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1. Introduction

In this article we study the behaviour of Néron models of abelian varieties with respect to tamely ramified extensions of discrete valuation rings. Let $D$ be a discrete valuation ring with field of fractions $K$ and residue field $k$, and let $A$ be an abelian variety over $K$. Then among all extensions of $A$ to a scheme over $S = \text{Spec}(D)$ there exists a canonical “best” one, the so-called Néron model $\mathcal{A}/S$, named after its discoverer A. Néron. It is characterized by the property that it is smooth over $S$ and that for every smooth morphism of schemes $T \to S$ the induced map $\mathcal{A}(T) \to A(T_K)$ is bijective. For more information about these models we refer to the book [1] on this subject.

Let $K'/K$ be a finite dimensional separable extension of fields, and let $D'$ be the localization at one of the maximal ideals of the integral closure of $D$ in $K'$. Then one can ask what the relations are between $\mathcal{A}$ and $\mathcal{A}'$, where $\mathcal{A}'$ is the Néron model over $S' = \text{Spec}(D')$ of $A_{K'}$. By the defining properties of $\mathcal{A}$ and $\mathcal{A}'$ we get a morphism $\mathcal{A}_S \to \mathcal{A}'$. It is this morphism that we want to understand, especially in the case where $A$ does not have semi-stable reduction over $D$.

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Let us assume that $D'$ is Galois over $D$ with group $G$ (by this we mean that $G$ acts on $D'$ and that $D$ is the subring of $G$-invariants). By the universal property of $\mathcal{A}'$ the right-action of $G$ on $A_{K'}$ extends to a $G$-equivariant right-action of $G$ on $\mathcal{A}'$ over $S'$. Let $X = \prod_{S'/S}(\mathcal{A}'/S')$ denote the Weil restriction of scalars of $\mathcal{A}'$ to $S$ (see Section 2). The group $G$ acts on $X$ in a natural way, and we have a morphism $\mathcal{A} \to X$. The key result of this article is that this morphism $\mathcal{A} \to X$ identifies $\mathcal{A}$ with the closed subscheme $X^G$ of fixed points of $X$, provided that the extension $D'/D$ is tamely ramified. This makes it possible to study $\mathcal{A}$ in terms of $\mathcal{A}'$ with its $G$-action. If $D'/D$ is not tame, then $\mathcal{A}$ is obtained from the closure of $A$ in $X$ by what is called the “smoothening process” in [1] (see [1] 7.2/4). We will restrict ourselves to tamely ramified extensions. Section 5 contains a detailed description of $\mathcal{A}_k$ in terms of $\mathcal{A}'_k$ with its $G$-action, and in Section 6 some criteria for exactness properties of Néron models to be true are given. I want to thank Hendrik Lenstra for suggestions concerning the proof of lemma 3.3, and René Schoof for his computations concerning elliptic curves.
2. Generalities on Weil restriction of scalars

Let \( \pi: S' \to S \) be a morphism of schemes, and let \( X \to S' \) be a scheme over \( S' \). Then the Weil restriction of \( X \) to \( S \), à la Grothendieck, is defined as the functor

\[
\prod_{S/S'} (\text{Sch}/S) \to (\text{Sets}), \quad (T \to S) \mapsto X(T'),
\]

where \( T' = T \times_S S' \) and \( X(T') = \text{Hom}_{S'}(T', X) \).

2.1. REMARK. If we consider \( X \to S' \) as a presheaf for some Grothendieck topology on \((\text{Sch}/S')\), then \( \Pi_{S'/S}(X/S') \) is just the push forward to \((\text{Sch}/S)\). It is proved in \([2]\), exp. 221, 4c, see also Proposition 5.7 and Section 7 of loc. cit., that \( \Pi_{S'/S}(X/S') \) is representable by an open subscheme of the Hilbert scheme of \( X \) over \( S \), if \( \pi: S' \to S \) is proper and flat, and \( X \to S \) quasi-projective. We will use this result only in the case where \( \pi \) is finite and flat, and \( X \to S' \) is quasi-projective. In that case, \( \Pi_{S'/S}(X/S') \to S \) is quasi-projective. It is clear from the definition that the formation of \( \Pi_{S'/S}(X/S') \) commutes with base change on \( S \): for all \( T \to S \), we have \( \Pi_{T'/T}(X_{T'/T}) = \Pi_{S'/S}(X/S') \times_S T \).

2.2. LEMMA. Let \( \pi: S' \to S \) be finite and flat and let \( X \to S' \) be quasi-projective and smooth. Then \( \Pi_{S'/S}(X/S') \) is smooth over \( S \).

Proof. By the remark above, \( \Pi_{S'/S}(X/S') \) is representable. By definition, see \([4]\), IV, 17.3.1, \( X \to S' \) is locally of finite presentation and formally smooth. It follows right from the definitions, see \([4]\), IV, 8.14.2 and 17.3.1, plus the fact that \( S' \to S \) is affine, that \( \Pi_{S'/S}(X/S') \) is locally of finite presentation and formally smooth over \( S \).

2.3. CONSTRUCTION. Suppose that \( X \) is obtained by base change via \( \pi: X = Y' = Y_S', \) for some \( Y \to S \). Then \( \Pi_{S'/S}(X/S')(T) = X(T') = Y_T(T') = Y(T') \). We have natural maps \( Y(T) \to Y(T') \), giving an \( S \)-morphism \( Y \to \Pi_{S'/S}(Y'/S') \). If \( \pi: S' \to S \) is faithfully flat, then all the maps \( Y(T) \to \Pi_{S'/S}(Y'/S')(T) \) are injections.

2.4. CONSTRUCTION. Suppose that a group \( G \) acts, on the right, equivariantly on \( \pi: S' \to S \) and on \( X \to S' \), with the trivial action on \( S \). Then we can define an equivariant right-action by \( G \) on \( \Pi_{S'/S}(X/S') \to S \) by:

\[
P \cdot g = \rho_X(g) \circ P \circ \rho_{T'}(g)^{-1},\]

where \( T \) is a scheme over \( S \), \( P \in \Pi_{S'/S}(X/S')(T) = \text{Hom}_S(T', X) \), \( g \in G \), \( \rho_X(g) \) the automorphism of \( X \) induced by \( g \), \( \rho_{T'}(g) \) the automorphism of \( S' \) induced by \( g \), and \( \rho_{T'}(g) = \rho_S(g) \times 1_T \).

Note that if \( X = Y' \), as in construction 2.3, with \( G \) acting equivariantly on \( Y \to S \), then the map \( Y \to \Pi_{S'/S}(Y'/S') \) of construction 2.3 is \( G \)-equivariant.
3. Generalities about fixed points

Let $X \to S$ be a morphism of schemes, and let $G$ be a finite group acting equivariantly on $X \to S$, with the trivial action on $S$. We define the functor $X^G$ of fixed points by:

$$X^G: (\text{Sch}/S) \to (\text{Sets}), \quad (T \to S) \mapsto X(T)^G.$$  

3.1. Proposition. The functor $X^G$ is represented by a subscheme of $X$. The formation of $X^G$ commutes with base change on $S$. If $X \to S$ is separated, then $X^G$ is a closed subscheme of $X$.

Proof. Let $Z$ be the fibered product, over $X \times_S X$, of the graphs $\Gamma_g: X \to X \times_S X$, where $g \in G$. The $\Gamma_g$ are immersions ([4], I, 5.1.4); hence $Z \to X \times_S X$ is an immersion ([4], I, 4.3.4). Since $Z$ is a subscheme of the diagonal (the graph of the unit element of $G$), we can consider it as a subscheme of $X$. As such, it represents $X^G$. If $X \to S$ is separated, then all the $\Gamma_g$ are closed immersions. Hence $Z \to X \times_S X$ is a closed immersion, and can be considered as a closed subscheme of $X$. \hfill \square

3.2. Proposition. Let $T_{X/S}$ and $T_{X^G/S}$ denote the tangent bundles of $X/S$ and $X^G/S$ (see [4], IV, 16.5.12.) Let $x \in X^G$; then we have:

$$T_{X^G/S}(x) = T_{X/S}(x)^G.$$  

Proof. For arbitrary $X \to S$, and $x \in X$ we have [4], IV, 16.5.13.1:

$$T_{X/S}(x) = \text{Hom}_{k(x)}(\Omega_{k(x)} \otimes \mathcal{O}_x, k(x), k(x)).$$

Let $Y_0 = \text{Spec}(k(x))$, and $Y = \text{Spec}(k(x)[\varepsilon])$, with $\varepsilon^2 = 0$, both considered as schemes over $S$. Let $u_0: Y_0 \to X$ be the canonical morphism with image $x$, and let $i: Y_0 \to Y$ denote the closed immersion of $Y_0$ into $Y$. Then [4], IV, 16.5.17 tells us that

$$T_{X/S}(x) = \{u \in X(Y)|u \circ i = u_0\}.$$  

Applying this to the situation mentioned in the proposition gives:

$$T_{X^G/S}(x) = \{u \in X^G(Y)|u \circ i = u_0\} = \{u \in X(Y)|u \circ i = u_0\}^G = T_{X/S}(x)^G. \hfill \square$$

3.3. Lemma. Let $A$ be a complete local ring with residue field $k$, $d$ a non-negative integer, $B = A[[T_1, \ldots, T_d]]$. Let $G$ be a finite group acting on the $A$-algebra $B$, and let $n = \# G$. Suppose that $A$ is a $\mathbb{Z}[1/n]$-algebra. Then there exist $S_1, \ldots, S_d \in B$, with $B = A[[S_1, \ldots, S_d]]$, such that the $A$-submodule $M = AS_1 \oplus \cdots \oplus AS_d$ is $G$-stable. Moreover, $M$ is then a direct sum of $A[G]$-
modules $M_i$ which are free as $A$-modules, and have the property that $M_i \otimes_A k$ is an irreducible $k[G]$-module.

Proof. Let $m_A$ and $m_B$ denote the maximal ideals of $A$ and $B$. Let $\mathcal{V}$ be a finitely generated $k[G]$-module. We claim that there exists an $A$-free $A[G]$-module $V$, unique up to isomorphism, such that $V \otimes_A k$ is isomorphic to $\mathcal{V}$. To prove this, it suffices to show that for any $m \geq 1$, a morphism of groups $G \to GL_d(A/m^n_A)$ can be lifted to a morphism of groups $G \to GL_d(A/m^n_A + 1)$, and that such a lift is unique up to an inner automorphism of $GL_d(A/m^n_A + 1)$. Now let $K_{m+1}$ be the kernel of $GL_d(A/m^n_A + 1) \to GL_d(A/m^n_A)$; then $K_{m+1}$ is the additive group of a $k$-vector space. Since $n = \# G$ is invertible in $k$, we have $H^i(G, H_{m+1}) = 0$, for all $i > 0$. For $i = 2$ this means that the required lift exists, and for $i = 1$ it means that the lift is unique up to inner automorphisms.

Let $\mathcal{V}$ be the $k[G]$-module $m_B/(Bm_A + m_B^2)$. Note that $\mathcal{V}$ has a $k$-basis consisting of the images of the $T_i$. We can write $\mathcal{V} = \oplus \mathcal{V}_i$, where the $\mathcal{V}_i$ are irreducible $k[G]$-modules. Let $V_i$ be a $A$-free $A[G]$-module with $V_i \otimes_A k \cong \mathcal{V}_i$, and let $V = \oplus V_i$. After changing the $(T_1, \ldots, T_d)$ by an element of $GL_d(A)$, we may assume that each $\mathcal{V}_i$ has a basis consisting of a subset of $\{T_1, \ldots, T_d\}$. We lift the $k$-basis of each $\mathcal{V}_i$ arbitrarily to a $A$-basis of $V_i$, and get a basis of $V$.

Now consider $\phi^i: V \to B$, sending the $i$th vector of the basis to $T_i$. Then we get a $G$-equivariant morphism $\phi: V \to B$ by:

$$\phi := \frac{1}{n} \sum_{g \in G} g \cdot \phi^i \cdot g^{-1}.$$ 

Let $S_i \in B$ be the image, under $\phi$, of the $i$th basis vector. Then $S_i$ and $T_i$ have the same image in $m_B/(Bm_A + m_B^2)$; hence $B = A[S_1, \ldots, S_d]$. \hfill $\square$

3.4. PROPOSITION. If the morphism $f: X \to S$ is smooth, and $n = \# G$ is invertible on $X$, then $X^G$ is smooth over $S$.

Proof. Let $x \in X^G$, and let $s = f(x)$. We have to show that $f$ is smooth at $x$. By an argument of [3] 3, exp. V, 5b, there exist an affine open $G$-stable neighborhood $U$ of $x$ in $X$, and an affine open neighborhood $V$ of $s$, with $fU \subset V_s$, such that $\mathcal{O}_x(U)$ is a $\mathcal{O}_s(V)$-algebra of finite presentation. We can now replace $X \to S$ by $U \to V$. By [4] IV, 8.8.3 and 17.7.8, we may assume that $S$ is noetherian. It is sufficient to show smoothness at $x$ after the (flat) base change $\text{Spec}(\mathcal{O}_{X,x}) \to S$. Then we have $k(x) = k(s)$. Let $A = \mathcal{O}_{S,x}$ and $B = \mathcal{O}_{X,x}$. By [4] IV, 17.5.3, $B$ is isomorphic, as an $A$-algebra, to $A[T_1, \ldots, T_d]$, where $d$ is the relative dimension of $X$ over $S$ at $x$. In order to test the formal smoothness of $X^G$ it is enough to consider only artinian local $A$-algebras ([4] IV, 17.5.4), which can be done in terms of the $G$-action on $B$.

By lemma 3.3, we may assume that the action of $G$ on $A[T_1, \ldots, T_d]$ is linearized: the $A$-submodule generated by the $T_i$ is $G$-stable. We may also
assume that $T_i$ is $G$-invariant for $i \leq e$, and that the $T_i$ with $i > e$ generate a $G$-stable $A$-submodule of $B$ that, modulo $m_A$, contains only non-trivial irreducible representations.

Let $I$ be the ideal in $B$ generated by the $(1 - g)b$, where $g \in G$ and $b \in B$. Then for every $A$-algebra $C$, the $G$-invariant morphisms $B \to C$ are precisely those that factor through $B/I$. Let $J$ be the ideal $\Sigma_{i > e} BT_i$. We want to show that $I = J$. It is immediate that $I \subset J$. For the other inclusion, we consider the morphism of finitely generated $B$-modules $\phi: \bigoplus_{g \in G} J \to J$, which is $1 - g$ on the $g$-component. Since $J/mBJ$ is a direct sum of nontrivial irreducible $k[G]$-modules, $\phi \otimes k$ is surjective. Then $\phi$ is surjective by Nakayama's lemma, which proves that $J \subset I$.

To finish the proof, we note that $B/I$ is a formally smooth $A$-algebra since it is isomorphic to $A[T_1, \ldots, T_e]$. □

3.5. PROPOSITION. Let $G$ be a finite group, acting equivariantly on a smooth morphism of schemes $f : X \to S$. If $\# G$ is invertible on $X$, then the induced morphism $f : X^G \to S^G$ is smooth.

Proof. Apply the preceding proposition to $X \times_SS^G \to S^G$. □

4. Application to Néron models

Let $D$ be a discrete valuation ring with field of fractions $K$, and residue field $k$. Let $K'$ be a finite separable extension of $K$, and let $D'$ be the integral closure of $D$ in $K'$. Note that $D'$ is not necessarily a discrete valuation ring, but that the localizations at its finitely many maximal ideals are. Let $S = \text{Spec}(D)$ and $S' = \text{Spec}(D')$. Let $A$ be an abelian variety over $K$, and let $\mathcal{A}'$ be the Néron model over $S'$ of $A_{K'}$.

4.1. PROPOSITION. $\Pi_{S'/S}(\mathcal{A}'/S')$ is the Néron model over $S$ of the abelian variety $\Pi_{K'/K}(A_{K'}/K')$ over $K$.

Proof. First of all, we have to prove that $\Pi_{K'/K}(A_{K'}/K')$ is an abelian variety over $K$. This is clear after base change to the separable closure $K^{sep}$ of $K$, since then we have:

$$\left(\prod_{K'/K} (A_{K'}/K')\right)_{K^{sep}} = \prod_{\phi: K' \to K^{sep}} A_{K', \phi}.$$

We have already seen in lemma 2.2 that $\Pi_{S'/S}(\mathcal{A}'/S') \to S$ is smooth. Let $T \to S$ be smooth, then:

$$\prod_{S'/S} (\mathcal{A}'/S')(T) = \mathcal{A}'(T') = A_K(T_K) = \prod_{K'/K} (A_{K'}/K')(T_K) = \left(\prod_{S'/S} (\mathcal{A}'/S')\right)_{K}(T_K).$$
Let \( \mathcal{A} \) be the Néron model over \( S \) of \( A \). Construction 2.3 gives us a closed immersion: \( A \hookrightarrow (\Pi_{S'/S}(\mathcal{A}'/S'))_K \). Since \( \mathcal{A} \) is smooth over \( S \), and \( \Pi_{S'/S}(\mathcal{A}'/S') \) has the Néronian property, this closed immersion extends to a morphism \( \mathcal{A} \to \Pi_{S'/S}(\mathcal{A}'/S') \).

4.2. THEOREM. If \( S' \to S \) is tamely ramified, then \( \mathcal{A} \to \Pi_{S'/S}(\mathcal{A}'/S') \) is a closed immersion. If moreover \( K' \) is a Galois extension of \( K \) with group \( G \), then this closed immersion induces an isomorphism between \( \mathcal{A} \) and the subscheme \( (\Pi_{S'/S}(\mathcal{A}'/S'))^G \) of fixed points, where \( G \) acts as in 2.4.

Proof. This can be checked after the base change from \( S \) to its strict henselization. Then \( S' \) is a finite disjoint union of schemes \( S'_i = \text{Spec}(D'_i) \), where the \( D'_i \) are discrete valuation rings, tamely ramified over \( D \). By definition, \( \Pi_{S'/S}(\mathcal{A}'/S') \) is the fibered product, over \( S \), of the schemes \( \Pi_{S'_i/S}(\mathcal{A}'_{S'_i}/S'_i) \). Since all the factors in this fibered product are separated over \( S \) (they are even quasi-projective over \( S \)), it is sufficient to prove that all the morphisms

\[
\mathcal{A} \to \prod_{S'/S} (\mathcal{A}'_{S'_i}/S'_i)
\]

are closed immersions. This means that we have reduced the proof of the theorem to the case where \( S \) is strictly henselian. From now on we assume that \( S \) is strictly henselian.

Let \( X \) denote \( \Pi_{S'/S}(\mathcal{A}'/S') \), and let \( \bar{A} \) be the scheme theoretic closure of the image of \( A = \mathcal{A}_k \) in \( X \) (see [6] 2.1). By definition, the morphism \( \mathcal{A} \to X \) factors through \( \bar{A} \). We will now show that \( \bar{A} \) is a Néron model of \( A \), which proves the theorem. It is clear that \( \bar{A} \) has the Néronian property: for all \( T \to S \) smooth, all elements of \( A(T_K) \) extend uniquely to an element of \( X(T) \), which must be in \( \bar{A}(T) \) by the definition of scheme theoretic closure. It remains to be seen that \( \bar{A} \) is smooth over \( S \). By construction, \( \bar{A} \) is a group scheme over \( S \).

Recall that \( S \) is strictly henselian, and that \( S' \) is tamely ramified. Hence \( K' \) is Galois over \( K \) with Galois group \( G = \mu_n \), where \( n = [K': K] \) is prime to the characteristic of \( k \), and \( \mu_n \) is the group of \( n \)th roots of unity of \( D \). We let \( G \) act, from the right, on \( A_K \) via its right-action on \( \text{Spec}(K') \). By the Néron property, this action extends uniquely to a right-action on \( \mathcal{A}' \), such that \( \mathcal{A}' \to S' \) is equivariant. As in construction 2.4, we get a right \( G \)-action, over \( S \), on \( X \). The morphism \( \mathcal{A} \to X \) is \( G \)-equivariant.

Since the \( G \)-action on \( A \) is trivial, \( \bar{A} \) is a closed subscheme of the scheme of fixed points \( X^G \). It is easy to check that \( X^G_K = A \). By proposition 3.4, \( X^G \) is smooth over \( S \). Now both \( \bar{A} \) and \( X^G \) are closed subschemes of \( X \), flat over \( S \), and they have the same generic fibre. It follows that \( \bar{A} = X^G \). This proves that \( \bar{A} \to S \) is smooth.

4.3. AN EXAMPLE. We will now give an example where \( S' \) is wildly ramified over \( S \), and where \( (\Pi_{S'/S}(\mathcal{A}'/S'))^G \) is not smooth over \( S \). Let \( S = \text{Spec}(\mathbb{Z}) \) and
\( S' = \text{Spec}(\mathbb{Z}[i]) \). In order to have simple equations, we will study the Néron model over \( S \) of a twisted form of the multiplicative group \( G_{m,S} \). To be precise, we will consider only the connected component of the Néron model involved. More information about Néron models of not necessarily proper group schemes can be found in [1] 10.

We write \( G = \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) = \{1, \sigma\} \), and we let \( A_{Q(i)} = G_{m,Q(i)} = \text{Spec}(\mathbb{Q}[i][X, X^{-1}]) \). We let \( G \) act on \( A_{Q(i)} \) by \( \sigma^*: i \mapsto -i, X \mapsto X^{-1} \), and we define \( A \) to be the corresponding quotient. Then \( A \) is a twisted form of \( G_{m,Q} \). According to [1] 10/5, \( A' = G_{m,\mathbb{Z}[i]} \) is the connected component of the Néron model of \( A_{Q(i)} \). Now let \( C \) be any \( \mathbb{Z} \)-algebra. Then we have:

\[
\left( \prod_{S'} \mathcal{A}'/S' \right)(C) = \text{Hom}_{\mathbb{Z}[\mathcal{I}]} \left( \mathbb{Z}[\mathcal{I}][X, X^{-1}], C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{I}] \right)
\]

\[
= (C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{I}])^*
\]

\[
= \{(x + yi, u + vi) \in C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{I}] | (x + yi)(u + vi) = 1\}.
\]

From this we see that \( \Pi_{S'/S}(\mathcal{A}'/S') \) is represented by \( \text{Spec}(\mathbb{Z}[X, Y, U, V]/(XU - YV = 1, XV + YU = 0)) \). Since \( \sigma \) acts by \( (x + yi, u + vi) \mapsto (u - vi, x - yi) \), the subscheme of \( G \)-invariants is given by the equations \( U = X \) and \( V = -Y \). It follows that \( (\Pi_{S'/S}(\mathcal{A}'/S'))^G \) is isomorphic to \( \text{Spec}(\mathbb{Z}[X, Y]/(X^2 + Y^2 = 1)) \), which has a non-smooth fibre at 2. To obtain the Néron model of \( A \), one has to apply the smoothening process of [1] 3 to \( (\Pi_{S'/S}(\mathcal{A}'/S'))^G \). In this case, that amounts to a blow-up in the two points with maximal ideals \( (2, X, Y - 1) \) and \( (2, X - 1, Y) \).

5. The special fibre in the totally ramified case

Let the notation and hypotheses be as in the preceding section. Let \( n = [K': K] \).

In this section we assume moreover that \( D \) contains the \( n \)th roots of unity, and that \( D' \) is a discrete valuation ring with uniformizer \( \pi' \), such that \( \pi = \pi'^n \) is a uniformizer of \( D \). Note that one always gets into this situation if \( D \) is strictly henselian. Then the group \( G = \mu_n \) acts on \( S' \) with quotient \( S \). To be precise, for any \( \zeta \in G \), we define \( \zeta^*(\pi') = \zeta \pi' \). This defines a right-action by \( G \) on \( S' \). Let \( X \) denote \( \Pi_{S'/S}(\mathcal{A}'/S') \); then \( G \) acts on \( X \to S \) and we have seen in the proof of theorem 4.2 that \( \mathcal{A}' = X^G \). This implies that \( \mathcal{A}_k \) is the closed subscheme of fixed points under \( G \) in the restriction of \( \mathcal{A}' \otimes_{D'} (D'/\pi D') \) to \( k = D/\pi D \). Note that \( D'/\pi D' = D[t]/(t^n - \pi, \pi) = k[t]/(t^n) \); hence for any \( k \)-algebra \( C \) we have:

\[
\mathcal{A}_k(C) = X_k^G(C) = X_k(C)^G = \mathcal{A}'(C[t]/(t^n))^G.
\]

According to this formula, in order to understand \( \mathcal{A}_k \) we must know what
$C[t]/(t^n)$-valued points of $\mathcal{A}'$ are, and how $G$ acts on them. Our aim in this section is to describe $\mathcal{A}'$ as accurately as we can in terms of $\mathcal{A}'_k$ together with its $G$-action. The fact that $\mathcal{A}'(C[t]/(t^n))$ depends not only on $\mathcal{A}'_k$, but on its $(n-1)^{st}$ infinitesimal neighborhood probably makes it impossible to describe $\mathcal{A}'_k$ itself. What we will do instead is to consider a natural filtration on $\mathcal{A}'_k$, and describe only the successive quotients.

5.1. The filtration on $X_k$

For any $k$-algebra $C$, and for any $i$ with $0 \leq i \leq n$ we define:

\[(F^i X_k)C = \ker(X_k(C) \to \mathcal{A}'(C[t]/(t^n)) \to \mathcal{A}'(C[t]/(t^i))).\]

This defines a filtration of $X_k$ by subfunctors: $X_k = F^0 X_k \supset F^1 X_k \supset \cdots \supset F^n X_k = 0$. The functor $C \mapsto \mathcal{A}'(C[t]/(t^i))$ is represented by the scheme $\Pi_{(k[t]/(t^i)_k, \mathcal{A}'_k[t]/(t^i))}$; hence the functors $F^i X_k$ are represented by closed subgroup schemes of $X_k$.

We will now investigate the successive quotients of the filtration. For $i$ with $0 \leq i \leq n - 1$ and $C$ any $k$-algebra we define $(\text{Gr}^i X_k)_C = (F^i X_k)_C / ((F^{i+1} X_k)_C)$. We have, for any $C$, that $(\text{Gr}^i X_k)_C = \mathcal{A}'_k(C)$; hence $\text{Gr}^0 X_k = \mathcal{A}'_k$.

To determine the $\text{Gr}^i(X_k)$ for $i > 0$, we choose parameters $T_1, \ldots, T_d$ for the formal group of $\mathcal{A}'$ over $D'$.

Let $P$ be an element of $(F^i X_k)(C)$, with $0 < i < n$; then $P$ corresponds to a morphism of rings $\phi: D'[T_1, \ldots, T_d] \to C[t]/(t^n)$ such that the $\phi(T_j)$ are 0 modulo $t^i$. This means that we can write $\phi(T_j) = \sum_{i=1}^{n} a_{ij} t^i$, with $a_{ij} \in C$. Now consider the $C$-module $T_{\mathcal{A}'_k,0} \otimes_k C$. Elements of this module correspond to morphisms of rings $\psi: D'[T_1, \ldots, T_d] \to C[\varepsilon]/(\varepsilon^2)$ which have $\psi(T_j) \in (\varepsilon)$ for all $j$. It follows that we can associate such a $\psi$ to $P$ by setting $\psi(T_j) = a_{ij} \varepsilon$. This gives us a map:

\[(F^i X_k)C \to (T_{\mathcal{A}'_k,0} \otimes_k (m/m^2)^{\oplus i}) \otimes_k C: P \mapsto \psi \otimes t^i, \tag{5.1.1}\]

where $m$ is the maximal ideal of $D'$. Both source and target of this map are groups; the group structure on the right hand side is induced from the $k$-vector space structure on $T_{\mathcal{A}'_k,0} \otimes_k (m/m^2)^{\oplus i}$. Since the group law on $D[T_1, \ldots, T_d]$ is of the form:

\[(a_1, \ldots, a_d) + (b_1, \ldots, b_d) = (a_1 + b_1, \ldots, a_d + b_d) + \text{higher terms},\]

the map 5.1.1 is actually a morphism of groups. It is clear that it is surjective, and that its kernel is $(F^{i+1} X_k)_C$. This means that we have an isomorphism of group schemes over $k$:

\[\text{Gr}^i X_k \to T_{\mathcal{A}'_k,0} \otimes_k (m/m^2)^{\oplus i}. \tag{5.1.2}\]
It is easy to check that this isomorphism does not depend on the choice of the parameters $T_1, \ldots, T_d$ and the uniformizers $\pi$ and $\pi'$.

5.2. Taking the $G$-invariants

Since $\mathcal{A}_k = X_G^0$, the $F^iX_k$ induce a filtration $F^i\mathcal{A}_k = (F^iX_k)^G$; we denote its successive quotients by $\text{Gr}^i\mathcal{A}_k$. Since the group schemes $F^iX_k$ are unipotent for $i > 0$, and $\#G$ is invertible in $k$, the short exact sequences:

$$0 \to F^{i+1}X_k \to F^iX_k \to \text{Gr}^iX_k \to 0$$

remain exact after taking the $G$-invariants. This means that $\text{Gr}^0\mathcal{A}_k$ is simply $(\mathcal{A}_k)^G$. For $i > 0$ it follows that $\text{Gr}^i\mathcal{A}_k = (T_{\mathcal{A}_k,0} \otimes_k (m/m^2)^{\otimes i})^G$. In order to make the isomorphism 5.1.2 $G$-equivariant, we have to let $G$ act on the target as follows. The right-action on $T_{\mathcal{A}_k,0}$ is induced by the action by $G$ on $\mathcal{A}_k$ from the right which is usually called the inertia action (as defined in the proof of theorem 4.2). The right-action of $\zeta \in G$ on $(m/m^2)^{\otimes i}$ is by multiplication by $\zeta^{-i}$ (note the sign). This means that for $i$ with $0 < i < n$, $\text{Gr}^i\mathcal{A}_k = T_{\mathcal{A}_k,0}[i] \otimes_k (m/m^2)^{\otimes i}$, where $T_{\mathcal{A}_k,0}[i]$ denotes the $k$-subspace of $T_{\mathcal{A}_k,0}$ on which the action of all $\zeta \in G = \mu_n$ is by multiplication by $\zeta^i$. We summarize these results in the following theorem.

5.3. THEOREM. Let $D$ be a discrete valuation ring with field of fractions $K$ and residue field $k$, and let $n$ be a positive integer. We suppose that $n$ is prime to the characteristic of $k$, and that $D$ contains the group $\mu_n$ of $n$th roots of unity. Let $K'/K$ be a totally ramified Galois extension of degree $n$, and let $D'$ be the integral closure of $D$ in $K'$. Let $m$ be the maximal ideal of $D'$, and let $G = \text{Gal}(K'/K)$. We identify $G$ with $\mu_n$ via its action on $m/m^2$: $\zeta \in \mu_n$ acts on $m/m^2$ by multiplication by $\zeta$. Let $A$ be an abelian variety over $K$; let $\mathcal{A}$ and $\mathcal{A}'$ be its Néron models over $\text{Spec}(D)$ and $\text{Spec}(D')$, respectively. By functoriality of the Néron model, $\mu_n$ acts from the right on the situation $\mathcal{A}'/\text{Spec}(D')$, and induces a right action on $\mathcal{A}_k$. The constructions above define a filtration by closed subgroup schemes on $\mathcal{A}_k$:

$$\mathcal{A}_k = F^0\mathcal{A}_k \supset F^1\mathcal{A}_k \supset \cdots \supset F^n\mathcal{A}_k = 0.$$ 

This filtration is functorial with respect to morphism of abelian varieties over $K$. For $0 \leq i < n$ let $\text{Gr}^i\mathcal{A}_k = F^i\mathcal{A}_k/F^{i+1}\mathcal{A}_k$. Then $\text{Gr}^0\mathcal{A}_k = (\mathcal{A}_k)^\mu$, and for $i > 0$ there are natural isomorphisms:

$$\text{Gr}^i\mathcal{A}_k \sim T_{\mathcal{A}_k,0}[i] \otimes_k (m/m^2)^{\otimes i},$$

where $T_{\mathcal{A}_k,0}[i]$ denotes the $k$-subspace of $T_{\mathcal{A}_k,0}$ on which the action of all $\zeta \in \mu_n$ is by multiplication by $\zeta^i$. 

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5.4. REMARKS. 1. Let $\Phi$ and $\Phi'$ denote the groups of connected components of $\mathcal{A}_k$ and $\mathcal{A}_k'$, respectively. Since $F^1\mathcal{A}_k$ is connected, and $Gr^0\mathcal{A}_k = \mathcal{A}_k'$, $\Phi$ is the group of connected components of $(\mathcal{A}_k')^{\mu}$. This induces an exact sequence:

$$0 \to \Phi_0 \to \Phi \to \Phi_1 \to 0,$$

where $\Phi_0$ is the set of connected components of $(\mathcal{A}_k')^{\mu}$ and where $\Phi_1$ denotes the subgroup of $\Phi^{\mu}$ consisting of those components that contain at least one fixed point.

2. Suppose that $D$ has residue characteristic $p > 0$. Let $T_1, \ldots, T_d$ be parameters for the formal group of $\mathcal{A}'$. Then multiplication by $p$ is given by a power series of the form $p f(T_1, \ldots, T_d) + g(T_1, \ldots, T_d)$. It follows that $F^1\mathcal{A}_k$ is annihilated by $p^a$, as soon as $p^a \nmid n$. In particular, if $\mathcal{A}'$ is semi-abelian, then the unipotent part of $\mathcal{A}_k$ is annihilated by $p^a$, as soon as $p^a \nmid n$.

3. Suppose that $k \supset \mathbb{Q}$, and that $ip > n$. Then the power series for the formal logarithm and exponential map induce an isomorphism:

$$F^i\mathcal{A}_k \cong F^i\left( \prod_{(D' \otimes \mathbb{Q}_k)/k} \mathcal{T}_{\mathcal{A}'(D' \otimes \mathbb{Q}_k)(0)}^{\mu_n} \right)$$

If $k \supset \mathbb{Q}$, then this formula is valid for all $i > 0$.

4. It is probably useful to note that every extension $D'$ of $D$, not necessarily tamely ramified, induces a filtration on $\mathcal{A}$. Namely, for every $D$-algebra $C$, consider the morphism $\mathcal{A}(C) \to \mathcal{A}'(C \otimes_D D')$, and for $i \geq 0$ let

$$F^i\mathcal{A}(C) = \ker(\mathcal{A}(C) \to \mathcal{A}'(C \otimes_D D'))$$

where $m$ denotes the maximal ideal of $D'$. The jumps of this filtration occur exactly at the $i$ for which there exists a $P \in \mathcal{A}(D)$ whose image in $\mathcal{A}'(D)$ is $0 \mod m$, but is not $0 \mod m^{i+1}$. For tamely ramified extensions, this gives the same filtration on $\mathcal{A}_k$ as discussed above. Let $\tilde{D}$ denote the minimal extension of $D$ over which $A$ acquires semi-stable reduction, let $\bar{n} = [\tilde{D}: D]$, and let $\tilde{F}$ denote the filtration induced by $\tilde{D}$. Then for every $D'$ containing $\tilde{D}$ we have $F^i\mathcal{A} = F^i[\bar{n}/n]$, where for any real number $x$, $[x]$ denotes the smallest integer $j$ with $j \geq x$.

5. If $D$ is strictly henselian and of residue characteristic $p$, then we have a whole tower of tamely ramified extensions, with Galois group $\varprojlim \mu_n$, where $n$ ranges through the positive integers that are prime to $p$. This induces a filtration $F^i$ on $\mathcal{A}_k$, with $i \in \mathbb{Z}/(p) \cap [0, 1]$, and $F^1\mathcal{A}_k = 0$. Namely, if $i = a/n$, with $n$ not divisible by $p$, then $F^i\mathcal{A}_k$ is the $F^a\mathcal{A}_k'$ induced by the Néron model of $A$ over the extension of degree $n$ of $K$ as above. It follows from our description of $\mathcal{A}_k$ that this does not depend on the choice of $a$ and $n$. It would be interesting to know
where the jumps in this filtration occur. If $A$ acquires semi-stable reduction over a tamely ramified extension $D'$ of degree $n$ of $D$, then for $i \in \mathbb{Z}_{(p)} \cap [0, 1]$ we have $F^{i}A_k = F^{[i]}A_k$ (where the latter filtration is the one induced by $D'$), and the jumps occur at indices $x \in (1/n)\mathbb{Z}/\mathbb{Z}$.

In general, we do not even know if the jumps occur at rational numbers. If $A$ is a jacobian we can say more. Suppose that $A = \text{Pic}^0_{X_K/k}$, where $X_K$ is a smooth, geometrically irreducible curve over $K$. Let $X$ be a regular model over $S$ of $X_K$. We assume in addition that $k = k$ and that the morphism $\text{Pic}^0_{X_K/S} \to \mathcal{A}^0$ is an isomorphism (this happens for example if $X_K(K) \neq \emptyset$, see [1] 9.5/4). After blowing up we may assume that $X_k$ is a divisor with normal crossings (locally for the Zariski topology). Let $l$ be the least common multiple of the multiplicities of the irreducible components of $X_k$. Then we claim that the filtration with indices in $\mathbb{Z}_{(p)} \cap [0, 1]$ as above depends only on the following data: the multiplicities and genera of the irreducible components of $X_k$, and their intersection graph. In particular, it does not depend on $p$. In order to prove this, one has to show that these data suffice to compute, for $n$ prime to $pl$ and $D'/D$ tamely ramified of degree $n$, the action of $\mu_n$ on $T_{\mathcal{A}^0/k}(0)$. The key point here is that one can compute the same combinatorial data for a similar model (i.e., regular, normal crossings) $X'/S'$ of $X_K$. Then one computes the character of the representation of $\mu_n$ on the formal difference $H^0(X'_k, \mathcal{O}_{X'_k}) - H^1(X'_k, \mathcal{O}_{X'_k})$. For elliptic curves we give the index where the (unique) jump in the filtration occurs in the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>$I_0$</th>
<th>$I_1$</th>
<th>$II$</th>
<th>$III$</th>
<th>$IV$</th>
<th>$I^*_0$</th>
<th>$I^*_1$</th>
<th>$IV^*$</th>
<th>$III^*$</th>
<th>$II^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>0</td>
<td>0</td>
<td>1/6</td>
<td>1/4</td>
<td>1/3</td>
<td>1/2</td>
<td>2/3</td>
<td>3/4</td>
<td>5/6</td>
<td></td>
</tr>
</tbody>
</table>

For the meaning of the entries of the first row (Kodaira symbols) we refer to [7].

6. The map $\mathcal{A}_{D'} \to \mathcal{A}'$ is given as follows. For every $D'$-algebra $C$, we have:

$$\mathcal{A}_{D'}(C) = \mathcal{A}(C) = (\mathcal{A}'(C \otimes_D D'))^{\mu_n} \subset \mathcal{A}'(C \otimes_D D') \to \mathcal{A}'(C).$$

In particular, the kernel of the natural map $\mathcal{A}_{k} \to \mathcal{A}_{k}$ is $F^1A_k$, and the image is $(\mathcal{A}'_{k})^{\mu_n}$. Let $\mathcal{X}_{D'}$ be the kernel of $\mathcal{A}_{D'} \to \mathcal{A}'$, so that we have an exact sequence:

$$0 \to \mathcal{X}_{D'} \to \mathcal{A}_{D'} \to \mathcal{A}'. $$

Let $\omega_{\mathcal{X}_{D'}/D'} = 0^{*}\Omega^1_{\mathcal{X}_{D'}/D'}$. Then we have an exact sequence:

$$0 \to \omega_{\mathcal{X}_{D'}/D'} \to \omega_{\mathcal{A}_{D'}/D'} \to \omega_{\mathcal{X}_{D'}/D'} \to 0.$$
By choosing parameters $T_i$ for the formal group of $\mathcal{A}'$ on which each $\zeta \in \mu_n$ acts by $\zeta^#(T_i) = \zeta^{a_i} T_i$, for some $a_i$ with $0 \leq a_i < n$, it can be seen that $\omega_{\mathcal{A}'/D'} \cong \bigoplus D'/(\pi')^a D'$.

6. Exactness and specialization

Let $D$ be a discrete valuation ring of with field of fractions $K$, and let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian varieties over $K$. Let $e$ be the absolute ramification index of $D$: $e$ is the valuation of $p \in D$. Passing to Néron models gives a sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of group schemes over $D$. Suppose that $e < p - 1$. Then it is known ([1] 7.5 Thm. 4) that:

1. If $\mathcal{A}$ is semi-abelian, then $\mathcal{A} \to \mathcal{B}$ is a closed immersion.
2. If $\mathcal{B}$ is semi-abelian, then $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact.
3. If $\mathcal{B}$ is abelian, then $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of abelian schemes over $D$.

In this section we will generalize this type of results to the case where $A, B$ and $C$ do not necessarily have semi-stable reduction over $D$. It turns out that assertions 1 and 2 are still true, provided that $p$ is sufficiently large with respect to $e$ and the dimensions of $A, B$ and $C$. After proving this result, we give an example where $p = 5, e = 1, A$ is an elliptic curve, $A \to B$ is a closed immersion, but $\mathcal{A} \to \mathcal{B}$ is not injective.

6.1. THEOREM. Consider the following statements:

1. $e < (p - 1)/n$, for all $n > 0$ with $\phi(n) \leq 2 \dim(A)$.
2. $A$ acquires semi-stable reduction over a tamely ramified extension $D'$ of $D$ with $e' < p - 1$.
3. $\mathcal{A} \to \mathcal{B}$ is a closed immersion.
4. $e < (p - 1)/n$, for all $n > 0$ with $\phi(n) \leq 2 \dim(B)$.
5. $B$ acquires semi-stable reduction over a tamely ramified extension $D'$ of $D$ with $e' < p - 1$.
6. $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact, and $\mathcal{B} \to \mathcal{C}$ is smooth.

Then we have the following implications: $1 \Rightarrow 2 \Rightarrow 3$ and $4 \Rightarrow 5 \Rightarrow 6$.

Proof. First of all, we may suppose that $D$ is complete, with separably closed residue field. In that case the theory of the preceding section applies. The implications $1 \Rightarrow 2$ and $4 \Rightarrow 5$ follow from the fact that for $n$ the order of an automorphism of a $d$-dimensional semi-abelian variety one has $\phi(n) \leq 2d$. Now assume that 2 holds. Then by the theorem from [1] cited above, $\mathcal{A}' \to \mathcal{B}'$ is a closed immersion. By proposition 7.6/2 of [1], the induced morphism of schemes

$$\prod_{D'/D} \mathcal{A}' \to \prod_{D'/D} \mathcal{B}'$$
is a closed immersion. Since $\mathcal{A}$ and $\mathcal{B}$ are closed subschemes of these, the morphism $\mathcal{A} \to \mathcal{B}$ is a closed immersion too.

Now assume that 5 holds. By the theorem of [1] cited above, the sequence $0 \to \mathcal{A}' \to \mathcal{B}' \to \mathcal{C}'$ is exact. Then the sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact since it is constructed from the sequence above by composing two left-exact functors (namely: push-forward of sheaves, and taking the subsheaf of invariants under a group action). Since passing from $\mathcal{A}$ to $\mathcal{T}_{\mathcal{A}/x}(0)$ is a left exact functor (it is taking the kernel of $\mathcal{A}(k[\epsilon]) \to \mathcal{A}(k)$), we find that

$$0 \to \mathcal{T}_{\mathcal{A}/x}(0) \to \mathcal{T}_{\mathcal{A}/x}(0) \to \mathcal{T}_{\mathcal{A}/x}(0)$$

is exact. By dimension considerations, it follows that $\mathcal{T}_{\mathcal{A}/x}(0) \to \mathcal{T}_{\mathcal{A}/x}(0)$ is surjective. From [4] IV 17.11.1 we can conclude that $\mathcal{B} \to \mathcal{C}$ is smooth. 

6.2. The connection with finite flat group schemes

Let $D, K$ and $e$ be as above. Suppose that we have a closed immersion $A \to B$ of abelian varieties over $K$. Then there exists an abelian subvariety $C$ of $B$ such that the induced morphism $A \times C \to B$ is an isogeny, and we get an exact sequence $0 \to G_K \to A \times C \to B \to 0$, with $G_K$ a finite group scheme over $K$. Let $G_{1,K}$ and $G_{2,K}$ be the images of $G$ under the projections to $A$ and $C$, respectively. After replacing $C$ by $C/(C \cap G_K)$ we may assume that the projections from $A \times C$ to $A$ and $C$ induce isomorphisms $G_K \to G_{1,K}$ and $G_K \to G_{2,K}$.

Now suppose that $B$ has semi-stable reduction over a tamely ramified extension $K'/K$ of degree $n$. Then $\mathcal{A}' \times_{D'} \mathcal{C}' \to \mathcal{B}'$ is flat, by the following argument. By the fibre-wise criterion for flatness ([4] IV 11.3.11), it suffices to show that $\mathcal{A}' \times_{D'} \mathcal{C}' \to \mathcal{B}'$ is flat. The morphism $(\mathcal{A}' \times_{D'} \mathcal{C}')_0 \to \mathcal{B}'_0$ is surjective (consider $l$-torsion, for some $l \neq p$), hence the open part where it is flat ([4] IV 11.1.1) is not empty. Since we deal with a morphism of group schemes, the result follows.

We have now an exact sequence $0 \to G \to \mathcal{A}' \times_{D'} \mathcal{C}' \to \mathcal{B}'$, where $G$ is the closure of $G_{K'}$. The image of $\mathcal{A}' \times_{D'} \mathcal{C}' \to \mathcal{B}'$, which is an open subscheme of $\mathcal{B}'$, represents $(\mathcal{A}' \times_{D'} \mathcal{C}')/G$. Let $G_1$ and $G_2$ denote the closures of $G_{K'}$ in $\mathcal{A}'$ and $\mathcal{C}'$, respectively, and let $\mathcal{H}'$ be the kernel of $\mathcal{A}' \to \mathcal{B}'$. From the diagram:

$$\begin{array}{ccc}
G & \to & G_2 \\
\downarrow & & \downarrow \\
0 \to \mathcal{A}' \to \mathcal{A}' \times_{D'} \mathcal{C}' \to \mathcal{C}' \to 0 \\
\downarrow & & \downarrow \\
\mathcal{B}'
\end{array}$$

it follows that $\mathcal{H}' = G \cap \mathcal{A}' = \ker(G \to G_2)$. Since $G_{K'} \to G_{2,K'}$ is an isomorph-
isomorphism, we can replace $G$ by its finite part $G^f$, $G_i (i = 1, 2)$ by the closure of $G_K^i$ in $G_i$, and still have that $\mathcal{H}' = \ker(G \to G_2)$. Now $G$, $G_1$, and $G_2$ are finite flat group schemes over $D$, all three extending $G^i_K$. In the terminology of Raynaud's article [6], we have that $G$ dominates $G_1$ and $G_2$, and that $G \subseteq G_1 \times_D G_2$. Therefore $G$ is the maximum of $G_1$ and $G_2$. Let $G\_ = G_1 \times_D G_2 / G$; then $G\_ = G_1$ is the minimum of $G_1$ and $G_2$, and it is not hard to see that $\mathcal{H}' = \ker(G_1 \to G\_)$.

Anyhow, given $G_1$ and $G_2$, and the isomorphism between their generic fibres, we know how to describe $\mathcal{H}'$. Also, note that $\mathcal{H}'$ is $p$-power torsion. Finally, we get an exact sequence over $D$:

$$0 \to \mathcal{H} \to \mathcal{A} \to \mathcal{B}, \text{ where } \mathcal{H} = \left( \prod_{D'/D} \mathcal{H}' \right)^{\mu_n}$$

Let us discuss one easy special case: suppose that $G^0_1$ is of multiplicative type (i.e., it is a direct sum of copies of $\mu_{p,D}$). This happens if the abelian variety part of $\mathcal{A}'$ is ordinary. Then $G^0_1 \to G\_ = G_1 \times_D G_2 / G$ is a closed immersion, since $G^0_1$ is minimal as a group scheme over $D$. Hence $\mathcal{H}'^0 = 0$, implying that $\mathcal{H}'$ is unramified over $D'$. Therefore $\mathcal{H}' \cong \Spec(D') \sqcup_i \Spec(D'/\pi'^n D')$, for some set of $n_i > 0$. Taking the quotient gives an exact sequence:

$$0 \to \mathcal{H}' \to G_1 / G^0_1 \to G\_ / G^0_1.$$ 

From this it is not hard to see that for all $i: n_i \leq e' / (p - 1)$ (one uses that $G_1 / G^0_1$ is étale, and that the statement does not change under base change so that one may assume that $G\_ = G_1$ is multiplicative). Now, if $\prod_{D'/D} \mathcal{H}' \neq 0$, then $\mathcal{H}'(D'/\pi'^n D') = (\prod_{D'/D} \mathcal{H}')(k) \neq 0$ (since $\prod_{D'/D} \mathcal{H}'$ is unramified over $D$). But then there exists $i$ such that $n_i \geq n = e'/e$, which implies that $e \geq p - 1$. So we have the following result.

6.3. PROPOSITION. If in the situation above, $e < p - 1$ and $\mathcal{A}_e[p]^0$ is multiplicative, then $\mathcal{A} \to \mathcal{B}$ is a monomorphism (i.e., its kernel is 0).

In this case, we do not know if $\mathcal{A} \to \mathcal{B}$ is a closed immersion.

6.4. AN EXAMPLE. In this example, we will have $\pi = p = 5$, and $n = 6$. Let $D$ be the ring of Witt vectors of $\mathbb{F}_p$, and let $D' = D[\pi']$, with $\pi'^n = \pi'$; hence $e = 1$ and $e' = 6$. Let $E$ be an elliptic curve over $D$ with $j$-invariant equal to 0. Let $\mu_6$ be the group of 6th roots of unity in $D$. For every $\zeta \in \mu_6$, let $g(\zeta)$ be the automorphism of $S' = \Spec(D')$ given by: $g(\zeta)(\pi') = \zeta \pi'$. Then $\zeta \mapsto g(\zeta)$ gives an isomorphism $\mu_6 \cong \Gal(K'/K)$. We define an action by $\mu_6$ on $\mathfrak{S}/D$ as follows: to $\zeta \in \mu_6$ we associate the element $a(\zeta) \in \Aut_D(\mathfrak{S})$ which acts by multiplication by $\zeta$ on the cotangent space at 0 of $\mathfrak{S}$. Let $E = \mathfrak{S}_k$.

We let $\zeta \in \mu_6$ act on $E = E \times \Spec(K') \Spec(K')$ by the automorphism $(a(\zeta)^{-1}, g(\zeta))$, and we let $A$ be the quotient. Then $A$ is an elliptic curve over $K$ (a
twist of $E$), having reduction type II*. To define $B$, we need to know more about $\mathcal{O}[p]$. It is well known (see for example [6] 3.4.7) that $B$ can be described by the equation $X^{p^2} - pX = 0$ in $D[X]$. In $D'[X]$, this equation can be factored: $X \Pi (X^{p^2} - \lambda \pi')$, where $\lambda$ ranges through $\mu_6$. For each $\lambda \in \mu_6$, we get a subgroup scheme $G_\lambda$ of rank $p$ of $B$ described by the equation $X^{p^2} - pX = 0$. We let $E_\lambda = E_K/G_{\lambda,K'}$, and $B_{K'} = \Pi E_\lambda$, where the product is the fibered product over $K'$, ranging over the $\lambda \in \mu_6$. Now we want to descend $B_{K'}$ to $K$, in such a way that the “diagonal” morphism $A_{K'} = E_{K'} \to \Pi E_\lambda$ descends too.

Let us first consider the action of the $a(z)$ on the $G_\lambda$, where $z$ and $\lambda$ are elements of $\mu_6$. From the theory in [6], it follows that $a(z)^* \phi \phi \phi$ acts on $D[X]/(X^{p^2} - pX)$ by $X \mapsto z^a X$, for some $a \in \mathbb{Z}/6\mathbb{Z}$. Since $dX$ generates the cotangent space at $0$ of $\mathcal{O}$, we must have that $a = -1$. It follows that the inverse image $a(z)^{-1}G_\lambda$ is described by the equation $(zX)^p - \pi' X = 0$. Hence $a(z)^{-1}G_\lambda = G_{\lambda^{-2}}$, and $a(z)G_\lambda = G_{\lambda^{-2}}$. This gives us a diagram:

$$
\begin{array}{ccc}
0 & \to & G_\lambda & \to & E_K, \\
\downarrow & & \downarrow \phi & & \downarrow a(z) \\
0 & \to & G_{\lambda^{-2}} & \to & E_K
\end{array}
$$

where the vertical arrows are isomorphisms. In particular, we have the formula $a(z)\phi = \phi_{\lambda^{-2}}a(z)$.

Then we need to know the action of the $g(z)$ on the $G_\lambda$. Since $G_\lambda$ is described by the equation $X^p - \lambda \pi' X = 0$, $g(z)^{-1}G_\lambda$ is given by $X^p - \lambda z' \pi X = 0$. From this we see that $g(z)G_\lambda = G_{\lambda^{-1}}$, and we get a diagram as above, and the formula $g(z)\phi = \phi_{\lambda^{-1}}g(z)$.

Finally, we can give the descent data for $B_{K'}$. For $z \in \mu_6$, we let $c(z)$ be the automorphism of $\Pi E_\lambda$ that sends the factor $E_\lambda$ to the factor $E_{\lambda^{-1}}$ via the automorphism $a(z)^{-1}g(z)$. It follows from the formulas above that this defines an action by $\mu_6$ on $B_{K'}$, compatible with its action on $\text{Spec}(K')$, such that the “diagonal” morphism $(\phi_\lambda)_*: E_K \to B_{K'}$ descends, say to a closed immersion $\phi: A \to B$ over $K$.

Let $\mathcal{X}$ be the kernel of $\phi: \mathcal{A} \to \mathcal{B}$. We have seen that $\mathcal{X} = (\Pi D'[\mathcal{X}])^\text{et}$, where $0 \to \mathcal{X} \to \mathcal{A} \to \mathcal{B}$. Now $\mathcal{A} = \mathcal{O}[\mathcal{X}]$ and $\mathcal{B} = \Pi \mathcal{O}[\mathcal{X}]$, from which it follows that $\mathcal{X} = \mathcal{A} \cap G_\lambda = \text{Spec}(D'[X]/(X^p, \pi' X))$. The next thing we do is to compute $\Pi \mathcal{O}[\mathcal{X}]$. Let $C$ be any $D$-algebra; then

$$
\left( \prod_{D'/D} \mathcal{X}' \right)(C) = \mathcal{X}'(C \otimes_D D') = \text{Hom}_{D'}(D'[X]/(X^p, \pi' X), C \otimes_D D')
$$

$$
= \{ a \in C \otimes_D D' | a^p = 0, \pi'a = 0 \}.
$$
Since $C \otimes_B D' = \bigoplus_{0}^{n-1} C \pi^i$, we can write every $a \in C \otimes_B D'$ uniquely as $a = \sum_{0}^{n-1} a_i \pi^i$. The condition $\pi^i a = 0$ is then equivalent to: $a_i = 0$ for $0 \leq i < n - 1$, and $pa_{n-1} = 0$. The other condition, $a^p = 0$, is then a consequence of $\pi^i a = 0$. So we see that $\mathcal{N} = \mathcal{N}$ is represented by $\text{Spec}(D[\mathcal{Y}]/(p \mathcal{Y}))$. Now let $X \mapsto \alpha \pi^5$ be in $(\Pi_{D'/D}[\mathcal{N}])^*(C)$, and let $\zeta \in \mu_p$. Then, since $a(\zeta^{-1})g(\zeta)$ sends $X$ to $\zeta^{-1}X$ and sends $\pi^5$ to $\zeta^5 \pi^5$, it follows that the action by $\mu_p$ on $\Pi_{D'/D}[\mathcal{N}']$ is trivial. The final conclusion is that $\mathcal{N} = \text{Spec}(D[\mathcal{Y}]/(p \mathcal{Y})) \neq 0$.

6.5. REMARK. In this example, $A$ has dimension 1 and $B$ has dimension 6. With a little bit more work, one can make an example where $A$ has dimension 1 and $B$ has dimension 2. Also, $5$ is the largest prime such that an example as above with $\epsilon = 1$ exists: by theorem 6.1 and proposition 6.3, $\mathcal{A} \rightarrow \mathcal{B}$ will be a closed immersion if $p > 7$, and is injective if $p = 7$.

References