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On the exceptional set for the 2k-twin primes problem

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1. Introduction

Let k be a positive integer, $L = \log N$, $e(x) = e^{2\pi i x}$,

$$\Psi(N,2k) = \sum_{\substack{m,n=N \\ n-m=2k}}^{2N} \Lambda(m)\Lambda(n), \quad \mathfrak{S}(2k) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|k \\ p>2}}\left(\frac{p-1}{p-2}\right)$$

where Λ denotes the von Mangoldt function and Π_p means product over prime numbers. A well known conjecture states that

$$\Psi(N,\ 2k) = \mathfrak{S}(2k)(N-2k)+0(NL^{-A}) \tag{1}$$

for any $A>0$. Conjecture (1) is still open, but several average versions of it are known to be true. For instance, using the circle method it is not difficult to prove that Huxley's density theorem [5] implies that

$$\sum_{k\leqslant H}\Psi(N,\ 2k) \sim 2HN \tag{2}$$

provided $N^{1/6}L^c < H < NL^{-c}$, $c>0$ suitable; see Heath-Brown [4] for a closely related result. Similarly, the Density Hypothesis in the form

$$N(\sigma,\ T) \ll T^{2(1-\sigma)+\varepsilon}$$

implies (2) for $N^{\varepsilon} < H < NL^{-c}$. Here $N(\sigma, T)$ denotes, as usual, the number of zeros $\rho=\beta+i\gamma$ of the Riemann zeta function with $\beta \geqslant \sigma$ and $|\gamma| \leqslant T$. A more precise average form of (1) has been recently investigated by Wolke [14], who proved that an asymptotic formula of the form (1) for the slightly different function $\Psi_2(N,2k) = \Sigma_{2k<n\leqslant N}\Lambda(n)\Lambda(n-2k)$ holds for all $k \leqslant H$ but $0(HL^{-B})$

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exceptions, with any $B > 0$, provided $N^{5/8+\varepsilon} < H < N/2$. Here the constants in the 0-symbol are ineffective. The relation between $\Psi(N, 2k)$ and $\Psi_2(N, 2k)$ is very simple. Indeed

$$\Psi(N, 2k) = \Psi_2(2N, 2k) - \Psi_2(N+2k, 2k).$$

The two functions $\Psi(N, 2k)$ and $\Psi_2(N, 2k)$ can be treated by the same methods, and we found $\Psi(N, 2k)$ slightly simpler to work with. Wolke's method seems to give in fact that every H with $N^{3/5+\varepsilon} < H < N/2$ is admissible. Apparently, the assumption of the Generalized Riemann Hypothesis (GRH), or even of the Generalized Density Hypothesis (GDH), easily implies that the choice of any $H \in [N^{1/2+\varepsilon}, N/2]$ is allowed.

In the present paper we show that the same result can be reached by using existing density estimates. Our result is the following:

THEOREM. *Let* $0 < \varepsilon < 1/2$ *and* $A > 0$ *be arbitrary constants and* $N^{1/2+\varepsilon} < H < N/4$. *Then for any* $0 \leqslant V < N/4$

$$\sum_{k=V}^{V+H} |\Psi(N, 2k) - \mathfrak{S}(2k)(N-2k)|^2 \ll_{\varepsilon,A} HN^2L^{-A}. \tag{3}$$

We remark that the constant in the $\ll$-symbol is ineffective, due to the use of Siegel's theorem. Our Theorem clearly implies:

COROLLARY. *Let* A, B *be arbitrary positive constants and* $0 \leqslant V < N/4$. *Then* (1) *holds for all* $V \leqslant k \leqslant V+H$ *but* $0(HL^{-B})$ *exceptions, provided* $N^{1/2+\varepsilon} < H < N/4$.

The arguments used in the proof of the theorem show that the same result holds for Goldbach's problem, that is, given any interval $I = [x, x+x^{1/2+\varepsilon}]$, for all even numbers $2n \in I$ but $0(|I|L^{-B})$ exceptions we have that

$$r(2n) = 2n\mathfrak{S}(2n) + 0(x \log^{-A} x),$$

where $r(2n) = \Sigma_{2n=h+k} \Lambda(h)\Lambda(k)$. It is interesting to note that, assuming GRH, Hardy and Littlewood [3] proved the estimate $E(X) \ll X^{1/2+\varepsilon}$, where $E(X)$ denotes the number of even numbers in $[1, X]$ which are not a sum of two primes. Hence a simple consequence of their work is the above mentioned result, which we can show without any unproved hypothesis.

It might be still worthwhile mentioning that although it is sufficient to use Montgomery's density estimates (39) and (40), any density theorem weaker than (39) near $\sigma = 1/2$ (i.e. having exponent $1-\sigma/(1/2+c(1/2-\sigma))$ with $c > 1/3$) would lead in our treatment (see (50)) to a final exponent larger than 1/2 in place of $N^{1/2+\varepsilon} < H < N/4$ in (3).

Our treatment is close to that of Hardy and Littlewood [3] and Linnik [6] in the sense that we use directly the connection with L-zeros, and no individual estimates like Vinogradov or Vaughan's are needed. Further, it will turn out from the proof that the Theorem is essentially equivalent to an estimate for an average of exponential sums over primes in short intervals. Namely we implicitly prove the following result (see Perelli [9]). Suppose

$$N^{1/2+\varepsilon} < H < N/2, \qquad \xi = \frac{a}{q} + \eta,\ |\eta| \leqslant \frac{1}{q\sqrt{N}},$$

where $q \leqslant N^{1/2}$ and $(a, q) = 1$. Then for any $A > 0$

$$\sum_{n=N}^{2N} \left| \sum_{m=n}^{n+H} \Lambda(m)e(m\xi) - \frac{\mu(q)}{\varphi(q)} \sum_{m=n}^{n+H} e(m\eta) \right|^2 \ll_{\varepsilon,A} H^2NL^{-A}$$

uniformly for $\xi \in [0, 1]$.

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2. Notation

We will use the following notation.

$$S(\alpha) = \sum_{n=N}^{2N} \Lambda(n)e(n\alpha), \qquad K(\alpha) = \sum_{k \leqslant H} e(-k\alpha),$$

$$Q = N^{1/2}/2, \qquad H = N^{1/2+\varepsilon},$$

$I_{a/q}$ = Farey arc with centre at $a/q = \{a/q + \eta : \eta \in \xi_{a/q}\}$ where

$$\xi_{a/q} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ} \right),$$

$$\xi'_{a/q} = \left(-\frac{L^B}{qN}, \frac{L^B}{qN} \right), \qquad I'_{a/q} = \left\{ \frac{a}{q} + \eta : \eta \in \xi'_{a/q} \right\},$$

$B = 2A + 100$, where A is as in the Theorem,

$$\mathfrak{M} = \bigcup_{q \leqslant L^{B/2}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I'_{a/q}, \qquad \mathfrak{m} = \left(\frac{1}{Q}, 1 + \frac{1}{Q} \right) \Big\backslash \mathfrak{M},$$

$$T(\eta) = \sum_{n=N}^{2N} e(n\eta), \qquad R(\eta, q, a) = S\left(\frac{a}{q} + \eta \right) - \frac{\mu(q)}{\varphi(q)} T(\eta),$$

$$\Psi(N, \chi, \eta) = \sum_{n=N}^{2N} \Lambda(n)\chi(n)e(n\eta),$$

$$\Psi'(N, \chi, \eta) = \Psi(N, \chi, \eta) - \delta_\chi T(\eta), \qquad \delta_\chi = \begin{cases} 0 & \chi \neq \chi_0 \\ 1 & \chi = \chi_0 \end{cases},$$

$$\psi'(t, \chi) = \sum_{n=N}^{N+t} \Lambda(n)\chi(n) - \delta_\chi t,$$

$$\sum_a{}^* = \sum_{\substack{a=1\\(a,q)=1}}^{q}, \qquad \sum_\chi{}^* = \sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\ \mathrm{primitive}}},$$

$$c_q(n) = \sum_a{}^* e\left(n\frac{a}{q}\right), \quad \text{Ramanujan's sum}$$

$$\tau(\chi) = \sum_a{}^* \chi(a)e\left(\frac{a}{q}\right), \quad \text{Gauss' sum}$$

$q \sim M$ means $M < q \leqslant 2M$,

$$N(\sigma, T, \chi) = |\{\rho = \beta + i\gamma : L(\rho, \chi) = 0,\ \beta \geqslant \sigma,\ |\gamma| \leqslant T\}|,$$

$$N_q(\sigma, T) = \sum_{\chi(\mathrm{mod}\, q)} N(\sigma, T, \chi),$$

$$\bar{N}_Q(\sigma, T) = \sum_{q \leqslant Q} \sum_\chi{}^* N(\sigma, T, \chi).$$

$|\mathscr{A}|$ denotes the cardinality of $\mathscr{A}$, c will denote a suitable explicitly computable positive absolute constant, whose value will not necessarily be the same at each occurrence. We will suppose that N is large enough with respect to A. The constants in the $\ll$ and 0-symbols might depend on ε and A, even in an ineffective way.

In the course of the proof of the Theorem it is clearly sufficient to prove the case $H = N^{1/2+\varepsilon}$ with ε sufficiently small, which we assume throughout the paper.

3. Outline of the method

We have

$$\sum_{k=V}^{V+H} \left|\Psi(N, 2k) - \mathfrak{S}(2k)(N-2k)\right|^2 \ll \sum_{k=V}^{V+H} \left|\int_{\mathfrak{m}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha\right|^2$$

$$+ \sum_{k=V}^{V+H} \left|\int_{\mathfrak{M}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha - \mathfrak{S}(2k)(N-2k)\right|^2$$

$$= \sum_1 + \sum_2, \quad \text{say.} \tag{4}$$

Clearly,

$$\sum_1 = \sum_{k=V}^{V+H} \int_{\mathfrak{m}} |S(\xi)|^2 e(-2k\xi)\,\mathrm{d}\xi \int_{\mathfrak{m}} |S(\alpha)|^2 e(2k\alpha)\,\mathrm{d}\alpha$$
$$= \int_{\mathfrak{m}} |S(\xi)|^2 \left(\int_{\mathfrak{m}} |S(\alpha)|^2 K(2(\xi-\alpha)) e(-2V(\xi-\alpha))\,\mathrm{d}\alpha \right) \mathrm{d}\xi.$$

Since

$$K(\alpha) \ll \min(H, |\alpha|^{-1}) \quad \text{and} \quad \int_0^1 |S(\alpha)|^2\,\mathrm{d}\alpha \ll NL$$

we get

$$\sum_1 \ll HNL \sup_{\xi \in \mathfrak{m}} \sum_{j=-1,0,1} \int_{(\xi+j/2-L^{A+2}/H,\, \xi+j/2+L^{A+2}/H)\cap\mathfrak{m}} |S(\alpha)|^2\,\mathrm{d}\alpha + HN^2L^{-A}. \tag{5}$$

By (4) and (5) and the fact that ε can be taken arbitrarily small, the proof of the Theorem is reduced to the proof of the following estimates:

$$\sum_{k=V}^{V+H} \left| \int_{\mathfrak{M}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha - \mathfrak{S}(2k)(N-2k) \right|^2 \ll HN^2L^{-A} \tag{6}$$

and

$$\int_{(\xi-1/H,\, \xi+1/H)\cap\mathfrak{m}} |S(\alpha)|^2\,\mathrm{d}\alpha \ll NL^{-A-1} \tag{7}$$

uniformly for $\xi \in [0,1]$.

4. The major arcs estimate

Clearly

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha$$
$$= \sum_{q\leqslant L^{B/2}} \sum_a{}^{*} e\left(-2k\frac{a}{q}\right) \int_{-L^B/qN}^{L^B/qN} \left| S\left(\frac{a}{q}+\eta\right) \right|^2 e(-2k\eta)\,\mathrm{d}\eta \tag{8}$$

and

$$\left|S\left(\frac{a}{q}+\eta\right)\right|^2=\frac{\mu(q)^2}{\varphi(q)^2}|T(\eta)|^2+0(|R(\eta, q, a)|^2)+0\left(\frac{1}{\varphi(q)}|T(\eta)R(\eta, q, a)|\right).$$

For $q \leqslant L^{B/2}$ we have

$$\begin{aligned}\int_{-LB/qN}^{LB/qN}|T(\eta)|^2e(-2k\eta)\,\mathrm{d}\eta &= \int_{-1/2}^{1/2}|T(\eta)|^2e(-2k\eta)\,\mathrm{d}\eta+0(qNL^{-B})\\ &= N+0(H+qNL^{-B}) \qquad (9)\end{aligned}$$

and

$$\begin{aligned}\int_{-LB/qN}^{LB/qN}|T(\eta)R(\eta, q, a)|\,\mathrm{d}\eta &\leqslant \left(\int_{-LB/qN}^{LB/qN}|T(\eta)|^2\,\mathrm{d}\eta\right)^{1/2}\left(\int_{-LB/qN}^{LB/qN}|R(\eta, q, a)|^2\,\mathrm{d}\eta\right)^{1/2}\\ &\ll N^{1/2}\left(\int_{-LB/qN}^{LB/qN}|R(\eta, q, a)|^2\,\mathrm{d}\eta\right)^{1/2} \qquad (10)\end{aligned}$$

From (8)–(10) we get

$$\int_{\mathfrak{M}}|S(\alpha)|^2e(-2k\alpha)\,\mathrm{d}\alpha=N\sum_{q\leqslant \mathrm{L}^{B/2}}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-2k)+R_1+R_2+R_3 \qquad (11)$$

where

$$R_1 \ll \sum_{q\leqslant L^{B/2}}\frac{1}{\varphi(q)}qNL^{-B} \ll NL^{-B/2+1}, \qquad (12)$$

$$R_2 \ll \sum_{q\leqslant L^{B/2}} q\sup_a \int_{-LB/qN}^{LB/qN}|R(\eta, q, a)|^2\,\mathrm{d}\eta,$$

$$R_3 \ll N^{1/2}\sum_{q\leqslant L^{B/2}}\sup_a\left(\int_{-LB/qN}^{LB/qN}|R(\eta, q, a)|^2\,\mathrm{d}\eta\right)^{1/2}$$

Now we write $S\left(\frac{a}{q}+\eta\right)$ in terms of characters. We have

$$S\left(\frac{a}{q}+\eta\right)=\sum_{b=1}^{q}e\left(\frac{ba}{q}\right)\sum_{\substack{n=N\\ n\equiv b(\mathrm{mod}\, q)}}^{2N}\Lambda(n)e(n\eta).$$

The contribution of the b's with $(b, q) > 1$ is

$$\ll \sum_{\substack{b=1 \\ (b,q)>1}}^{q} \left(\sum_{\substack{p=N \\ p \equiv b (\bmod q)}}^{2N} \log p + \sum_{\substack{n=N \\ n \equiv b (\bmod q) \\ n \neq p}}^{2N} \Lambda(n) \right).$$

Since $q \leqslant Q < N$, the sum over p is empty, so that the total contribution of $(b, q) > 1$ is

$$\ll \sum_{\substack{b=1 \\ (b,q)>1}}^{q} \sum_{\substack{n=N \\ n \equiv b (\bmod q) \\ n \neq p}}^{2N} \Lambda(n) \ll \sum_{\substack{n=N \\ n \neq p}}^{2N} \Lambda(n) \ll N^{1/2}.$$

Hence

$$S\left(\frac{a}{q} + \eta\right) = \sum_{b}{}^{*} e\left(\frac{ba}{q}\right) \sum_{\substack{n=N \\ n \equiv b (\bmod q)}}^{2N} \Lambda(n) e(n\eta) + O(N^{1/2})$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \chi(a) \tau(\bar{\chi}) \Psi(N, \chi, \eta) + O(N^{1/2})$$

uniformly in q, a and η. Hence

$$R(\eta, q, a) = \frac{1}{\varphi(q)} \sum_{\chi} \chi(a) \tau(\bar{\chi}) \Psi'(N, \chi, \eta) + O(N^{1/2}). \tag{13}$$

By partial summation we get

$$\Psi'(N, \chi, \eta) \ll |\psi'(N, \chi)| + \frac{L^B}{q} \sup_{1 \leqslant t \leqslant N} |\psi'(t, \chi)| \tag{14}$$

uniformly for $-\dfrac{L^B}{qN} \leqslant \eta \leqslant \dfrac{L^B}{qN}$. Hence

$$\Psi'(N, \chi, \eta) \ll \frac{L^B}{q} \sup_{t \leqslant N} |\psi'(t, \chi)|. \tag{15}$$

By the Siegel–Walfisz theorem (see [1] Ch. 22, eqs (2) and (3)) we have, for any $C > 0$, that

$$\psi'(x, \chi) \ll x \exp(-c_1(C)\sqrt{\log x}),\ c_1(C) > 0, \tag{16}$$

uniformly for $q \leqslant (\log x)^C$ and $\chi\,(\bmod q)$.

From (13), (15) and (16) we get

$$R(\eta, q, a) \ll N \exp(-c_2(B)L^{1/2}) + N^{1/2}, \; c_2(B) > 0, \tag{17}$$

uniformly for $q \leqslant L^{B/2}$, $a \pmod q$ and $-\dfrac{L^B}{qN} \leqslant \eta \leqslant \dfrac{L^B}{qN}$. Hence from (17) we have that

$$R_2, R_3 \ll N \exp(-c_3(B)L^{1/2}), \; c_3(B) > 0, \tag{18}$$

and from (11), (12) and (18) we get

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha = N \sum_{q \leqslant L^{B/2}} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2k) + O(NL^{-B/2+1}). \tag{19}$$

By (6.14) of [8] we have

$$\sum_{q > L^{B/2}} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2k) \ll L^{-B/2} \frac{k\,\mathrm{d}(2k)}{\varphi(2k)}.$$

Moreover it is well known that

$$\sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2k) = \mathfrak{S}(2k)$$

(see e.g. Vaughan [13], Ch. 3), hence by (19) we get

$$\sum_{k=V}^{V+H} \left| \int_{\mathfrak{M}} |S(\alpha)|^2 e(-2k\alpha)\,\mathrm{d}\alpha - \mathfrak{S}(2k)(N-2k) \right|^2$$
$$\ll N^2 L^{-B+2} \sum_{k=V}^{V+H} \mathrm{d}(2k)^2 + HN^2L^{-A}. \tag{20}$$

By Theorem 2 of [12] we have that the first term in the right-hand side of (20) is

$$\ll HN^2L^{-A}$$

and (6) follows.

5. Preparation for the minor arcs estimate

Let us consider an arbitrary $\xi \in [0, 1]$, which we fix for the remaining part of the paper. All our estimates will be uniform in ξ.

For every $q \leqslant Q$ there is at most one $a, a(\text{mod } q)$, such that $I_{a/q}$ intersects $\left(\xi - \frac{1}{H}, \xi + \frac{1}{H}\right)$. Indeed, the distance between $I_{a_1/q}$ and $I_{a_2/q}$ is at least

$$\frac{1}{q} - \frac{2}{qQ} > \frac{1}{2q} > \frac{2}{H}.$$

Let $1 \leqslant M \leqslant Q$ and

$$Y_M^\xi = \left\{q \sim M : \exists\ a \text{ such that } I_{a/q} \cap \left(\xi - \frac{1}{H}, \xi + \frac{1}{H}\right) \neq \varnothing\right\}.$$

Since

$$\left|\frac{a_1}{q_1} - \frac{a_2}{q_2}\right| \geqslant \frac{1}{q_1 q_2},$$

it follows that

$$|Y_M^\xi| \ll \frac{M^2}{H} + 1. \tag{21}$$

Let $q \leqslant L^{B/2}$. If $I_{a/q} \cap \left(\xi - \frac{1}{H}, \xi + \frac{1}{H}\right) \neq \varnothing$ we have

$$\int_{\xi_{a/q} \setminus \xi'_{a/q}} \left|S\left(\frac{a}{q} + \eta\right)\right|^2 \mathrm{d}\eta \ll \frac{L^2}{q^2} \int_{L^B/qN}^{1/qQ} |T(\eta)|^2 \,\mathrm{d}\eta + \int_{-1/qQ}^{1/qQ} |R(\eta, q, a)|^2 \,\mathrm{d}\eta. \tag{22}$$

Analogously, if $L^{B/2} \leqslant q \leqslant Q$ we have

$$\int_{\xi_{a/q}} \left|S\left(\frac{a}{q} + \eta\right)\right|^2 \mathrm{d}\eta \ll \frac{L^2}{q^2} \int_{-1/qQ}^{1/qQ} |T(\eta)|^2 \,\mathrm{d}\eta + \int_{-1/qQ}^{1/qQ} |R(\eta, q, a)|^2 \,\mathrm{d}\eta. \tag{23}$$

By the observation at the beginning of the paragraph we have that the total contribution of the first integral in the right-hand side of (22) and (23) to the quantity in (7) is

$$\ll \sum_{q \leqslant L^{B/2}} \frac{L^2}{q^2} NqL^{-B} + \sum_{L^{B/2} \leqslant q \leqslant Q} \frac{L^2}{q^2} N \ll NL^{-A-1}. \tag{24}$$

From (22)–(24) we get

$$\int_{(\xi-1/H,\xi+1/H)\cap\mathfrak{m}} |S(\alpha)|^2\,\mathrm{d}\alpha$$

$$\ll L \sup_{1\leqslant M\leqslant Q} \sum_{q\in Y_M^\xi} \int_{-1/qQ}^{1/qQ} |R(\eta, q, a')|^2\,\mathrm{d}\eta + NL^{-A-1}, \tag{25}$$

where a' denotes that $a(\bmod q)$ such that $I_{a/q}$ intersects $\left(\xi-\dfrac{1}{H}, \xi+\dfrac{1}{H}\right)$. Further, the total contribution to (25) of the term $0(N^{1/2})$ in (13) is

$$\ll L \sum_{q\sim M} \frac{N}{qQ} \ll N^{1/2}L \ll NL^{-A-1}. \tag{26}$$

Clearly,

$$\frac{1}{\varphi(q)^2}\int_{-1/qQ}^{1/qQ}\left|\sum_{\chi}\chi(a)\tau(\bar{\chi})\Psi'(N,\chi,\eta)\right|^2\mathrm{d}\eta$$

$$\ll \frac{L^2}{q^2}\sum_{\chi_1}\sum_{\chi_2}|\tau(\chi_1)\tau(\chi_2)|\left(\int_{-1/qQ}^{1/qQ}|\Psi'(N,\chi_1,\eta)|^2\,\mathrm{d}\eta\right)^{1/2}\left(\int_{-1/qQ}^{1/qQ}|\Psi'(N,\chi_2,\eta)|^2\,\mathrm{d}\eta\right)^{1/2},$$

and by Gallagher's lemma (see [2]) this is

$$\ll \frac{L^c}{q^2}\sum_{\chi_1}\sum_{\chi_2}|\tau(\chi_1)\tau(\chi_2)|\left(\int_{N/2}^{2N}(qQ)^{-2}\left|\sum_{\substack{n=x\\ n\in[N,2N]}}^{x+2qQ}(\Lambda(n)\chi_1(n)-\delta_{\chi_1})\right|^2\mathrm{d}x\right)^{1/2}$$

$$\times\left(\int_{N/2}^{2N}(qQ)^{-2}\left|\sum_{\substack{n=x\\ n\in[N,2N]}}^{x+2qQ}(\Lambda(n)\chi_2(n)-\delta_{\chi_2})\right|^2\mathrm{d}x\right)^{1/2}.$$

Put

$$W(\chi)=\left((qQ)^{-2}\frac{1}{N}\int_{N/2}^{2N}\left|\sum_{\substack{n=x\\ n\in[N,2N]}}^{x+2qQ}(\Lambda(n)\chi(n)-\delta_{\chi})\right|^2\mathrm{d}x\right)^{1/2}.$$

From (25) and (26) we have that

$$S=\frac{1}{N}\int_{(\xi-1/H,\xi+1/H)\cap\mathfrak{m}}|S(\alpha)|^2\,\mathrm{d}\alpha$$

$$\ll L^c \sup_{1\leqslant M\leqslant Q}\frac{1}{M^2}\sum_{q\in Y_M^\xi}\sum_{\chi_1}\sum_{\chi_2}|\tau(\chi_1)\tau(\chi_2)|W(\chi_1)W(\chi_2)+L^{-A-1}. \tag{27}$$

At this point we may use the arguments of Saffari and Vaughan [11], Lemmas 5 and 6. First, by the arguments of Lemma 6 of [11], we see that

$$W(\chi) \ll \left(\frac{1}{N(qQ)^2} \sup_{qQ/20N \leqslant \Theta \leqslant 20qQ/N} \int_{N/2}^{3N} \left| \sum_{n=z}^{x+\Theta x} (\Lambda(n)\chi(n) - \delta_\chi) \right|^2 \mathrm{d}x \right)^{1/2} = \tilde{W}(\chi).$$

In $\tilde{W}(\chi)$ we use the following explicit formula (see [1], Ch. 19):

$$\sum_{n \leqslant x} \Lambda(n)\chi(n) - \delta_\chi x = - \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} + E(x, T, \chi),$$

where the sum is over the zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$ and

$$E(x, T, \chi) \ll \frac{NL^2}{T} + N^{1/4}L \tag{28}$$

uniformly for $q \leqslant Q$, $\chi\,(\mathrm{mod}\, q)$, $N/2 \leqslant x \leqslant 3N$ and $2 \leqslant T \leqslant N/2$. Choose

$$T = N^{1/2}L^{A/2+c}. \tag{29}$$

By (28), the contribution of $E(x, T, \chi)$ to $\tilde{W}(\chi)$ is

$$\ll \left(\frac{1}{N} \int_{N/2}^{2N} \frac{NL^{-A-c}}{(qQ)^2} \mathrm{d}x \right)^{1/2} \ll \frac{L^{-A/2-c}}{q},$$

hence its total contribution to (27) is

$$\ll L^c \sup_{1 \leqslant M \leqslant Q} \frac{1}{M^2} \sum_{q \in Y_M^\xi} \frac{M^{-A-c}}{q} \ll L^{-A-1}. \tag{30}$$

By (30) we have that (27) still holds with the $W(\chi)$'s replaced by

$$\tilde{\tilde{W}}(\chi) = \left(\frac{1}{N(qQ)^2} \sup_{qQ/20N \leqslant \Theta \leqslant 20qQ/N} \int_{N/2}^{3N} \left| \sum_{|\gamma| \leqslant T} \frac{(x+\Theta x)^\rho - x^\rho}{\rho} \right|^2 \mathrm{d}x \right)^{1/2} \tag{31}$$

where T is as in (29). Now we use the arguments of Lemma 5 of [11]. We square out the sum in (31), integrate over $[N/2, 3N)$ and use the inequality

$$\frac{(1+\Theta)^\rho - 1}{\rho} \leqslant \min\left(\Theta, \frac{1}{|\rho|} \right),$$

thus getting

$$\tilde{\tilde{W}}(\chi)^2 \ll L^c \sum_{|\gamma| \leqslant Q/q} \sum_{|\gamma'| \leqslant Q/q} N^{\beta+\beta'-2} \min\left(1, \frac{1}{|\gamma-\gamma'|}\right)$$
$$+ \frac{L^c}{(qQ)^2} \sup_{Q/q \leqslant K \leqslant T} K^{-2} \sum_{K<|\gamma|\leqslant 2K} \sum_{K<|\gamma'|\leqslant 2K} N^{\beta+\beta'} \min\left(1, \frac{1}{|\gamma-\gamma'|}\right). \tag{32}$$

We will deal only with the first sum in (32). However, it will be clear from our treatment that, due to the factor K^{-2}, the most critical part in the second sum is when K is minimal, that is $K = Q/q$, which can be estimated in exactly the same way as the first sum.

Put $\delta = 1 - \beta, \delta' = 1 - \beta'$. We may clearly suppose $\delta \leqslant \delta'$. Observing that there are uniformly $\ll L$ zeros of $L(s, \chi)$ in any vertical interval of length 1 we obtain, after subdividing $(0, 1/2]$ into $\ll L$ intervals of the form $\left(\frac{m-1}{L}, \frac{m}{L}\right]$, $1 \leqslant m \leqslant [L/2]+1$, that

$$\tilde{\tilde{W}}(\chi)^2 \ll L^c \sum_{\delta \in J} N\left(1-\delta, \frac{Q}{q}, \chi\right) N^{-2\delta}$$

and hence

$$\tilde{\tilde{W}}(\chi) \ll L^c \sum_{\delta \in J} \left(N\left(1-\delta, \frac{Q}{q}, \chi\right)\right)^{1/2} N^{-\delta} \ll L^c \sum_{\delta \in J} N\left(1-\delta, \frac{Q}{q}, \chi\right) N^{-\delta}, \tag{33}$$

where $\Sigma_{\delta \in J}$ means that the summation is extended to a set J of representatives of the intervals $\left(\frac{m-1}{L}, \frac{m}{L}\right]$, $1 \leqslant m \leqslant [L/2]+1$. By the Cauchy-Schwarz inequality and (33) we get

$$\sum_{\chi} \tilde{\tilde{W}}(\chi) \ll L^c \sum_{\delta \in J} \min\left(N_q\left(1-\delta, \frac{Q}{q}\right), \left(qN_q\left(1-\delta, \frac{Q}{q}\right)\right)^{1/2}\right) N^{-\delta}. \tag{34}$$

We will start from (27), using the estimates (33) and (34) with χ replaced by χ_1 and χ_2. We will denote by δ (resp. δ') the quantity, defined as above, connected with the zeros of $L(s, \chi_1)$ (resp. $L(s, \chi_2)$). We consider the contribution $S_{\delta,\delta'}$ of an arbitrary pair $\delta = \frac{m}{L}, \delta' = \frac{m'}{L}$, that is the contribution of zeros with $\delta \in \left(\frac{m-1}{L}, \frac{m}{L}\right]$ and $\delta' \in \left(\frac{m'-1}{L}, \frac{m'}{L}\right]$. Since the formula (27) is symmetric in χ_1

and χ_2, we can clearly assume $\delta \leqslant \delta'$. Thus from (27), (33) and (34) we obtain

$$S_{\delta,\delta'} \ll L^c \sup_{1 \leqslant M \leqslant Q} \frac{1}{M^2} \sum_{q \in Y_M^\xi} \sum_{\chi} |\tau(\chi)| N\left(1-\delta, \frac{Q}{q}, \chi\right) N^{-\delta}$$

$$\times M^{1/2} \min\left(N_q\left(1-\delta', \frac{Q}{M}\right), \left(MN_q\left(1-\delta', \frac{Q}{M}\right)\right)^{1/2}\right) N^{-\delta'} + L^{-A-3}. \tag{35}$$

6. The minor arcs density estimate

In order to apply the density estimates in their full force we have to reduce the sum in (35) to primitive characters.

A given $\chi (\bmod q)$ is induced by a suitable primitive $\chi^* (\bmod r)$. Since a primitive χ^* (mod r) can induce at most one character $\chi (\bmod q)$ for every fixed q, the number of χ (mod q), $q \in Y_M^\xi$, induced by $\chi^* (\bmod r)$ is

$$\ll |\{q \in Y_M^\xi : q \equiv 0 \ (\bmod\ r)\}| \ll \frac{M}{r}.$$

Hence

$$\sum_{q \in Y_M^\xi} \sum_{\chi} |\tau(\chi)| N\left(1-\delta, \frac{Q}{M}, \chi\right) N^{-\delta}$$

$$\ll L^c \sup_{1 \leqslant R \leqslant M} \sum_{\substack{r \sim R \\ \exists \chi (\bmod q), q \in Y_M^\xi \\ \text{induced by some} \\ \chi^* (\bmod r), \\ \chi^* \text{ primitive}}} \frac{M}{R} R^{1/2} \sideset{}{^*}\sum_{\chi^* (\bmod r)} N\left(1-\delta, \frac{Q}{M}, \chi^*\right) N^{-\delta}. \tag{36}$$

Since a modulus r appears in the sum over r in (36) only if $r \mid q$ for some $q \in Y_M^\xi$, the number of such r is, by (21),

$$\ll \sum_{q \in Y_M^\xi} \mathrm{d}(q) \ll \left(\frac{M^2}{H} + 1\right) M^{\varepsilon'} \tag{37}$$

for every $\varepsilon' > 0$. From (35)–(37) we have that

$$S_{\delta,\delta'} \ll L^c \sup_{1 \leqslant M \leqslant Q} \sup_{1 \leqslant R \leqslant M} (MR)^{-1/2} \min\left(\bar{N}_{2R}\left(1-\delta, \frac{Q}{M}\right),\right.$$

$$\left.\left(\frac{M^2}{H} + 1\right) M^{\varepsilon'} \max_{r \sim R} N_r\left(1-\delta, \frac{Q}{M}\right)\right)$$

$$\times N^{-\delta} \min\left(\max_{q \sim M} N_q\left(1-\delta', \frac{Q}{M}\right), \left(M \max_{q \sim M} N_q\left(1-\delta', \frac{Q}{M}\right)\right)^{1/2}\right) N^{-\delta'} + L^{-A-3} \tag{38}$$

where $\delta \leqslant \delta'$ and $\varepsilon' > 0$.

We will use the following Ingham's type density estimates in the range $1/2 \leqslant \sigma \leqslant 4/5$

$$\begin{aligned} &\sum_{\chi(\mathrm{mod}\, q)} N(\sigma, T, \chi) \ll (qT)^{3(1-\sigma)/(2-\sigma)}(\log qT)^c \\ &\sum_{q \leqslant Q} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (Q^2T)^{3(1-\sigma)/(2-\sigma)}(\log QT)^c, \end{aligned} \tag{39}$$

and in the range $4/5 \leqslant \sigma < 1$ the estimates

$$\begin{aligned} &\sum_{\chi(\mathrm{mod}\, q)} N(\sigma, T, \chi) \ll (qT)^{\frac{5}{2}(1-\sigma)}(\log qT)^c \\ &\sum_{q \leqslant Q} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (Q^2T)^{\frac{5}{2}(1-\sigma)}(\log QT)^c, \end{aligned} \tag{40}$$

see Montgomery [7], Ch. 12. We will use also the following zero-free region (see Prachar [10], Ch. 8)

$$L(s, \chi) \neq 0 \quad \text{for } \sigma > 1 - \frac{c}{\max(\log q, \log^{4/5}(|t|+2))} \tag{41}$$

except for the possible Siegel zero β_0 satisfying $1-\beta_0 \ll_{\varepsilon''} q^{-\varepsilon''}$, for every $\varepsilon'' > 0$.

According to the above density estimates we distinguish two main cases.

Case 1. $\delta \geqslant 1/5$. By (38) and (39) we get

$$\begin{aligned} S_{\delta,\delta'} \ll L^c &\sup_{1 \leqslant M \leqslant Q} \sup_{1 \leqslant R \leqslant M} (MR)^{-1/2} \\ &\times \min\left(\left(\frac{R^2Q}{M}\right)^{\frac{3\delta}{1+\delta}}, \left(\frac{M^2}{H}+1\right) M^{\varepsilon'} \left(\frac{RQ}{M}\right)^{3\delta/(1+\delta)}\right) N^{-\delta} \\ &\times \min(Q^{3\delta'/(1+\delta')}, M^{1/2}Q^{3\delta'/2(1+\delta')})N^{-\delta'} + L^{-A-3}. \end{aligned} \tag{42}$$

Here we substitute the value $Q = N^{1/2}/2$. Since the exponent of R in (42) is non-negative, the supremum over R is attained at $R = M$. Hence let us take $R = M$ in the following. Now we show that is always possible to assume $\delta = \delta'$ in (42). The exponent of N in the δ'-variable is

$$f_1(\delta') = \frac{3\delta'}{2(1+\delta')} - \delta' \quad \text{or} \quad f_2(\delta') = \frac{3\delta'}{4(1+\delta')} - \delta'$$

according to which value of the minimum is taken. It is clear that $f_1(\delta')$ increases from 1/5 to $\sqrt{3/2}-1$ and decreases from $\sqrt{3/2}-1$ to 1/2, while $f_2(\delta')$ decreases from 1/5 to 1/2. Hence

$$\sup_{\delta'\in[\delta,1/2]} f_2(\delta') = f_2(\delta).$$

Concerning $f_1(\delta')$, there are three possibilities for the mutual position of δ and δ':

(i) $\sqrt{3/2}-1 \leqslant \delta \leqslant \delta'$. In this case, we decrease δ' to δ, and $f_1(\delta')$ increases. Hence we may assume $\delta=\delta'$ in this case.

(ii) $\delta \leqslant \delta' \leqslant \sqrt{3/2}-1$. In this case we raise δ to δ'. The exponent of N in the δ-variable is again $f_1(\delta)$, which increases during this process. The exponent of M in the δ-variable (considering $R = M$) is $3\delta/(1+\delta)$ or constant, which again does not decrease in this process. Hence we may assume $\delta=\delta'$ also in this case.

(iii) $\delta \leqslant \sqrt{3/2}-1 \leqslant \delta'$. In this case we raise δ to $\sqrt{3/2}-1$ and decrease δ' to $\sqrt{3/2}-1$. By (i) and (ii) nothing is decreasing in this process, and so we may again assume $\delta=\delta'$ $(=\sqrt{3/2}-1)$.

In conclusion, we may assume $R=M$ and $\delta=\delta'$ in (42).
Suppose now

$$M \geqslant H^{1/2}. \tag{43}$$

In this case from (42) we get

$$\begin{aligned} S \ll L^c \sup_{\sqrt{H}\leqslant M\leqslant\sqrt{N}} \sup_{\delta\in[1/5,1/2]} N^{3\delta/2(1+\delta)-\delta+3\delta/4(1+\delta)-\delta} M^{-1} \\ \times \min(M^{3\delta/(1+\delta)}, M^{2+\varepsilon'}N^{-(1/2+\varepsilon)}) \min(N^{3\delta/4(1+\delta)}, M^{1/2}) + L^{-A-1}. \end{aligned} \tag{44}$$

We will assume, as we may, from now on that

$$\varepsilon' \ll \varepsilon^3. \tag{45}$$

Let us consider first the case when the first minimum in (44) is attained by the first term, that is

$$N^{(1+2\varepsilon)(1+\delta)/(2(2-\delta)+2\varepsilon'(1+\delta))} \leqslant M \leqslant N^{1/2}. \tag{46}$$

In this case we choose the first term in the first minimum and the first term in the second minimum of (44), thus obtaining

$$S \ll L^c \sup_{M} \sup_{\delta\in[1/5,1/2]} M^{(2\delta-1)/(1+\delta)} N^{3\delta/(1+\delta)-2\delta} + L^{-A-1},$$

where the $\sup_M$ is restricted by (46). The worst case is when M is minimal, hence

$$S \ll L^c \sup_{\delta\in[1/5,1/2]} N^{3\delta/(1+\delta)-2\delta+(1+2\varepsilon)(2\delta-1)/(2(2-\delta)+2\varepsilon'(1+\delta))} + L^{-A-1}. \tag{47}$$

We observe that (45) and (46) imply

$$\delta \leqslant \frac{1}{2} - c_1(\varepsilon, \varepsilon'), \qquad c_1(\varepsilon, \varepsilon') = \frac{3}{2}\varepsilon - \frac{3}{2}\varepsilon^2 + 0(\varepsilon^3) > 0 \tag{48}$$

for ε sufficiently small. By (45), the exponent in (47) is equal to

$$\frac{3\delta}{1+\delta} - 2\delta + \frac{(1+2\varepsilon)(2\delta-1)}{2(2-\delta)} + 0(\varepsilon^3). \tag{49}$$

Denoting by $g_\varepsilon(\delta)$ the function of δ in (49), we have

$$2(2-\delta)(1+\delta)g_\varepsilon(\delta) \leqslant 2(2-\delta)(1+\delta)g_0(\delta) = (2\delta-1)^2(\delta-1), \tag{50}$$

and from (47)–(50) we deduce that

$$S \ll L^c N^{-c_2(\varepsilon,\varepsilon')} + L^{-A-1}$$

with a suitable $c_2(\varepsilon, \varepsilon') > 0$. Hence the result follows in Case 1 under the assumption (43) and (46).

Suppose now

$$1 \leqslant M \leqslant \min(N^{(1+2\varepsilon)(1+\delta)/(2(2-\delta)+2\varepsilon'(1+\delta))}, N^{1/2}). \tag{51}$$

In this case we choose the second term in the first minimum and still the first term in the second minimum of (44), thus obtaining

$$S \ll L^c \sup_M \sup_{\delta\in[1/5,1/2]} N^{3\delta/(1+\delta)-2\delta-(1/2+\varepsilon)} M^{1+\varepsilon'} + L^{-A-1}, \tag{52}$$

where $\sup_M$ is restricted by (51). By (48) we know that the first term in the minimum in (51) is smaller if $\delta \leqslant 1/2 - c_1(\varepsilon, \varepsilon')$. Thus, as the exponent of M in (52) is positive, we get

$$\begin{aligned} S \ll L^c &\sup_{\delta\in[1/5,1/2-c_1(\varepsilon,\varepsilon')]} N^{3\delta/(1+\delta)-2\delta-(1/2+\varepsilon)+(1+\varepsilon')(1+2\varepsilon)(1+\delta)/(2(2-\delta)+2\varepsilon'(1+\delta))} \\ &+ L^c \sup_{\delta\in[1/2-c_1(\varepsilon,\varepsilon'),1/2]} N^{3\delta/(1+\delta)-2\delta-(1/2+\varepsilon)+1/2(1+\varepsilon')} + L^{-A-1}. \end{aligned} \tag{53}$$

The exponent of N in the first term in (53) is

$$g_\varepsilon(\delta) + O(\varepsilon^3), \tag{54}$$

so that by (50) and (54) the first term in (53) is

$$\ll L^c N^{-c_3(\varepsilon,\varepsilon')}, \quad c_3(\varepsilon, \varepsilon') > 0.$$

The exponent of N in the second term in (53) is equal to

$$\frac{3\delta}{1+\delta} - 2\delta - \varepsilon + O(\varepsilon^3),$$

whose maximum in $[1/2 - c_1(\varepsilon, \varepsilon'), 1/2]$ is attained at $\delta = 1/2 - c_1(\varepsilon, \varepsilon')$, giving the estimate

$$\ll L^c N^{-3\varepsilon^2 + O(\varepsilon^3)}$$

to the second term in (53). The result follows then in Case 1 under the assumption (43).

Suppose now $M \leqslant H^{1/2}$. In this case we choose the second term in the first minimum and the second term in the second minimum of (42) (with $R = M$ and $\delta = \delta'$), thus obtaining

$$S \ll L^c \sup_{1 \leqslant M \leqslant \sqrt{H}} \sup_{\delta \in [1/5, 1/2]} M^{\varepsilon' - 1/2} N^{9\delta/4(1+\delta) - 2\delta} + L^{-A-1}.$$

For $\delta \geqslant 1/5$ the exponent of N is $\leqslant -1/40$. The result follows then in Case 1.

Case 2. $0 < \delta \leqslant 1/5$. We choose the first term in the first minimum and the second term in the second minimum of (38). By (39) and (40) we get from $\delta \leqslant \delta'$ and $R \leqslant M$ that

$$S_{\delta,\delta'} \ll L^c \sup_{1 \leqslant M \leqslant Q} \sup_{1 \leqslant R \leqslant M} \sup_{\delta \in (0,1/5]} \sup_{\delta' \in [\delta, 1/2]} M^{-5\delta/2} R^{5\delta - 1/2} N^{5\delta/4 - \delta + 5\delta'/8 - \delta'} + L^{-A-3}$$

$$\ll L^c \sup_{1 \leqslant R \leqslant \sqrt{N}} \sup_{\delta \in (0,1/5]} R^{5\delta/2 - 1/2} N^{-\delta/8} + L^{-A-3}. \tag{55}$$

If $1/10 \leqslant \delta \leqslant 1/5$ the result follows easily from (55). If $0 < \delta \leqslant 1/10$ let $D = 4(A + 3 + c)$, where c is as in (55). If $R \geqslant L^D$ then (55) gives

$$S_{\delta,\delta'} \ll L^c L^{-D/4} + L^{-A-3} \ll L^{-A-3}.$$

If $R \leqslant L^D$ then (41) implies, choosing $\varepsilon'' = 1/2D$, that

$$\delta \geqslant \min\left(\frac{c_4(D)}{L^{1/2}}, \frac{c}{\max(D \log L, L^{4/5})}\right) = \frac{c}{L^{4/5}},$$

hence from (55) we obtain

$$S_{\delta,\delta'} \ll L^c N^{-c/8L^{4/5}} + L^{-A-3} \ll L^{-A-3},$$

and the Theorem is proved.

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