(j)} \geqslant r.$$

Moreover, if $s \neq j$

$$\phi_{s\sigma(j)}H(f_{\sigma(j)})\phi_{j\sigma(j)}^t = 0 \Rightarrow \phi_{s\sigma(j)}H(f_{\sigma(j)}) = 0$$

Now, since $H(g_s) \neq 0$.

$$\psi_{s\sigma(j)}H(g_s) = \phi_{s\sigma(j)}H(f_{s\sigma(j)})\phi_{s\sigma(j)}^t = 0 \Rightarrow \psi_{s\sigma(j)} = 0$$

if $s \neq j$. In other words, all the elements in the $\sigma(j)$ column of the matrix ψ are 0 except $\psi_{j\sigma(j)}$. Since ψ is a unit in the ring of matrices over $\mathbb{Z}_{(p)}$, it follows that $\psi_{j\sigma(j)}$ is a unit in $\mathbb{Z}_{(p)}$ and that the map

$$j \mapsto \sigma(j)$$

is 1-1. In particular, the number t' of elements f_i such that rank $H(f_i) = r$ is greater than or equal to the number t of g_j's with rank $H(g_j) = r$. By symmetry we get $t' = t$. We shall suppose that rank $H(f_i) = r$ if and only if $1 \leqslant i \leqslant t$.

Let

$$\varphi^{-1} = (C_{ij}),$$

where the C_{ij} are $k \times k$ matrices with entries in $\mathbb{Z}_{(p)}$.

Thus

$$C_{\sigma(j)j}\phi_{j\sigma(j)} + \sum_{s\neq j} C_{\sigma(j)s}\phi_{s\sigma(j)} = I$$

and, from Lemma 6.4 in [1], it follows that there is a matrix B_j such that

$$\bar{\phi}_{j\sigma(j)} = \phi_{j\sigma(j)} + B_j\left(\sum_{s\neq j} C_{\sigma(j)s}\phi_{s\sigma(j)}\right)$$

is a unit in the ring of $k\times k$ matrices over $\mathbb{Z}_{(p)}$. Now, by (I),

$$\bar{\phi}_{j\sigma(j)}H(f_{\sigma(j)})\bar{\phi}^t_{j\sigma(j)} = \phi_{j\sigma(j)}H(f_{\sigma(j)})\phi^t_{j\sigma(j)} = \psi_{j\sigma(j)}H(g_j)$$

since

$$\begin{aligned}\phi_{s\sigma(j)}H(f_{\sigma(j)})\phi^t_{u\sigma(j)} &= 0 && \text{if } u\neq s\\ &= \psi_{s\sigma(j)}H(g_s) && \text{if } u = s\end{aligned}$$

and $\psi_{s\sigma(j)} = 0$ for $s\neq j$. In addition

$$\begin{aligned}\bar{\phi}_{j\sigma(j)}\Sigma f_{\sigma(j)} &= \phi_{j\sigma(j)}\Sigma f_{\sigma(j)} + \sum_{s\neq j} B_j C_{\sigma(j)s}\phi_{s\sigma(j)}\Sigma f_{\sigma(j)}\\ &= \psi_{j\sigma(j)}\Sigma g_j + \sum_{s\neq j} B_j C_{\sigma(j)s}(\psi_{s\sigma(j)}\Sigma g_s)\\ &= \psi_{j\sigma(j)}\Sigma g_j\end{aligned}$$

Hence, the diagram

$$\begin{array}{ccc} S^{4n-1}_{(p)} & \xrightarrow{f_{\sigma(j)}} & \vee^k S^{2n}_{(p)} \\ {\scriptstyle \psi_{j\sigma(j)}}\downarrow & & \downarrow{\scriptstyle \bar{\phi}_{j\sigma(j)}} \\ S^{4n-1}_{(p)} & \xrightarrow{g_j} & \vee^k S^{2n}_{(p)} \end{array}$$

is homotopy commutative and induces

$$C_{g_j} \simeq C_{f_{\sigma(j)}} \quad \text{if } 1\leqslant j\leqslant t.$$

Finally, write

$$\varphi = \begin{pmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{pmatrix}$$

where Φ_1 is a $tk \times tk$ matrix. Again by Lemma 6.4 in [1] there is a matrix D such that

$$\bar{\Phi}_4 = \Phi_4 + D\Phi_2$$

is a unit. That is to say, there are matrices D_{il} such that, if $\Phi_4 = (\bar{\phi}_{ij})$,

$$\bar{\phi}_{ij} = \phi_{ij} + \sum_{1 \leqslant l \leqslant t} D_{il}\phi_{lj} \qquad t < i, j \leqslant H.$$

Let $\bar{\Psi}$ denote the minor of ψ formed by the columns $i > t$ and the rows $j > t$. $\bar{\Psi}$ is clearly a unit. Now, using (I) and (II) and arguing as before, it can be easily checked that the diagram

$$\begin{array}{ccc} \vee_{t+1}^{H} S_{(p)}^{4n-1} & \xrightarrow{\vee_{t+1}^{H} f_i} & \vee_{t+1}^{H}(\vee^k S_{(p)}^{2n}) \\ \bar{\Psi}\downarrow & & \downarrow \bar{\Phi}_4 \\ \vee_{t+1}^{H} S_{(p)}^{4n-1} & \xrightarrow{\vee_{t+1}^{H} g_j} & \vee_{t+1}^{H}(\vee^k S_{(p)}^{2n}) \end{array}$$

is homotopy commutative. Therefore, it induces a homotopy equivalence

$$\bigvee_{t+1}^{H} C_{f_i} \simeq \bigvee_{t+1}^{H} C_{g_j}$$

By induction we obtain (iii). This ends the proof of Theorem 1. □

3. Two Corollaries

COROLLARY 1. *Given maps α_i, β_j: $S^{4n-1} \to \vee^k S^{2n}$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant M$, such that α_i and β_j represent elements of infinite order in $\pi_{4n-1}(\vee^k S^{2n})$ if and only if $i \leqslant m$ and $j \leqslant m'$, then*

$$\bigvee_{1}^{H} C_{\alpha_i} \quad \text{and} \quad \bigvee_{1}^{M} C_{\beta_i}$$

are in the same genus.

If and only if

(i) $H = M$, $m = m'$
(ii) $\vee_{m+1}^{H} C_{\alpha_i}$ *and* $\vee_{m+1}^{H} C_{\beta_j}$ *are in the same genus*
(iii) $\vee^{m} C_{\alpha_i}$ *and* $\vee^{m} C_{\beta_i}$ *are in the same genus*

Recall that two finite *CW*-complexes X and Y are in the same genus if and only if their p-localizations at each prime p are homotopy equivalent $X_{(p)} \simeq Y_{(p)}$.

Observe that the spaces C_{α_i}, C_{β_i}, $i = 1, \ldots, m$, in Corollary 1 needn't be in the same genus, since the permutation in Theorem 1 (iii) depends on the prime p. Take, for instance, α_i, β_i: $S^{4n-1} \to S^{2} \vee S^{2n}$, $i = 1, 2$, such that $\Sigma\alpha_i = 0 = \Sigma\beta_i$, $i = 1, 2$ and

$$H(\alpha_1) = \begin{pmatrix} 2p & 0 \\ 0 & 2q \end{pmatrix}, \quad H(\alpha_2) = \begin{pmatrix} 2p^2q & 0 \\ 0 & 2pq^2 \end{pmatrix},$$

$$H(\beta_1) = \begin{pmatrix} 2p^2 & 0 \\ 0 & 2pq \end{pmatrix}, \quad H(\beta_2) = \begin{pmatrix} 2pq & 0 \\ 0 & 2q^2 \end{pmatrix}$$

where $p \neq q$ are prime integers.

Then, for any prime $r \neq p$,

$$C_{\alpha_1(r)} \simeq C_{\beta_1(r)}, \quad C_{\alpha_2(r)} \simeq C_{\beta_2(r)}$$

and

$$C_{\alpha_1(p)} \simeq C_{\beta_2(p)}, \quad C_{\alpha_2(p)} \simeq C_{\beta_1(p)}$$

but

$$C_{\alpha_1(p)} \not\simeq C_{\beta_1(p)}, \quad C_{\alpha_1(q)} \not\simeq C_{\beta_2(q)}$$

so that C_{α_1} is not in the genus of C_{β_i}, $i = 1, 2$.

COROLLARY 2. *Let* f_i: $S^{4n-1}_{(p)} \to \vee^{k_i} S^{2n}_{(p)}$, $1 \leqslant i \leqslant H$, *and* g_j: $S^{4n-1}_{(p)} \to \vee^{k_j} S^{2n}_{(p)}$, $1 \leqslant j \leqslant M$, *and suppose that* f_i *and* g_j *represent elements of infinite order in the corresponding homotopy groups if* $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m'$. *Then*

$$\bigvee^{H} C_{f_i} \simeq \bigvee^{M} C_{g_j}$$

$\Rightarrow$ (i) $H = M$, $m = m'$.

(ii) $\vee^{H}_{m+1} C_{f_i} \vee T \simeq \vee^{H}_{m+1} C_{g_j} \vee T'$, *for some wedges of p-localized 2n-spheres T and T'.*

(iii) $C_{g_j} \vee T_j \simeq C_{f_{\sigma(j)}} \vee T'_j$, *for some wedges of p-localized 2n-spheres* T_j *and* T'_j, $j = 1, \ldots, m$, *and for some permutation* σ *of* $\{1, 2, \ldots, m\}$.

Proof. Comparing the homology of the two given wedges, we easily get $H = M$ and $\Sigma k_i = \Sigma h_j$. Now, take $k = \max\{k_i, h_j;\ 1 \leqslant i \leqslant H, 1 \leqslant j \leqslant H\}$ and consider

$$\bar{f}_i: S^{4n-1}_{(p)} \xrightarrow{f_i} \bigvee^{k_i} S^{2n}_{(p)} \hookrightarrow \bigvee^{k} S^{2n}_{(p)}$$

$$\bar{g}_j: S^{4n-1}_{(p)} \xrightarrow{g_j} \bigvee^{h_j} S^{2n}_{(p)} \hookrightarrow \bigvee^{k} S^{2n}_{(p)}.$$

Clearly $C_{\bar{f}_i} \simeq C_{f_i} \vee \vee^{k-k_i} S^{2n}_{(p)}$, $C_{\bar{g}_i} \simeq C_{g_j} \vee \vee^{k-k_j} S^{2n}_{(p)}$.

Hence,

$$\bigvee^{H} C_{\bar{f}_i} \simeq \bigvee^{H} C_{\bar{g}_j}$$

since $\Sigma_i(k - k_i) = \Sigma_i(k - k_j)$, and we can apply Theorem 1. □

4. Some results in the non-local case

The following example shows that Theorem 1 fails for non-p-local spaces.

Consider maps α_i, $\beta_j S^{4n-1} \to S^{2n} \vee S^{2n} i, j = 1, 2$, such that

$$H(\alpha_1) = H(\beta_1) = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}, \quad H(\alpha_2) = H(\beta_2) = 0,$$

$$\Sigma\alpha_1 = \Sigma\beta_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \Sigma\alpha_2 = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \Sigma\beta_2 = \begin{pmatrix} 3y \\ z \end{pmatrix}$$

where y and z generate two cyclic subgroups of order 8 in $\pi_{4n}(S^{2n+1})$ which intersection is the unit element. It is easy to see that the diagram

$$\begin{array}{ccc} S^{4n-1} \vee S^{4n-1} & \xrightarrow{\alpha_1 \vee \alpha_2} & \vee^2 S^{2n} \vee \vee^2 S^{2n} \\ \downarrow \psi & & \downarrow \varphi \\ S^{4n-1} \vee S^{4n-1} & \xrightarrow{\beta_1 \vee \beta_2} & \vee^2 S^{2n} \vee \vee^2 S^{2n} \end{array}$$

where

$$\psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \varphi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 8 & 8 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is homotopy commutative by checking the conditions (I) and (II) in Section 1.

Hence,

$$C_{\alpha_1} \vee C_{\alpha_2} \simeq C_{\beta_1} \vee C_{\beta_2}.$$

Obviously $C_{\alpha_1} \simeq C_{\beta_1}$, since $\alpha_1 \simeq \beta_1$. However, for any homotopy commutative diagram

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{\alpha_2} & S^{2n} \vee S^{2n} \\ {\scriptstyle \pm 1}\big\downarrow & & \big\downarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \bar{\varphi} \\ S^{4n-1} & \xrightarrow{\beta_2} & S^{2n} \vee S^{2n} \end{array}$$

we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \pm\begin{pmatrix} 3y \\ z \end{pmatrix} \Rightarrow \det \bar{\varphi} \equiv +3 \text{ module } 8.$$

Therefore $\bar{\varphi}$ is not a unit, and $C_{\alpha_2} \not\simeq C_{\beta_2}$.

Some special results hold, however, in the non-local case.

THEOREM 2. *Suppose that* α_i, β_j: $S^{4n-1} \to \vee^k S^{2n}$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant M$, *are given and*

rank $H(\alpha_i) = k =$ rank $H(\beta_j)$ *if and only if* $i \leqslant m, j \leqslant m'$.

Then

$$\bigvee_{i=1}^{H} C_{\alpha_i} \simeq \bigvee_{j=1}^{M} C_{\beta_j}$$

if and only if

(i) $H = H$, $m = m'$
(ii) $\vee^H_{m+1} C_{\alpha_i} \simeq \vee^H_{m+1} C_{\beta_j}$
(iii) $C_{\beta_j} \simeq C_{\alpha_{\sigma(j)}}$, $j = 1, \ldots, m$, *for some permutation* σ *of* $\{1, 2, \ldots, m\}$.

Proof. $\Leftarrow$ is obvious. We prove $\Rightarrow$.

Clearly $H = M$. Consider the homotopy commutative diagram

$$\begin{array}{ccc} \vee^H S^{4n-1} & \xrightarrow{\alpha = \vee \alpha_i} & \vee^H(\vee^k S^{2n}) \\ {\scriptstyle \psi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \vee^H S^{4n-1} & \xrightarrow{\beta = \vee \beta_j} & \vee^H(\vee^k S^{2n}) \end{array}$$

where ψ and φ are homotopy equivalences which induce the given homotopy equivalence $C_\alpha \simeq C_\beta$.

Since ψ is unimodular, for each $j \leqslant m'$ there is an integer $\sigma(j)$ such that $\psi_{j\sigma(j)} \neq 0$. But $\det H(\beta_j) \neq 0$ and from (I) and (II) in Section 1 it follows that

$$\det H(\alpha_{\sigma(j)}) \neq 0 \quad \text{and} \quad \det \phi_{j\sigma(j)} \neq 0.$$

Thus $\sigma(j) \leqslant m$ and $\phi_{s\sigma(j)} = 0$ if $s \neq j$. Arguing now as in the proof of Theorem 1 we get $m = m'$ and $\psi_{s\sigma(j)} = 0$ if $s \neq j$. So σ must be 1-1 and ψ and φ are of the following form

$$\psi = \begin{pmatrix} \Psi_1 & 0 \\ \Psi_3 & \Psi_4 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \Phi_1 & 0 \\ \Phi_3 & \Phi_4 \end{pmatrix}$$

with $\Psi_1, \Psi_4, \Phi_1, \Phi_4$ unimodular matrices. Now it can be easily checked that the diagrams

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{\alpha_{\sigma(j)}} & \vee^k S^{2n} \\ \downarrow{\scriptstyle \psi_{j\sigma(j)}} & & \downarrow{\scriptstyle \phi_{j\sigma(j)}} \\ S^{4n-1} & \xrightarrow{\beta_j} & \vee^k S^{2n} \end{array},$$

$$\begin{array}{ccc} \vee_{m+1}^{H} S^{4n-1} & \xrightarrow{\vee_{m+1}^{H} \alpha_i} & \vee_{m+1}^{H} (\vee^k S^{2n}) \\ \downarrow{\scriptstyle \Psi_4} & & \downarrow{\scriptstyle \Phi_4} \\ \vee_{m+1}^{H} S^{4n-1} & \xrightarrow{\vee_{m+1}^{H} \beta_j} & \vee_{m+1}^{H} (\vee^k S^{2n}) \end{array}$$

are homotopy commutative and induce homotopy equivalences between the corresponding mapping cones

$$C_{\alpha_{\sigma(j)}} \simeq C_{\beta_j} \quad \text{if} \quad j \leqslant m \qquad \text{and} \qquad \bigvee_{m+1}^{H} C_{\alpha_i} \simeq \bigvee_{m+1}^{H} C_{\beta_i}. \qquad \square$$

The Factorisation Theorem in [2] is the case $k = 1$ of Theorem 1.

THEOREM 3. *Suppose that* $\alpha_i, \beta_j : S^{4n-1} \to S^{2n} \vee S^{2n}$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant M$, *represent elements of infinite order in* $\pi_{4n-1}(S^{2n} \vee S^{2n})$. *Then*

$$\bigvee^{H} C_{\alpha_i} \simeq \bigvee^{M} C_{\beta_j}$$

if and only if $H = M$ *and*

$$C_{\alpha_i} \simeq C_{\beta_{\sigma(i)}} \quad \textit{for some permutation } \sigma \textit{ of } \{1, 2, \ldots, H\}.$$

Proof. By Theorem 2 the mapping cones corresponding to maps α_i, β_j with rank $H(\alpha_i) = 2 = \operatorname{rank} H(\beta_j)$ are homotopy equivalent and can be cancelled. Hence, we may assume that rank $H(\alpha_i) = 1 = \operatorname{rank} H(\beta_j)$ for all i and j.

Each of the matrices $H(\alpha_i)$ (and $H(\beta_i)$) is a symmetric matrix of rank 1; that is

$$H(\alpha_i) = \begin{pmatrix} a_i & \lambda a_i \\ \lambda a_i & \lambda^2 a_i \end{pmatrix}$$

Hence $\bar{\alpha}_i \colon S^{4n-1} \xrightarrow{\alpha_i} S^{2n} \vee S^{2n} \xrightarrow{\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}} S^{2n} \vee S^{2n}$ has

$$H(\bar{\alpha}_i) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} a_i & \lambda a_i \\ \lambda a_i & \lambda^2 a_i \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix}$$

and $C_{\bar{\alpha}_i} \simeq C_{\alpha_i}$. Therefore, we may assume that

$$H(\alpha_i) = \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad H(\beta_j) = \begin{pmatrix} b_j & 0 \\ 0 & 0 \end{pmatrix}$$

$a_i \neq 0$, $b_j \neq 0$, $1 \leqslant i \leqslant H$, $1 \leqslant j \leqslant M$. Clearly $H = M$.

Now the homotopy equivalence $\vee^H C_{\alpha_i} \simeq \vee^H C_{\beta_j}$ arises from a homotopy commutative diagram

$$\begin{array}{ccc} \vee^H S^{4n-1} & \xrightarrow{\vee \alpha_i} & \vee^H (\vee^2 S^{2n}) \\ \Big\downarrow \psi & & \Big\downarrow \varphi \\ \vee^H S^{4n-1} & \xrightarrow{\vee \beta_j} & \vee^H (\vee^2 S^{2n}) \end{array}$$

where ψ and φ are homotopy equivalences. Write $\psi = (\psi_{ij})$ and $\varphi = (\phi_{ij})$ with ϕ_{ij} 2×2 integer matrices. For each j there is an integer $\sigma(j)$ such that $\psi_{\sigma(j)j} \neq 0$ then, from (I) in Section 1, it follows

$$\phi_{\sigma(j)j} H(\alpha_j) \phi^t_{\sigma(j)j} = \psi_{\sigma(j)j} H(\beta_{\sigma(j)}) \Rightarrow \phi_{\sigma(j)j} = \begin{pmatrix} r & u \\ 0 & v \end{pmatrix} \text{ with } r \neq 0$$

$$\phi_{sj} H(\alpha_j) \phi^t_{\sigma(j)j} = 0, \quad s \neq \sigma(j) \quad \Rightarrow \phi_{sj} = \begin{pmatrix} 0 & u' \\ 0 & v' \end{pmatrix}, \quad s \neq \sigma(j)$$

$$\Rightarrow 0 = \phi_{sj} H(\alpha_j) \phi^t_{sj} = \psi_{sj} H(\beta_s) \Rightarrow \psi_{sj} = 0, \quad s \neq \sigma(j).$$

In particular, $r = \pm 1$ and $\psi_{\sigma(j)j} = \pm 1$, since ψ and φ are unimodular.

On the other hand, if $\Sigma\alpha_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$, from (II) in Section 2 it follows that

$$\phi_{\sigma(j)j}\Sigma\alpha_j = \begin{pmatrix} rx_j + uy_j \\ vy_j \end{pmatrix} = \psi_{\sigma(j)j}\Sigma\beta_{\sigma(j)}$$

$$\phi_{sj}\Sigma\alpha_j = \begin{pmatrix} u'y_j \\ v'y_j \end{pmatrix} = \psi_{sj}\Sigma\beta_s = 0 \quad \text{if } s \neq \sigma(j)$$

Thus $u' \equiv 0$, $v' \equiv 0$ modulo $|y_j|$. That is to say the matrices ϕ_{sj}, $s \neq \sigma(j)$, are 0 modulo $|y_j|$ and, hence, $\det\varphi \equiv \det\phi_{\sigma(j)j} = rv$ modulo $|y_j|$. Thus $v \equiv \pm 1$ modulo $|y_j|$ since φ is unimodular and $r = \pm 1$. Now it is easy to see that the diagram

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{\alpha_j} & S^{2n} \vee S^{2n} \\ {\scriptstyle \psi_{\sigma(j)j}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} r & u \\ 0 & \pm 1 \end{pmatrix}} \\ S^{4n-1} & \xrightarrow{\beta_{\sigma(j)}} & S^{2n} \vee S^{2n} \end{array}$$

is homotopy commutative and induces a homotopy equivalence $C_{\alpha_j} \simeq C_{\beta_{\sigma(j)}}$. □

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